Convergence Acceleration of Implicit Schemes in the Presence of High Aspect Ratio Grid Cells

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The performance of Navier-Stokes codes are influenced by several phenomena. For example, the robustness of the code may be compromised by the lack of grid resolution, by a need for more precise initial conditions or because all or part of the flowfield lies outside the flow regime in which the algorithm converges efficiently. A primary example of the latter effect is the presence of extended low Mach number and/or low Reynolds number regions which cause convergence deterioration of time marching algorithms. Recent research into this problem by several workers including the present authors has largely negated this difficulty through the introduction of time-derivative preconditioning. In the present paper, we employ the preconditioned algorithm to address convergence difficulties arising from sensitivity to grid stretching and high aspect ratio grid cells.

Strong grid stretching is particularly characteristic of turbulent flow calculations where the grid must be refined very tightly in the dimension normal to the wall, without a similar refinement in the tangential direction. High aspect ratio grid cells also arise in problems that involve high aspect ratio domains such as combustor coolant channels. In both situations, the high aspect ratio cells can lead to extreme deterioration in convergence. It is the purpose of the present paper to address the reasons for this adverse response to grid stretching and to suggest methods for enhancing convergence under such circumstances.

Numerical algorithms typically possess a maximum allowable or optimum value for the time step size, expressed in non-dimensional terms as a CFL number or von-Neumann number (VNN). In the presence of high aspect ratio cells, the smallest dimension of the grid cell controls the time step size causing it to be extremely small, which in turn results in the deterioration of convergence behaviour. For explicit schemes, this time step limitation cannot be exceeded without violating stability restrictions of the scheme. On the other hand, for implicit schemes, which are typically unconditionally stable, there appears to be room for improvement through careful tailoring of the time-step definition based on results of linear stability analyses. In the present paper, we focus on the central-differenced alternating direction implicit (ADI) scheme. The understanding garnered from this analyses can then be applied to other implicit schemes.

In order to systematically study the effects of aspect ratio and the methods of mitigating the associated problems, we use a two pronged approach. We use stability analyses as a tool for predicting numerical convergence behavior and numerical experiments on simple model problems to verify predicted trends. Based on these analyses, we determine that efficient convergence may be obtained at all aspect ratios by getting a combination of things right. Primary among these are the proper definition of the time step size, proper selection of viscous preconditioner and the precise treatment of boundary conditions. These algorithmic improvements are then applied to a variety of test cases to demonstrate uniform convergence at all aspect ratios.
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Philosophy of Grid Aspect Ratio Study

- Assessment of High Aspect Ratio Problem
  - Disparate propagation speeds in X and Y

- Stability Theory
  - Scalar Convection-Diffusion Equation
  - Euler Equations
  - Navier-Stokes Equations

- Numerical Convergence Studies
  - Simple Model Problems
  - Realistic Flow Problems

- Improved Algorithm to Provide Aspect Ratio Control
  - Precise Time-Step Definition
  - Viscous Preconditioning
  - Boundary Condition Implementation
The Navier-Stokes Equations

\[ \Gamma \frac{\partial Q_v}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = H + L(Q_v) \]

- Solution Vector

\[ Q_v = ( p, u, v, T )^T \]

- Preconditioning Matrix

\[ \Gamma = \begin{pmatrix}
\frac{1}{\epsilon c^2} & 0 & 0 & 0 \\
\frac{u}{\epsilon c^2} & \rho & 0 & 0 \\
\frac{v}{\epsilon c^2} & 0 & \rho & 0 \\
\frac{h + \frac{1}{2}(u^2 + v^2)}{\epsilon c^2} - 1 & \rho u & \rho v & \rho C_p
\end{pmatrix} \]

- Parameter \( \epsilon \)
  - Activates Inviscid and Viscous Preconditioning
  - Value Depends on Local Mach Number and Cell Reynolds Number
Numerical Solution Procedure

• Central-Differenced ADI Algorithm

\[
\left[ S + \frac{\partial A}{\partial x} - \frac{\partial}{\partial x} R_{xx} \right] S^{-1} \left[ S + \frac{\partial B}{\partial y} - \frac{\partial}{\partial y} R_{yy} \right] \Delta Q_v = -R^n
\]

• Approximate factorization errors control convergence behavior.

• Optimum $CFL_{u+c}$ is typically between 1 and 10.
  — Other inviscid and viscous time scales are optimized by the preconditioning matrix.
Time-Step Definition

• Local Time-Stepping or Constant $CFL$ Condition

$$\text{Max}(CFL_x, CFL_y) = CFL$$

• For high aspect ratios, $CFL_x$ and $CFL_y$ become disparate

$$CFL_y = CFL, \quad CFL_x = CFL/AR$$
Euler Stability Analysis

- Aspect Ratio ($AR$) of Unity
Euler Stability Analysis

- Aspect Ratio (AR) of 100

\[
\begin{align*}
CFL_y &= 1 \\
CFL_x &= 0.01
\end{align*}
\]

\[
\begin{align*}
CFL_y &= 10 \\
CFL_x &= 0.1
\end{align*}
\]
Euler Stability Analysis

