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OSCILLATOR-LIKE COHERENT STATES FOR THE JAYNES-CUMMINGS MODEL

Y. Bérubé-Lauzière, V. Hussin, and L.M. Nieto

CRM, Université de Montréal, C.P. 6128, succ. centre ville

Montréal (Québec), H3C 3J7, Canada

Abstract

A new way of diagonalizing the Jaynes-Cummings Hamiltonian is proposed, which allows the definition of annihilation operators and coherent states for this model. Mean values and dispersions over these states are computed and interpreted.

1 Introduction

The Jaynes-Cummings (J.C.) model [1], which is extensively used in Quantum Optics, describes, in its simplest version, the interaction of a cavity mode with a two-level system. In the rotating-wave approximation, it may be described by the Hamiltonian [1, 2]

$$H_{JC} = \omega\left(a^\dagger a + \frac{1}{2}\right)\sigma_0 + \frac{\omega_0}{2}\sigma_3 + \kappa(a^\dagger\sigma_- + a\sigma_+)\,, \tag{1}$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ are Pauli matrices, σ_0 is the identity, $\sigma_\pm = \sigma_1 \pm i\sigma_2$, and $a^\dagger$ and a are the photon creation and annihilation operators. Moreover, κ is the coupling constant, ω is the field mode frequency, and ω_0 is the atomic frequency. Let us also introduce the detuning $\Delta = \omega - \omega_0$. The exact solvability of this model is well-known. Working in the Fock space

$$\mathcal{F} = \mathcal{F}_b \otimes \mathcal{F}_f = \left\{|n,-\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix},\ |n,+\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix},\ n = 0,1,2,\ldots\right\}, \tag{2}$$

the energy eigenstates take the form (for $n = 0, 1, 2, \ldots$)

$$|E_0^-\rangle = |0,-\rangle\,, \tag{3}$$

$$|E_{n+1}^-\rangle = \frac{1}{R(n+1)}\left(\kappa\sqrt{n+1}|n,+\rangle + \left(\frac{\Delta}{2} + \kappa\, r(n+1)\right)|n+1,-\rangle\right), \tag{4}$$

$$|E_n^+\rangle = \frac{1}{R(n+1)}\left(\left(\frac{\Delta}{2} + \kappa\, r(n+1)\right)|n,+\rangle - \kappa\sqrt{n+1}|n+1,-\rangle\right), \tag{5}$$

where

$$r(n) = (\delta + n)^{1/2}, \quad \delta = \left(\frac{\Delta}{2\kappa}\right)^2, \quad R(n) = \left[\left(\frac{\Delta}{2} + \kappa\, r(n)\right)^2 + \kappa^2 n\right]^{1/2}. \tag{6}$$

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In the expression of $r(n)$ we have introduced the parameter δ which will be important in the following. The corresponding energy eigenvalues are

$$E_n^- = \omega n + \kappa\, r(n), \quad E_n^+ = \omega(n+1) - \kappa\, r(n+1). \tag{7}$$

The interest of this model, its solvability and its applications have long been discussed. More precisely, dynamical properties have been obtained through the use of states which are initially harmonic oscillator coherent states [3], but that evolve according to the J.C. Hamiltonian [2, 4]. Here, we construct new coherent states which correspond to eigenstates of an annihilation operator for H_{JC}. To do that, we have to find first such an annihilation operator through the diagonalization of the Jaynes-Cummings Hamiltonian (1). Second, we use the theoretical approach, based on the direct product of the Weyl-Heisenberg group with SU(2), to evaluate those coherent states. Finally, we exhibit some of their properties. More details can be found in [5].

