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Splines and Control Theory

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Abstract

In this work, the relationship between splines and the control theory has been analyzed. We show that spline functions can be constructed naturaly from the control theory. By establishing a framework based on control theory, we provide a simple and systematic way to construct splines. We have constructed the traditional spline functions including the polynomial splines and the classical exponential spline. We have also discovered some new spline functions such as trigonometric splines and the combination of polynomial, exponential and trigonometric splines. The method proposed in this paper is easy to implement. Some numerical experiments are performed to investigate properties of different spline approximations.

1. Introduction.

Spline functions are well known and are widely used for practical approximation of functions or more commonly for fitting smooth curves through preassigned points. Spline techniques have the advantage over most approximation and interpolation techniques in that they are computational feasible. Most of the published spline algorithms are for polynomial splines and the vast preponderance are for cubic splines. There is a small but excellent literature on the so called exponential splines and there is an even smaller literature on splines with more or less arbitrary nodal functions, [9, 3].

In this paper we will present a common frame work for splines that includes polynomial splines of all orders and generalized exponential splines of all orders. This common frame work is based on ideas from linear control theory. Let's recall some basic ideas from control theory. A linear control system is a differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

where $\vec{x} \in \mathbb{R}^n$, $\vec{u} \in \mathbb{R}^m$ and the matrices A and B are constant matrices of compatible dimension. The vector $\vec{x}$ is the state of the system and the vector $\vec{u}$ is the control. The idea is that we can use the control $\vec{u}$ to steer the state from point to point in the state space $\mathbb{R}^n$. We can think of the first component of $\vec{x}$ as representing the position of the system and for appropriate A the second coordinate is the velocity, the third acceleration, etc. A common situation, for example in air traffic control, is to specific the position that the system must be in at a sequence of times. So in fact what we have is a set of points through which the system must traverse at specified times. One could fit these points with a spline curve and then ask for the control that would move the system along that trajectory. In fact this can be done but we will show that the control law can be developed from natural control theoretic principles that will move the system through the points at the desired times and the resulting curve will be piecewise analytic and will have $2n-1$ continuous derivatives, i.e. a generalized spline. With this framework we can construct a wide variety of spline functions. If the matrix A is nilpotent then the resulting construction is just that for polynomial splines. If the matrix is 2×2 and one eigenvalue is zero and the other is a nonzero real number then the spline is the usual exponential spline. In general the nodal functions are the coordinate functions of the matrix function e^{At}.

In this paper we give a unified treatment of all of the common one dimensional spline functions using simple ideas from control theory. It is coming to be understood that there is a large overlap between linear control theory and elementary numerical analysis. Eigenvalue methods are know to be closely related to the theory of the matrix Riccati equation [2], there are close relations between observability and quadrature techniques [8], system identification and Prony's method are very similar [1] and now we see that the spline constructions and basic linear controllability are manifestations of the same phenomena.

In Section 2 we review basic material from the theory of linear control systems that is needed for the development and give a condition that characterizes the optimal control law that generates the spline functions. In Section 3 we give the details of the construction of spline functions using control theory and in Section 4 we classify the possible classes of spline functions that arise from the control theoretic construction. In Section 5 we examine in detail some of the particular classes from Section 4 and finally in Section 6 we present a series of numerical examples comparing the various classes.

2. Some results from the control theory.

In this section we collect a series of results from linear control theory. Most can be found in any control theory textbook. See, for example, the book by Brockett, [5].

