1995116589

N95- 23006

KRAVCHUK FUNCTIONS FOR THE FINITE OSCILLATOR APPROXIMATION[1]

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Abstract

Kravchuk orthogonal functions — Kravchuk polynomials multiplied by the square root of the weight function — simplify the inversion algorithm for the analysis of discrete, finite signals in harmonic oscillator components. They can be regarded as the best approximation set. As the number of sampling points increases, the Kravchuk expansion becomes the standard oscillator expansion.

1 Introduction

In a harmonic oscillator environment, such as Fourier optics in a multimodal parabolic index-profile fiber, **sampling on a finite set of discrete observation points** reconstructs the wavefunction through partial wave synthesis. For the harmonic oscillator eigenfunctions, one must invert a nondiagonal matrix with the dimension of the number of data.

We show that Kravchuk orthogonal functions optimize the algorithm for the expansion coeffients, because the matrix **is already** diagonal.

- Kravchuk functions [1, 2] are solutions of the difference analogue of the Schrödinger equation describing a discrete harmonic oscillator system.
- Kravchuk functions have a well-defined analytical structure *inside* the measurement interval.
- Kravchuk functions become the standard oscillator wavefunctions, as the number of sampling points increases.

This contribution is a résumé of Ref. [3].

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2 Harmonic oscillator expansions over a lattice of sampling points

The standard harmonic oscillator eigenfuctions are

$$\psi_n(\xi) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}\, H_n(\xi)\, e^{-\xi^2/2}, \qquad n = 0, 1, 2, \ldots,$$

where $H_n(\xi)$ are the Hermite polynomials, $\xi = \sqrt{m\omega/\hbar}\, x$, m is oscillator mass, ω is oscillation frequency, and the position coordinate is x. This function set is orthonormal under the $\mathcal{L}^2(\Re)$ inner product:

$$(\psi_m, \psi_n)_\Re = \int_{-\infty}^{\infty} d\xi\, \psi_m(\xi)\, \psi_n(\xi) = \delta_{m,n} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases}$$

Thus, an arbitrary function $f(\xi) \in \mathcal{L}^2(\Re)$ can be approximated in the norm as

$$f(\xi) = \sum_{n=0}^{\infty} c_n\, \psi_n(\xi),$$

where the expansion coefficients $\{c_n\}_{n=0}^{\infty}$ are determined by

$$c_n = (\psi_n, f)_\Re = \int_{-\infty}^{\infty} d\xi\, \psi_n(\xi)\, f(\xi).$$

When the $N+1$ values $\{f(\xi_j)\}_{j=0}^{N}$ of function $f(\xi)$ are sampled on the points

$$\xi_0 = -\tfrac{1}{2}Nh, \ldots\ \xi_j = \left(-\tfrac{1}{2}N + j\right) h, \ldots\ \xi_N = \tfrac{1}{2}Nh,$$

then

$$f(\xi_j) = \sum_{n=0}^{N} c_n^{(N)}\, \psi_n(\xi_j), \quad j = 0, 1, \ldots N. \tag{2.1}$$

The task to determine the $N+1$ coefficients $\{c^{(N)}\}_{n=0}^{N}$ is formulated in matrix form as

$$\mathbf{f} = \mathbf{\Psi}^{(N)}\, \mathbf{c}^{(N)},$$

where

$$\mathbf{f} = \begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_N) \end{pmatrix}, \qquad \mathbf{c}^{(N)} = \begin{pmatrix} c_0^{(N)} \\ c_1^{(N)} \\ \vdots \\ c_N^{(N)} \end{pmatrix},$$

and

$$\mathbf{\Psi}^{(N)} = \begin{pmatrix} \psi_0(\xi_0) & \psi_0(\xi_1) & \ldots & \psi_0(\xi_N) \\ \psi_1(\xi_0) & \psi_1(\xi_1) & \ldots & \psi_1(\xi_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(\xi_0) & \psi_N(\xi_1) & \ldots & \psi_N(\xi_N) \end{pmatrix}$$

This is the $N \times N$ matrix that **has to be inverted** to find the coefficients in (2.1).

3 Kravchuk functions are difference analogs of the oscillator eigenfunctions

Kravchuk **polynomials** $k_n^{(p)}(x,N)$ are:

- polynomials of degree $0 \le n \le N$,
- in the variable $x \in [0,N]$,
- of the parameter $0 < p < 1$.

These polynomials are related to the binomial distribution of probability theory [4, 5]. They form an orthogonal set

$$\sum_{j=0}^{N} \varrho(j)\, k_m^{(p)}(j,N)\, k_n^{(p)}(j,N) = d_n^2 \delta_{m,n}$$

with respect to a discrete binomial weight function

$$\varrho(x) = C_N^x\, p^x (1-p)^{N-x}.$$

Kravchuk functions are defined as

$$\phi_n^{(p)}(x,N) = d_n^{-1}\, k_n^{(p)}(Np+x,N)\, \varrho^{1/2}(x+Np),$$

$$0 \le n \le N, \qquad -Np \le x \le (1-p)N$$

(*cf.* definition of the Hermite functions in Ref. [6]). They obey the three-term recurrence relation

$$[x - n - p(N-2n)]k_n^{(p)}(x,N) = (n+1)k_{n+1}^{(p)}(x,N) + p(1-p)(N-n+1)k_{n-1}^{(p)}(x,N),$$

and satisfy the equation

$$\mathbf{H}^{(N)}(x)\, \phi_n^{(p)}(x,N) = (n+\tfrac{1}{2})\phi_n^{(p)}(x,N),$$

with the difference **Hamiltonian**

$$\mathbf{H}^{(N)}(x) = (1-2p)x + 2p(1-p)N + \tfrac{1}{2} - \sqrt{p(1-p)}[\alpha(x)\, e^{-\partial_x} + \alpha(x+1)\, e^{\partial_x}],$$

namely

$$[(1-2p)x - n + 2p(1-p)N]\phi_n^{(p)}(x,N) = \sqrt{p(1-p)}\,[\alpha(x)\phi_n^{(p)}(x-1,N) + \alpha(x+1)\phi_n^{(p)}(x+1,N)].$$