2 Annihilation operators and coherent states for H_{JC}

The diagonalization of H_{JC} is performed by the unitary operator $\mathcal{O}$, so that

$$H_d \equiv \mathcal{O}^\dagger H_{JC} \mathcal{O} = \begin{pmatrix} \omega(N+1) - \kappa\, r(N+1) & 0 \\ 0 & \omega N + \kappa\, r(N) \end{pmatrix}, \tag{8}$$

where $N = a^\dagger a$, and the definition of $r(N)$ is given in Eq. (6). This operator $\mathcal{O}$ has the form

$$\mathcal{O} = \begin{pmatrix} \frac{1}{R(N+1)}\left[\frac{\Delta}{2} + \kappa\, r(N+1)\right] & \frac{\kappa}{R(N+1)}\, a \\ -a^\dagger \frac{\kappa}{R(N+1)} & \frac{1}{R(N)}\left[\frac{\Delta}{2} + \kappa\, r(N)\right] \end{pmatrix}. \tag{9}$$

Clearly, an annihilation operator for H_d is given by $A_d = a\,\sigma_0$. Since the states depend also on the spin index $\pm$, we introduce the spinorial annihilation and creation operators:

$$\Sigma_{-d} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_{+d} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

We then obtain candidates to be annihilation and creation operators for H_{JC} as:

$$A = \mathcal{O} A_d \mathcal{O}^\dagger, \quad A^\dagger = \mathcal{O} A_d^\dagger \mathcal{O}^\dagger, \quad \Sigma_\pm = \mathcal{O} \Sigma_{\pm d} \mathcal{O}^\dagger. \tag{11}$$

For the determination of coherent states, the situation is particularly simple when we work with H_d in (8), since the energy eigenstates are the Fock-space basis vectors (2). The group theoretical approach to coherent states leads us to define

$$|z,\beta\rangle_d = T(z,\beta)_d |0,-\rangle, \qquad z,\, \beta \in \mathbf{C}, \tag{12}$$

in terms of the unitary representation of the direct product of the Weyl-Heisenberg group with $SU(2)$

$$T(z,\beta)_d = \exp[z A_d^\dagger - \bar{z} A_d + \beta \Sigma_{+d} - \bar{\beta} \Sigma_{-d}]. \tag{13}$$

The coherent states for $H_{\rm JC}$ are then given by

$$|z,\beta\rangle = \mathcal{O}|z,\beta\rangle_d = T(z,\beta)|E_0^-\rangle = \mathcal{O}T(z,\beta)_d\mathcal{O}^\dagger|E_0^-\rangle\,. \tag{14}$$

From the harmonic oscillator coherent states we easily get

$$|z,\beta\rangle_d = \cos(\theta/2)\begin{pmatrix} 0 \\ |z\rangle \end{pmatrix} + e^{i\phi}\sin(\theta/2)\begin{pmatrix} |z\rangle \\ 0 \end{pmatrix} = \cos(\theta/2)\,|z-\rangle_d + e^{i\phi}\sin(\theta/2)\,|z+\rangle_d\,, \tag{15}$$

with $\beta = (\theta/2)e^{i\phi}$, and $|z\rangle$ the normalized state $|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}(z^n/\sqrt{n!})|n\rangle$. The state $|z,\beta\rangle_d$ is a linear combination of the "fundamental coherent states" $|z+\rangle_d$ and $|z-\rangle_d$, which are both eigenstates of A_d (but not of Σ_{-d}). Similar fundamental coherent states are defined by using such a decomposition of $|z,\beta\rangle$ in (14).

To be complete, we give the time evolution of such states; the one of the general state (15) will be obtained easily by linear combination. We start with the diagonal case, for which the evolution operator is

$$U_d(t) = e^{-itH_d} = \begin{pmatrix} e^{it[\omega(N+1)-\kappa r(N+1)]} & 0 \\ 0 & e^{it[\omega N+\kappa r(N)]} \end{pmatrix}, \tag{16}$$

and we compute $|z,\pm,t\rangle = \mathcal{O}U_d(t)|z,\pm\rangle_d = U_{\rm JC}(t)\mathcal{O}|z,\pm\rangle_d$. More precisely, we get

$$\begin{aligned} |z,+,t\rangle &= e^{-|z|^2/2}e^{-i\omega t}\sum_{n=0}^{\infty}\frac{\left(ze^{-i\omega t}\right)^n}{\sqrt{n!}}\,e^{it\kappa r(n+1)}|E_n^+\rangle, \\ |z,-,t\rangle &= e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{\left(ze^{-i\omega t}\right)^n}{\sqrt{n!}}\,e^{-it\kappa r(n)}|E_n^-\rangle\,. \end{aligned} \tag{17}$$