Consider the linear system:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \quad t \in [0,T], \tag{2.1}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{m-1}(t) \\ x_m(t) \end{pmatrix}, \tag{2.2}$$

and the observation function

$$y(t) = \vec{c}^T\vec{x}(t), \quad \vec{c}^T = (1,0,\cdots,0). \tag{2.3}$$

Let us divide $[0,T]$ into n subintervals as

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

and define $h_k = t_k - t_{k-1}$, the length of the rth subinterval. Our goal is to find a control law $u \in C^{m-2}[0,T]$ that drives the system (2.1) from $\vec{x}(0) = \vec{x}^0$ to $\vec{x}(T) = \vec{x}^T$ such that the observation function $y(t)$ satisfies the interpolation conditions

$$y(t_k) = \alpha_k, \quad k = 1,\cdots,n-1. \tag{2.4}$$

Furthermore, $u(t)$ minimizes the functional

$$\int_0^T u(s)^2 ds. \tag{2.5}$$

Such a control is called an optimal control.

Definition. The system (2.1) is called controllable if for any $\vec{x}^0$, $\vec{x}^\tau$, and $\tau > 0$, there is a $u(t)$ such that,

$$\vec{x}^\tau = \vec{x}(\tau) = e^{A\tau}\vec{x}^0 + \int_0^\tau e^{A(\tau-s)}\vec{b}u(s)ds.$$

Theorem 2.1 : *The system (2.1) is controllable if and only if*

$$rank\,(\vec{b}, A\vec{b}, \cdots, A^{m-1}\vec{b}) = m. \tag{2.6}$$

For the special matrix A as in (2.2), it is easy to verify that

$$(\vec{b}, A\vec{b}, \cdots, A^{m-1}\vec{b}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & * & * \\ 1 & * & \cdots & * & * \end{pmatrix}, \tag{2.7}$$

and hence the condition (2.6) is satisfied. From Theorem 2.1, the system (2.1) is controllable.

Theorem 2.2 : *The system (2.1) is controllable, if and only if the matrix* $\int_0^T e^{-As}\vec{b}\vec{b}^T e^{-A^T s} ds$ *is invertible.*

For the special matrix in (2.2), we then define

$$M(t) = (\int_0^t e^{-As}\vec{b}\vec{b}^T e^{-A^T s} ds)^{-1}. \tag{2.8}$$

Theorem 2.3 : *When the system (2.1) is controllable, a control that moves the system from* $\vec{x}(\underline{t}) = \vec{\rho}_L$ *to* $\vec{x}(\bar{t}) = \vec{\rho}_R$ *given by*

$$u(t) = \vec{b}^T e^{-A^T t}(\int_{\underline{t}}^{\bar{t}} e^{-As}\vec{b}\vec{b}^T e^{-A^T s} ds)^{-1}(e^{-A\bar{t}}\vec{\rho}_R - e^{-A\underline{t}}\vec{\rho}_L), \tag{2.9}$$

minimizes the functional $J(v) = \int_{\underline{t}}^{\bar{t}} v^2(s)ds$ *among all controls that move the system from* $\vec{x}(\underline{t}) = \vec{\rho}_L$ *to* $\vec{x}(\bar{t}) = \vec{\rho}_R$.

Theorem 2.4 : *When the system (2.1) is controllable, a control* $u \in C^{m-2}[0,T]$ *that moves the system from* $\vec{x}(0) = \vec{x}^0$*, passing through* $\vec{c}^T\vec{x}(t_k) = \alpha_k$*, to* $\vec{x}(T) = \vec{x}^T$ *is given by*

$$u(t) = \sum_{k=1}^{n-1} \beta_k f_k(t) + \sum_{i=1}^{m} \gamma_i g_i(t), \tag{2.10}$$

with

$$f_k(t) = \begin{cases} \vec{e}_1^T e^{A(t_k - t)}\vec{b} & t < t_k, \\ 0 & t \geq t_k, \end{cases} \quad k = 1, \cdots, n-1,$$

$$g_i(t) = \vec{e}_i^T e^{A(t_n - t)}\vec{b}, \quad i = 1, \cdots, m,$$

where $\vec{e}_1^T = (1, 0, \cdots, 0), \cdots, \vec{e}_m^T = (0, \cdots, 0, 1)$*, and* β_k*'s,* γ_i*' are determined by* $n-1$ *interpolation conditions* $\vec{c}^T\vec{x}(t_k) = \alpha_k$ *and* m *boundary conditions* $\vec{x}(T) = \vec{x}^T$*. Moreover, the control*

(2.10) minimizes the functional $J(v) = \int_0^T v^2(s)ds$ *among all functions* $v \in C^{m-2}[0,T]$ *that drives the system (2.1) from* $\vec{x}(0) = \vec{x}^0$*, passing through* $\vec{c}^T\vec{x}(t_k) = \alpha_k$*, to* $\vec{x}(T) = \vec{x}^T$.