The oscillator equation of motion in the Schrödinger representation [7] is $[H,[H,x]] = (\hbar\omega)^2 x$. The difference analogue of this relation satisfied by this Hamiltonian is [1]

$$[\mathbf{H}^{(N)}(x), [\mathbf{H}^{(N)}(x), x]] = x.$$

Finally, the limit $N \to \infty$ of Kravchuk functions is

$$\lim_{N\to\infty} h^{-1/2}\phi_n^{(p)}(h^{-1}\xi, N) = \psi_n(\xi).$$

The set of Kravchuk functions $\phi_n^{(p)}(x,N)$, $n=0,1,\ldots,N$ forms a basis for irreducible representations of the rotation group $SO(3)$ [8], corresponding to the eigenvalues $\ell=\frac{1}{2}N$ of the invariant Casimir operator; the eigenvalues of generator J_z are the integer $m=n-\frac{1}{2}N=n-\ell$. The representations corresponding to different values of the parameter p turn out to be unitarily equivalent [1], so it is sufficient to consider a set of functions $\phi_n^{(p)}(x,N)$ with some fixed value of this parameter. It is convenient to choose the value $p=\frac{1}{2}$, since these Kravchuk functions have definite parity with respect to reflections of x,

$$\phi_n^{(1/2)}(-x,N)=(-1)^n\phi_n^{(1/2)}(x,N).$$

We thus use henceforth the **symmetric Kravchuk functions**

$$\phi_n(x,N)=2^{n-N/2}k_n(x+\frac{1}{2}N,N)\sqrt{\frac{n!\,(N-n)!}{\Gamma(\frac{1}{2}(N+x+1)\,\Gamma(\frac{1}{2}(N-x+1)}}.$$

4 Finite approximation by Kravchuk functions

A function $f(\xi)$ that 'lives in a harmonic oscillator environment', of which the values on $N+1$ equidistant points ξ_j are known, is meaningfully expanded in symmetric Kravchuk functions as

$$f(\xi_j)=\frac{1}{\sqrt{h}}\sum_{n=0}^{N}\kappa_n^{(N)}\,\phi_n(\xi_j/h,N),\quad j=0,1,\ldots N.$$

To find the expansion coefficients $\{\kappa_n^{(N)}\}_{n=0}^{N}$, we multiply the above equation by $\phi_m(\xi_j/h,N)$ and sum over the sample points ξ_j:

$$\kappa_n^{(N)}=\sqrt{h}\sum_{j=0}^{N}\phi_n(\xi_j/h,N)\,f(\xi_j).$$

We thus have only to multiply the sampled values $f(\xi_j)$ by the (numerically calculated) values of the Kravchuk functions at the points $x_j=\xi_j/h$, for $n=0,1,\ldots,N$, to find the expansion coefficients. **No matrix inversion is necessary.**

The sum

$$f(\xi,N)=\frac{1}{\sqrt{h}}\sum_{n=0}^{N}\kappa_n^{(N)}\,\phi_n(\xi/h,N),\quad j=0,1,\ldots,N,$$

interpolates the original function defined on discrete points to the interval $[\xi_0,\xi_N]$ and is a **finite approximation** to the square-integrable function $f(\xi)$. This approximant is *finite* because for any fixed N it has a *finite support* $(-h-h^{-1},h+h^{-1})$ with $h=\sqrt{2/N}$. When N grows, the approximation to $f(\xi)$ becomes better. The time evolution of the approximating function multiplies each $\phi_n(\xi_j,N)$ by the usual time dependence $\exp(-iE_nt/\hbar)$, with the equally spaced energy eigenvalues $E_n=\hbar\omega(n+\frac{1}{2})$.

5 Position and momentum functions

The canonical vector basis of **position** functions

$$\mathbf{\Lambda}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{\Lambda}_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \mathbf{\Lambda}_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is interpolated to

$$f(\xi_j) = \sum_{k=0}^{N} f_k^{(N)} \Lambda_k(\xi_j, N) = f_j^{(N)}, \quad j = 0, 1, \ldots N.$$

These basis functions can be expanded in terms of Kravchuk functions as

$$\Lambda_k(\xi_j, N) = \frac{1}{\sqrt{h}} \sum_{n=0}^{N} \lambda_{k,n}^{(N)} \phi_n(\xi_j/h, N), \quad j = 0, 1, \ldots N,$$

where the coefficients are

$$\lambda_{k,n}^{(N)} = \sqrt{h}\, \phi_n(\xi_k/h, N)$$

for $k = 0, 1, \ldots [N/2]$ and continuous ξ. These functions are the localized states of the discrete oscillator.

Momentum basis functions are defined in the same way, because Kravchuk functions are self-reproducing under the discrete Fourier transformation [2], *i.e.*

$$\tilde{\Lambda}_k(\xi, N) = \sum_{n=0}^{N} i^n\, \phi_n(\xi_k/h, N)\, \phi_n(\xi/h, N).$$

Acknowledgments

We are grateful to Mesuma K. Atakishiyeva and Guillermo Krötzsch for supporting computation and graphics. One of us (N.M.A.) would like to thank Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, UNAM (Cuernavaca) for hospitality. This work is partially supported by the DGAPA Project N 104293.

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