These are similar to the states obtained by the evolution of the harmonic oscillator coherent states, except for the supplementary oscillation for each n in the sum. This implies that the coherent state does not evolve in time to another coherent state, unlike to the case of the harmonic oscillator. Moreover, our states (17) are different from those considered in other approaches [2, 4]. Indeed, all these authors deal with the states $|z,\pm\rangle_d$ (or a mixture of them) and their evolution is

$$U_{\rm JC}(t)|z,\pm\rangle_d = e^{-itH_{\rm JC}}|z,\pm\rangle_d\,. \tag{18}$$

3 Relevant physical quantities

Let us recall the introduction of the parameter δ in the expression of $r(n)$ in (6). It will be used as a variable in the following. It contains both the detuning Δ and the coupling parameter κ and leads to the exact resonance case when $\delta = 0$ or to the weak coupling limit when $\delta \to \infty$. The parameter $x = |z|^2$ is also introduced, and will be proved to be a good approximation of the number of photons. We will deal with the function

$$G(\delta,x) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\,r(n+1) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\sqrt{\delta+n+1}\,. \tag{19}$$

Its asymptotic behaviour is $G(\delta,x) \sim \sqrt{x}$, which is independent of δ. The calculations will be done explicitly over the states (17), and for a general state $|z,\beta,t\rangle$. We can use (15) to write the mean value and dispersion of an operator X. Indeed, defining $\langle X\rangle_\pm = \langle z,t,\pm|X|z,t,\pm\rangle$ and $\langle X\rangle_{\pm\mp} = \langle z,t,\pm|X|z,t,\mp\rangle$, the mean value of X over a general coherent state is

$$\langle X\rangle = \frac{1-\cos\theta}{2}\langle X\rangle_+ + \frac{1+\cos\theta}{2}\langle X\rangle_- + \frac{\sin\theta}{2}\left(e^{i\phi}\langle X\rangle_{+-} + e^{-i\phi}\langle X\rangle_{-+}\right). \tag{20}$$

In the case of having $\langle X\rangle_{+-} = \langle X\rangle_{-+} = 0$, the square of the dispersion is simply

$$(\Delta X)^2 \equiv \langle X^2\rangle - \langle X\rangle^2 = \frac{1-\cos\theta}{2}(\Delta X)_+^2 + \frac{1+\cos\theta}{2}(\Delta X)_-^2 + \frac{\sin^2\theta}{4}[\langle X\rangle_+ - \langle X\rangle_-]^2. \tag{21}$$

Let us now compute the relevant physical magnitudes. The operator $\mathcal{N} = (a^\dagger a + 1/2)\sigma_0 + \sigma_3/2$ corresponds to the total number of particles. It is a constant of motion and is invariant under the transformation by $\mathcal{O}$. We then get

$$\langle\mathcal{N}\rangle_+ = x+1, \quad \langle\mathcal{N}\rangle_- = x, \quad (\Delta\mathcal{N})_+^2 = (\Delta\mathcal{N})_-^2 = x\,. \tag{22}$$

(Those are known results in connection with the susy harmonic oscillator.)

The evaluation of the mean values of the number of photons $N = a^\dagger a$ is less trivial, and gives

$$\langle N\rangle_+ = x + \frac{1}{2} - \frac{\Delta}{4\kappa}e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{1}{r(n+1)}, \quad \langle N\rangle_- = x - \frac{1}{2} + \frac{\Delta}{4\kappa}e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{1}{r(n)}. \tag{23}$$

Comparing with the harmonic oscillator, where $\langle N\rangle = x$, we have a correction due to the interaction. Since κ is usually small, a good approximation to the average number of photons is x. Indeed, the contribution of the terms containing the series is, in this case, approximately $1/2$.