The construction of the optimal control is based on Hilbert space techniques and is based on writing the interior constraints in terms of a linear variety defined in terms of the functions $f_k(t)$ and the terminal constraints in terms of the functions $g_i(t)$. Once the constraints are written in terms of linear varieties the form of the optimal control is clear based on the orthogonal complement of the intersection of the varieties. this is a standard technique and is found for example in [7] or [6]. The proof of Theorem 2.4 is based on two facts: (i) $u(t)$ defined by (2.10) is $m-2$ times continuously differentiable; (ii) f_k's and g_i's are $n+m-1$ linearly independent functions. In the following, we verify (i) and (ii).

(i) From the construction (2.10), we only need to show that

$$f_k^{(r)}(t_k) = 0, \quad r = 0, \cdots, m-2, \quad k = 1, \cdots, n-1.$$

Indeed, for all $k = 1, \cdots, n-1$, $r = 0, \cdots, m-2$,

$$\lim_{t \to t_k - 0} f_k^{(r)}(t) = \lim_{t \to t_k - 0} \vec{e}_1^T(-A)^r e^{A(t_k - t)}\vec{b} = (-1)^r \vec{e}_1^T A^r \vec{b} = 0,$$

by virtue of (2.7). Hence $f_k(t)$ is $m-2$ times continuously differentiable, and so is $u(t)$.

(ii) Set $\sum_{i=1}^m \gamma_i g_i(t) = 0$ for $t \in [0,T]$, i.e.,

$$F(t) = \sum_{i=1}^m \gamma_i \vec{e}_i^T e^{A(t_n - t)}\vec{b} = \vec{\rho}^T e^{A(t_n - t)}\vec{b} = 0,$$

with $\vec{\rho} = \sum_{i=1}^m \gamma_i \vec{e}_i$. Then we have $F^{(r)}(t) = 0$ on $[0,T]$, especially

$$F^{(r)}(t_k) = \vec{\rho}^T(-A)^r\vec{b} = 0, \quad r = 0, \cdots m-1.$$

Therefore,

$$\vec{\rho}(\vec{b}, A\vec{b}, \cdots, A^{m-1}\vec{b}) = \vec{0}^T.$$

In light of (2.7), $\vec{\rho}$ is a zero vector and consequently, $\gamma_i = 0$, $i = 1, \cdots, m$. So g_i's are m linearly independent functions.

Next we set

$$\beta_{n-1} f_{n-1}(t) + \sum_{i=1}^m \gamma_i g_i(t) = 0, \quad t \in [0,T]. \tag{2.11}$$

If $\beta_{n-1} \neq 0$, then

$$f_{n-1}(t) = -\frac{1}{\beta_{n-1}} \sum_{i=1}^{m} \gamma_i g_i(t). \tag{2.12}$$

By the definition, $f_{n-1}(t) = 0$ on (t_{n-1}, t_n), So $\sum_{i=1}^{m} \gamma_i g_i(t) = 0$ for $t \in (t_{n-1}, t_n)$ which yields $\gamma_i = 0$, $i = 1, \cdots, m$, and hence $f_{n-1}(t) \equiv 0$ by (2.12). This is a contradiction. Then (2.11) yields $\beta_{n-1} = 0$, and consequently $\gamma_i = 0$, $i = 1, \cdots, m$. So f_{n-1} and g_i's are $m+1$ linearly independent functions.