• Aspect Ratio (AR) of 100

\[ CFL_y = 100 \]
\[ CFL_x = 1 \]

\[ CFL_y = 1000 \]
\[ CFL_x = 10. \]
New Time-Step Definition

• Conclusions from Stability Analysis:
  — Min-\(CFL\) Preferable to Max-\(CFL\)
  — Efficient Convergence at all \(AR\)

• Minimum-\(CFL\) Definition

\[
\text{Min} \ (\ CFL_x, \ CFL_y \ ) = CFL
\]

• For high aspect ratios,

\[
CFL_y = CFL \times AR, \quad CFL_x = CFL
\]
Implementation of Boundary Conditions

- Extrapolation vs Characteristic
  - Both work well for small $CFL$'s
  - Characteristics usually superior at high $CFL$'s

- Proper $MOC$ Implementation:
  - Implicit procedures
  - Boundary conditions applied before approximate factorization
  - Consistent order of accuracy: $LHS / RHS$
High Aspect Ratio Convergence—Inviscid Duct

![Graph showing the relationship between Grid Aspect Ratio and No. of Iterations to Machine Accuracy for MOC I, min CFL, MOC I, max CFL, and MOC II, min CFL.]
Navier-Stokes Analysis

- Parameter $\epsilon$ controls low Re number convergence
  - Viscous terms limit time step at high $AR$
  - $\epsilon$ chosen to optimize inviscid and viscous modes simultaneously

- Obvious choice: $\epsilon = f$ (Max-$CFL$, Max-$VNN$)

- Scalar Stability Results: $\epsilon = f$ (Min-$CFL$, Min-$VNN$)
Navier-Stokes Analysis

- Min-$CFL$, Min-$VNN$ Stability Result

$$AR = 1000$$

$$CFL_x = 1, \; CFL_y = 1000$$

$$VNN_x = 1, \; VNN_y = 1 \times 10^6$$
Navier-Stokes Analysis

- Conclusions from Stability Results:
  - Vector system different from scalar equation
  - Approximate factorization error $CFL_x \ast VNN_y$ limits convergence

- Viscous preconditioner, $\epsilon = f \left( \text{Min-CFL}, \text{Max-VNN} \right)$
  - Maintains Min-CFL for 'inviscid' modes
  - Uses traditional Max-VNN definition for 'viscous' modes
Navier-Stokes Analysis

- Min-$CFL$, Max-$VNN$ Stability Result

\[ AR = 1000 \]

\[ CFL_x = 1, \quad CFL_y = 1000 \]

\[ VNN_x = 1 \times 10^{-6}, \quad VNN_y = 1 \]
High Aspect Ratio Convergence—Viscous Duct

![Graph showing grid aspect ratio vs. number of iterations to machine accuracy for different CFL conditions.]
Hydrogen/Oxygen Shear-Layer
Stretched Grid and Solution

Hydrogen

Oxygen

Re=20000
Re=2000
Re=200

HYDROGEN MASS FRACTION

RADIAL COORDINATE

HYDROGEN MASS FRACTION
Hydrogen/Oxygen Shear-Layer Convergence

L₂ Norm of the Residual

No. of Iterations

Re=200, 2000 & 20000

I min-CFL, max-VNN
II max-CFL Time Step
III min-CFL, No V.Prec.
IV min-CFL, No Prec.
High Reynolds Number Boundary Layer
Convergence—Stretched Grid

![Graph showing L2 Norm of the Residual vs. No. of Iterations for different Re values.]

- Re=4.0e6 (AR=8000)
- 4.5e5 (200)
- 4.4e5 (30)
- 4.0e5 (10)

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High Reynolds Number Boundary Layer
Blasius Solution

\[ \eta \]

\[ u/U \]

Re=4.e5
High Reynolds Number Boundary Layer
Turbulent Velocity Profile

Re=4.e6

\[ u^+ \]

\[ \log_{10}(y^+) \]
Turbulent Nozzle Computation
Stretched Grid and Solution
Turbulent Nozzle Computation
Convergence

![Graph showing convergence of L_2 norm of residuals for different grids.](image)
Turbulent Nozzle Computation

Wall Heat Flux

Heat Flux, W/m²

Axial Distance
Conclusions—High Aspect Ratio Study

• High Aspect Ratio Analysis
  — Stability Theory
  — Numerical Convergence Studies

• Convergence Control:
  — Min-$CFL$ Time Step
  — Max-$VNN$ Viscous Preconditioner
  — Correct implementation of boundary conditions

• Uniform Convergence Demonstrated for All $AR$’s
  — Above issues addressed in combination
  — Efficient convergence for variety of test cases
Conclusions (Contd.)

- Present results are for two-dimensional central-differenced 
  \textit{ADI} scheme

- Explicit Schemes:
  - Optimum time step causes poor convergence at high 
    \textit{AR's}

- Upwind Schemes Also Suffer at High \textit{AR's}
  - Present improvements may be incorporated
  - Rich variety of approximate factorization methods

- Three-dimensional computations:
  - \textit{ADI} scheme is conditionally stable
  - Two kinds of high aspect ratio grids
  - Algorithmic improvements appear promising