We can compute the mean values and dispersions of the energy in the fundamental states and study their behaviour with respect to both x and δ. Since $\langle H_{\rm JC}\rangle_{+-} = \langle H_{\rm JC}\rangle_{-+} = 0$, the calculations over the general coherent states through Eqs. (20)–(21) do not give anything new with respect to the results for the fundamental states. Indeed, we can see from (21) that the dispersion attains his minimum over the pure states. The mean values are easily computed and take the simple form

$$\langle H_{\rm JC}\rangle_+ = \omega\Big[(x+1) - \lambda G(\delta,x)\Big], \quad \langle H_{\rm JC}\rangle_- = \omega\Big[x + \lambda G(\delta-1,x)\Big], \tag{24}$$

while the values of the dispersion are more complicated and present interesting features

$$(\Delta H_{\rm JC})_+^2 = \omega^2\left[\lambda^2(1+\delta) + (1+\lambda^2)x + 2\lambda x(G(\delta,x) - G(\delta+1,x)) - \lambda^2(G(\delta,x))^2\right], \tag{25}$$

$$(\Delta H_{\rm JC})_-^2 = \omega^2\left[\lambda^2\delta + (1+\lambda^2)x - 2\lambda x(G(\delta-1,x) - G(\delta,x)) - \lambda^2(G(\delta-1,x))^2\right]. \tag{26}$$

We have introduced $\lambda = \kappa/\omega$. When a large number of photons is considered, we can use the asymptotic behaviour of $G(\delta,x)$ to see that

$$\frac{(\Delta H_{\rm JC})_\pm}{\langle H_{\rm JC}\rangle_\pm} \sim \frac{1}{\sqrt{x}}, \tag{27}$$

as in the harmonic oscillator case.

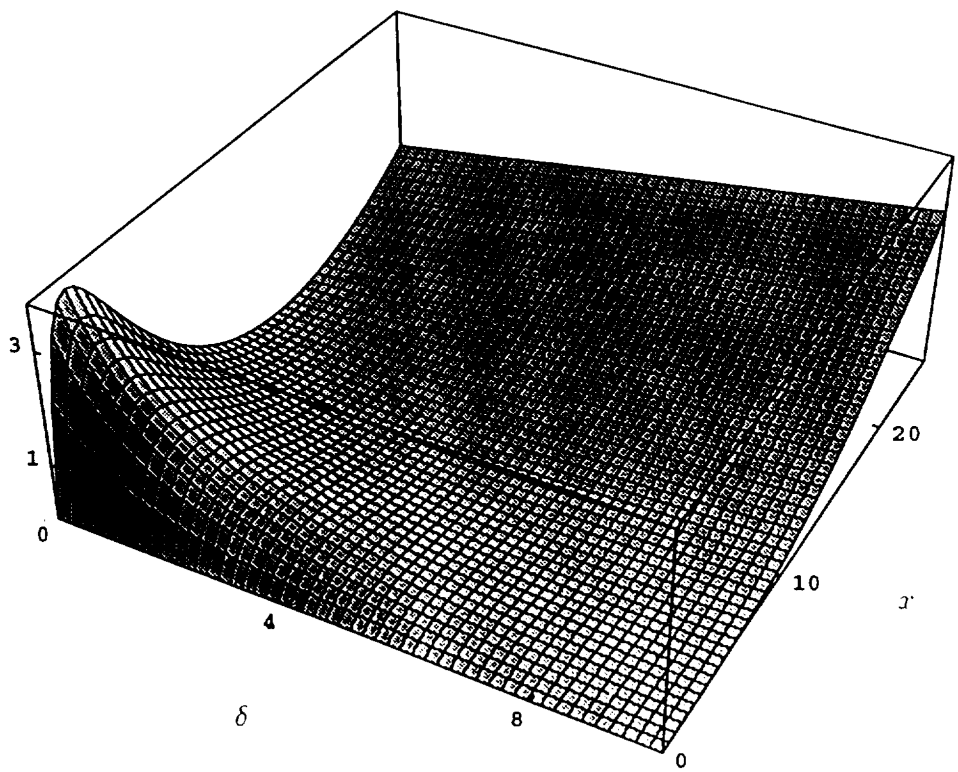

Fig. 1. Behaviour of $(\Delta H_{\rm JC})^2_+/\omega^2$ as a function of δ and x for $\lambda = 8$.

If we want to see how the dispersion evolves with respect to a variation of the characteristics of the system δ and λ, we analyse the form of $(\Delta H_{\rm JC})^2_+$, since $(\Delta H_{\rm JC})^2_-$ shows a similar qualitative behaviour. If we fix λ, the typical behaviour of $(\Delta H_{\rm JC})^2_+$ is as in Fig. 1. It can be proved that for fixed values of δ smaller than a certain δ_0, $(\Delta H_{\rm JC})^2_+$ has a minimum for $x \neq 0$.