Continue the above procedure by adding f_k's one by one, we are able to show that f_k's and g_i's are $m+n-1$ linearly independent functions.

3. Construction of splines by the control theory.

Theorem 2.4 implies that an optimal control for the system (2.1) (with A given by (2.2)) is unique. But in general, β_k's and γ_i's in (2.10) are difficult to find, we then introduce a practical procedure to construct a control law that satisfies all the requirements. This control law actually leads us to a construction of spline functions.

By the existence of a control law, there exists a set of points $\vec{x}^1, \cdots, \vec{x}^{n-1}$ with $x_1^k = \alpha_k$, $k = 1, \cdots, n-1$ such that the solution of the system (2.1) satisfies $\vec{x}(t_k) = \vec{x}^k$, $k = 0, 1, \cdots, n-1, n$. By virtue of Theorem 2.3, a control law that satisfies all the requirement can be defined piecewise as

$$u(t)|_{[t_{k-1}, t_k]} = u_k(t), \quad k = 1, \cdots, n, \tag{3.1}$$

where $u_k(t)$ is given by (2.9) with $\underline{t} = t_{k-1}$, $\bar{t} = t_k$, $\vec{\rho}_L = \vec{x}^{k-1}$, and $\vec{\rho}_R = \vec{x}^k$. Then equations to find $(n-1)(m-1)$ unknowns in $\vec{x}^1, \cdots, \vec{x}^{n-1}$ (recall that $x_1^k = \alpha_k$ $k = 1, \cdots, n-1$, are known) come from $(n-1)(m-1)$ continuity conditions on $u(t)$, i.e.,

$$u_k^{(r)}(t_k) = u_{k+1}^{(r)}(t_k), \quad r = 0, \cdots, m-2, \quad k = 1, \cdots, n-1. \tag{3.2}$$

From (2.9),

$$u_k(t_k) = (A^r \vec{b})^T e^{-A^T t_k} \left(\int_{t_{k-1}}^{t_k} e^{-As} \vec{b}\vec{b}^T e^{-A^T s} ds\right)^{-1} (e^{-At_k} \vec{x}^k - e^{-At_{k-1}} \vec{x}^{k-1}); \tag{3.3}$$

$$u_{k+1}(t_k) = (A^r \vec{b})^T e^{-A^T t_k} \left(\int_{t_k}^{t_{k+1}} e^{-As} \vec{b}\vec{b}^T e^{-A^T s} ds\right)^{-1} (e^{-At_{k+1}} \vec{x}^{k+1} - e^{-At_k} \vec{x}^k). \tag{3.4}$$

Next, we shall simplify (3.3) and (3.4). Toward this end, we introduce a change of variable $s = t_{k-1} + s'$ into

$$\left(\int_{t_{k-1}}^{t_k} e^{-As} \vec{b}\vec{b}^T e^{-A^T s} ds\right)^{-1} = \left(e^{-At_{k-1}} \int_0^{h_k} e^{-As'} \vec{b}\vec{b}^T e^{-A^T s'} ds' e^{-A^T t_{k-1}}\right)^{-1}$$

$$= \quad e^{A^T t_{k-1}} M(h_k) e^{A t_{k-1}}, \tag{3.5}$$

where $M(h_k)$ is defined by (2.8). Substituting (3.5) into (3.3), we have

$$u_k^{(r)}(t_k) = (A^r \vec{b})^T e^{-A^T h_k} M(h_k)(e^{-Ah_k} \vec{x}^k - \vec{x}^{k-1}). \tag{3.6}$$

Similarly,

$$u_{k+1}^{(r)}(t_k) = (A^r \vec{b})^T M(h_{k+1})(e^{-Ah_{k+1}} \vec{x}^{k+1} - \vec{x}^k). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.2) yields a linear system for $(n-1)(m-1)$ unknowns in $\vec{x}^1, \cdots, \vec{x}^{n-1}$:

$$\begin{aligned} &-(A^r \vec{b})^T e^{-A^T h_k} M(h_k) \vec{x}^{k-1} + (A^r \vec{b})^T [e^{-A^T h_k} M(h_k) e^{-A h_k} + M(h_{k+1})] \vec{x}^k \\ &-(A^r \vec{b})^T M(h_{k+1}) e^{-A h_{k+1}} \vec{x}^{k+1} = 0, \quad r = 0, \cdots, m-2, \quad k = 1, \cdots, n-1. \end{aligned} \tag{3.8}$$

By virtue of the existence and uniqueness of the optimal control, the linear system (3.8) has a unique solution and hence its coefficient matrix is invertible.

In order to solve (3.8), The following quantities needs to be calculated, A^r, e^{-Ah} $(e^{-A^T h})$, and $M(h)$. Sometimes it is easier to use the Jordan matrix of A, denoted by Λ. There exists an invertible matrix Q such that $A = Q \Lambda Q^{-1}$, and hence

$$A^r = Q \Lambda^r Q^{-1}, \quad e^{-Ah} = Q e^{-\Lambda h} Q^{-1}, \quad e^{-A^T h} = Q^{-T} e^{-\Lambda^T h} Q^T. \tag{3.9}$$

$$M(h) = (\int_0^h Q e^{-\Lambda s} Q^{-1} \vec{b} \vec{b}^T Q^{-T} e^{-\Lambda^T s} Q^T ds)^{-1} = Q^{-T} \hat{M}(h) Q^{-1}, \tag{3.10}$$

where

$$\hat{M}(h) = (\int_0^h e^{-\Lambda s} Q^{-1} \vec{b} (Q^{-1} \vec{b})^T e^{-\Lambda^T s} ds)^{-1}. \tag{3.11}$$

Substituting (3.9) and (3.10) into (3.8), we then have

$$\begin{aligned} &-(\Lambda^r Q^{-1} \vec{b})^T e^{-\Lambda^T h_k} \hat{M}(h_k) Q^{-1} \vec{x}^{k-1} + (\Lambda^r Q^{-1} \vec{b})^T [e^{-\Lambda^T h_k} \hat{M}(h_k) e^{-\Lambda h_k} + \hat{M}(h_{k+1})] Q^{-1} \vec{x}^k \\ &-(\Lambda^r Q^{-1} \vec{b})^T \hat{M}(h_{k+1}) e^{-\Lambda h_{k+1}} Q^{-1} \vec{x}^{k+1} = 0, \quad r = 0, \cdots, m-2, \quad k = 1, \cdots, n-1. \end{aligned} \tag{3.12}$$

Solving (3.8) or (3.12) for $(n-1)(m-1)$ unknowns in $\vec{x}^1, \cdots, \vec{x}^{n-1}$, we then have the control $u(t)$ defined piecewise by (3.1). The solution of the system (2.1) is thus given by

$$\begin{aligned} \vec{x}(t) &= e^{At} \vec{x}^0 + \int_0^t e^{A(t-s)} \vec{b} u(s) ds \\ &= Q(e^{\Lambda t} Q^{-1} \vec{x}^0 + \int_0^t e^{\Lambda(t-s)} Q^{-1} \vec{b} u(s) ds). \end{aligned} \tag{3.13}$$

Note that $x_i'(t) = x_{i+1}(t)$, $i = 1, \cdots, m-1$. So the continuity of $x_i'(t)$ is continuity of $x_{i+1}(t)$ for $i < m$. Further, continuity of $x_1^{(m+r)}(t)$ is continuity of $u^{(r)}(t)$, $r = 0, \cdots, m-2$. Therefore, the observation function $y(t) = \vec{c}^T \vec{x}(t) = x_1(t)$ is a $2m-2$ times continuously differentiable function that satisfies the boundary conditions

$$y^{(r)}(0) = x_{r+1}^0, \quad y^{(r)}(T) = x_{r+1}^T, \quad r = 0, \cdots, m-1 \tag{3.14}$$

and the interpolation conditions

$$y(t_k) = x_1^k, \quad k = 1, \cdots, n-1. \tag{3.15}$$

Hence $y(t)$ is a spline function. We see that from the control theory, we can derive quite general spline functions. Summing up, we have proved