The atomic inversion is the last quantity we will consider. Over a general coherent state, we have

$$\langle\sigma_3\rangle = \frac{1}{2}e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\left[\frac{\Delta}{2\kappa}\left(\frac{1-\cos\theta}{r(n+1)} - \frac{1+\cos\theta}{r(n)}\right) + 2\sin\theta\frac{\cos\varphi_n(t)}{r(n+1)}\right], \tag{28}$$

where $\varphi_n(t) = \phi + 2t\kappa\, r(n+1)$. If we take $\Delta = 0$ and $\theta = -\phi = \pi/2$, we have a temporal behaviour which is similar to the one obtained by Narozhny *et al.* [2] (let us recall that their states are different from ours). It consists of Rabi oscillations, as shown in Fig. 2:

$$\langle\sigma_3\rangle = \frac{1}{2}e^{-x}\left[-1 + 2\sum_{n=0}^{\infty}\frac{x^n}{n!}\,\frac{\sin(2t\kappa\sqrt{n+1})}{\sqrt{n+1}}\right]. \tag{29}$$

The derivative of this function with respect to t is essentially the value obtained in Ref. [2]. In Fig. 2, we show the graph of our $\langle\sigma_3\rangle$ for $x = 20$. It is similar to those obtained in many other papers, and that although the expression of $\langle\sigma_3\rangle$ is not exactly the same in all the cases.

These results indicate that it is reasonable to analyze the coherent states associated to the J.C. Hamiltonian in the way we are doing, but we do not know for the present if they could be interesting for the experiments.

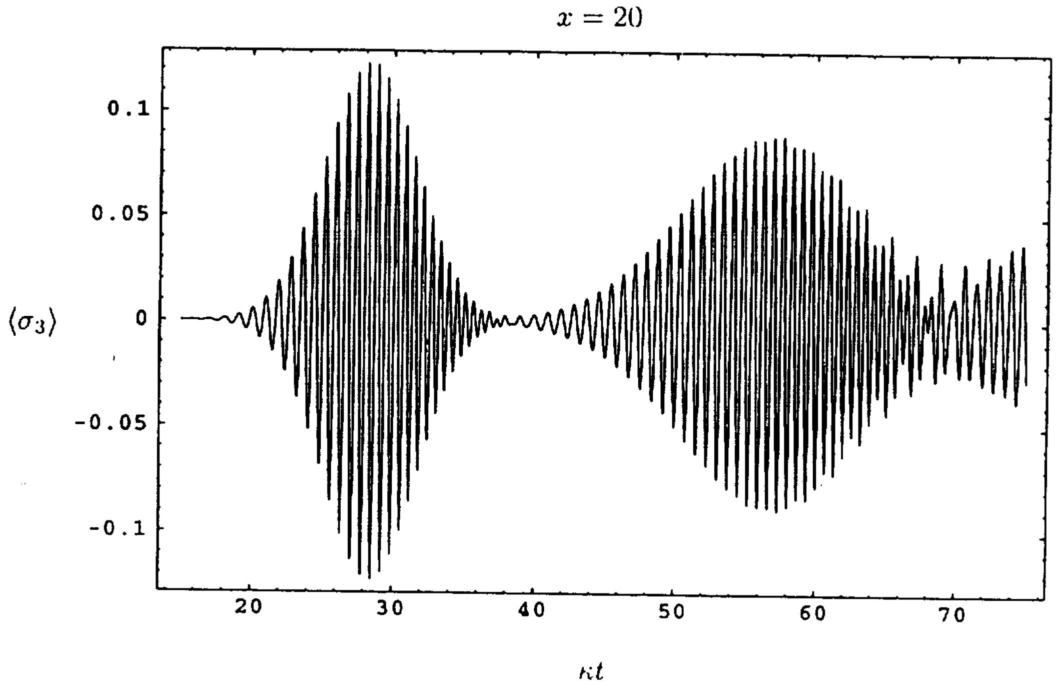

Fig. 2. Collapses and revivals of the atomic inversion in a general coherent state for $x=20$ and $\delta=0$.

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