Theorem 3.1 : *(1) There exists a unique function $y(t) \in C^{m-2}[0,T]$ that satisfies the boundary conditions (3.14) and the interpolation conditions (3.15); (2) $y(t)$ is the first component of the vector function $\vec{x}(t)$ given by (3.13) in which $u(s)$ is defined piecewise on each subinterval $[t_{k-1}, t_k]$, $k = 1, \cdots, n$, by*

$$\begin{aligned} u_k^{(r)}(t_k) &= \vec{b}^T e^{-A^T(t-t_{k-1})} M(h_k)(e^{-Ah_k}\vec{x}^k - \vec{x}^{k-1}) \\ &= (Q^{-1}\vec{b})^T e^{-\Lambda^T(t-t_{k-1})} \hat{M}(h_k)(e^{-\Lambda h_k} Q^{-1}\vec{x}^k - Q^{-1}\vec{x}^{k-1}), \end{aligned} \tag{3.16}$$

where $\vec{x}^k$, $k = 1, \cdots, n-1$ are determined by solving the linear systems (3.8) or (3.12) (Note that $\vec{x}^0$, $\vec{x}^n$ and x_1^k, $k = 1, \cdots, n-1$ are given by the boundary conditions (3.14) and the interpolation conditions (3.15)).

In the next section, we will see that these splines can be piecewise polynomials, trigonometric functions, exponentials or any combination. As special cases, we are able to recover classical polynomial splines (odd order) and exponential splines by properly selecting parameters $a_1, \cdots, a_m$ in (2.2) for the matrix A.

4. Classification of splines.

The type of the splines is determined by its nodal shape functions. From the control theory, we are able to construct the nodal shape functions of splines.

In order to see the kind of interpolation functions in $\vec{x}(t)$, we only need to consider one subinterval. Without loss of generality, we use the first interval $(t_0, t_1) = (0, h)$ where the solution of the system (2.1) is given by

$$\vec{x}(t) = e^{At}\vec{x}^0 + \int_0^t e^{A(t-s)}\vec{b}u(s)ds. \tag{4.1}$$

From Theorem 2.3,

$$u(t) = \vec{b}^T e^{-A^T t}\left(\int_0^h e^{-As}\vec{b}\vec{b}^T e^{-A^T s}ds\right)^{-1}(e^{-Ah}\vec{x}^1 - \vec{x}^0). \tag{4.2}$$

Substituting (4.2) into (4.1), we have

$$\begin{aligned}
\vec{x}(t) &= e^{At}\vec{x}^0 + \int_0^t e^{A(t-s)}\vec{b}\vec{b}^T e^{-A^T s}dsM(h)(e^{-Ah}\vec{x}^1 - \vec{x}^0) \\
&= Qe^{\Lambda t}[Q^{-1}\vec{x}^0 + \hat{M}(t)^{-1}\hat{M}(h)(e^{-\Lambda h}Q^{-1}\vec{x}^1 - Q^{-1}\vec{x}^0)],
\end{aligned} \tag{4.3}$$

where $M(h)$ and $\hat{M}(h)$ are defined by (2.8) and (3.11), respectively.

Theorem 4.1 : *Let A be given by (2.2), let $(p_1(t), \cdots, p_m(t))$ be the first row of the matrix*

$$e^{At}[I - M(t)^{-1}M(h)] = Qe^{\Lambda t}[I - \hat{M}(t)^{-1}\hat{M}(h)]Q^{-1}, \tag{4.4}$$

and let $(q_1(t), \cdots, q_m(t))$ be the first row of the matrix

$$e^{At}M(t)^{-1}M(h)e^{-Ah} = Qe^{\Lambda t}\hat{M}(t)^{-1}\hat{M}(h)e^{-\Lambda h}Q^{-1}. \tag{4.5}$$

Then for $r = 0, \cdots, m-1$,

$$p_i^{(r)}(0) = \delta_{i,r+1}, \quad p_i^{(r)}(h) = 0, \quad i = 1, \cdots, m, \tag{4.6}$$

$$q_j^{(r)}(0) = 0, \quad q_j^{(r)}(h) = \delta_{j,r+1}, \quad j = 1, \cdots, m. \tag{4.7}$$

Proof : From Theorem 2.1 and (2.7), the system (2.1) is controllable. By virtue of Theorem 3.3, a control that moves $\vec{x}(t)$ from $\vec{x}(0) = \vec{x}^0$ to $\vec{x}(h) = \vec{x}^1$ is given by (2.9) (with $\underline{t} = 0$, $\bar{t} = h$, $\vec{\rho}_L = \vec{x}^0$, $\vec{\rho}_R = \vec{x}^1$), and consequently

$$\begin{aligned}
y(t) &= \vec{e}_1^T \vec{x}(t) \\
&= \vec{e}_1^T(e^{At}\vec{x}^0 + \int_0^t e^{A(t-s)}\vec{b}u(s)ds) \\
&= \vec{e}_1^T e^{At}[I - M(t)^{-1}M(h)]\vec{x}^0 + \vec{e}_1^T e^{At}M(t)^{-1}M(h)e^{-Ah}\vec{x}^1 \\
&= (p_1(t), \cdots, p_m(t))\vec{x}^0 + (q_1(t), \cdots, q_m(t))\vec{x}^1 \\
&= \sum_{i=1}^m x_i^0 p_i(t) + \sum_{j=1}^m x_j^1 q_j(t).
\end{aligned} \tag{4.8}$$

Choose $\vec{x}^0 = \vec{e}_i$ $\vec{x}^1 = \vec{0}$ in (4.8), and we have

$$p_i^{(r)}(t) = y^{(r)}(t), \quad r = 0, \cdots, m-1.$$

Therefore for $r = 0, \cdots, m-1$,

$$\begin{aligned} p_i^{(r)}(0) &= y^{(r)}(0) = x_1^{(r)}(0) = x_{r+1}(0) = x_{r+1}^0 = \delta_{i,r+1}, \\ p_i^{(r)}(h) &= y^{(r)}(h) = x_1^{(r)}(h) = x_{r+1}(h) = x_{r+1}^1 = 0, \quad i = 1, \cdots, m. \end{aligned}$$

Then we have proved (4.6). The proof of (4.7) is similar. ■

We call p_i, q_i nodal shape functions by the characteristics (4.6) and (4.7). From (4.4) and (4.5), We see that the nodal shape functions are linear combinations of function entries of matrices $e^{\Lambda t}$ and $e^{\Lambda t}\hat{M}(t)^{-1}$. In order to see the type of functions in the spline, we only need to examine the entries of these two matrices.

In the following, we classify the spline functions derived from control theory. This classification is based on the spectrum of the coefficient matrix A of the system (2.1) under different circumstances. We shall concentrate on the case $m = 2$. The reasons are: (1) The general situation for large m is very complicated and is difficult to describe precisely. (2) The case $m = 2$ has almost all features for the general case. (3) From the practical point of view, the case $m = 2$ is the most useful and important case. Let

$$A = \begin{pmatrix} 0 & 1 \\ \beta & 2\gamma \end{pmatrix}, \quad \beta, \gamma \in \mathbb{R}^1.$$

The eigenvalues of A are $\lambda_1 = \gamma + \sqrt{\gamma^2 + \beta}$, $\lambda_2 = \gamma - \sqrt{\gamma^2 + \beta}$.

1. $\gamma^2 + \beta > 0$. There are two distinct real eigenvalues. Then the Jordan matrix Λ of A, the transformation matrix Q and its inverse are given by

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad Q^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}. \tag{4.9}$$

Then

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \tag{4.10}$$

From (3.11) (by changing h to t), we have

$$\begin{aligned} \hat{M}(t)^{-1} &= \frac{1}{(\lambda_2 - \lambda_1)^2} \int_0^t \begin{pmatrix} e^{-\lambda_1 s} & 0 \\ 0 & e^{-\lambda_2 s} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} (-1, 1) \begin{pmatrix} e^{-\lambda_1 s} & 0 \\ 0 & e^{-\lambda_2 s} \end{pmatrix} ds \\ &= \frac{1}{(\lambda_2 - \lambda_1)^2} \begin{pmatrix} (1 - e^{-2\lambda_1 t})/2\lambda_1 & (e^{-(\lambda_1+\lambda_2)t} - 1)/(\lambda_1 + \lambda_2) \\ (e^{-(\lambda_1+\lambda_2)t} - 1)/(\lambda_1 + \lambda_2) & (1 - e^{-2\lambda_2 t})/2\lambda_2 \end{pmatrix}, \end{aligned} \tag{4.11}$$

and hence

$$e^{\Lambda t}\hat{M}(t)^{-1} = \frac{1}{(\lambda_2 - \lambda_1)^2} \begin{pmatrix} (e^{\lambda_1 t} - e^{-\lambda_1 t})/2\lambda_1 & (e^{-\lambda_2 t} - e^{\lambda_1 t})/(\lambda_1 + \lambda_2) \\ (e^{-\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 + \lambda_2) & (e^{\lambda_2 t} - e^{-\lambda_2 t})/2\lambda_2 \end{pmatrix}. \tag{4.12}$$

1.a. $\gamma \neq 0$, $\beta \neq 0$. In this case $\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$ are all distinct. We then have the exponential spline with basis functions given by linear combinations of $e^{\lambda_1 t}, e^{-\lambda_1 t}, e^{\lambda_2 t}, e^{-\lambda_2 t}$.

1.b. $\gamma = 0$, $\beta > 0$. In this case $\lambda_1 = -\lambda_2 = \sqrt{\beta}$, the basis functions in 1.a. degenerate. However, by applying the following limits

$$\lim_{\lambda_2 \to -\lambda_1} \frac{e^{-\lambda_2 t} - e^{\lambda_1 t}}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 + \lambda_2 \to 0} \frac{e^{\lambda_1 t}(e^{-(\lambda_1+\lambda_2)t} - 1)}{\lambda_1 + \lambda_2} = -te^{\lambda_1 t},$$

$$\lim_{\lambda_2 \to -\lambda_1} \frac{e^{-\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 + \lambda_2 \to 0} \frac{e^{-\lambda_1 t}(1 - e^{(\lambda_1+\lambda_2)t})}{\lambda_1 + \lambda_2} = -te^{-\lambda_1 t},$$

(4.12) becomes

$$e^{\Lambda t}\hat{M}(t)^{-1} = \frac{1}{4\lambda_1^2} \begin{pmatrix} (e^{\lambda_1 t} - e^{-\lambda_1 t})/2\lambda_1 & -te^{\lambda_1 t} \\ -te^{-\lambda_1 t} & (e^{\lambda_1 t} - e^{-\lambda_1 t})/2\lambda_1 \end{pmatrix}. \tag{4.13}$$

Hence we have the exponential spline with basis functions given by linear combinations of $e^{\sqrt{\beta}t}, e^{-\sqrt{\beta}t}, te^{\sqrt{\beta}t}, te^{-\sqrt{\beta}t}$.

1.c. $\beta = 0$, $\gamma \neq 0$. In this case, $\lambda_1 = 0$ (if $\gamma < 0$), or $\lambda_2 = 0$ (if $\gamma > 0$). Again the basis functions in 1.a. degenerate. Assume that $\lambda_1 = 0$, then $\lambda_2 = \lambda = 2\gamma$. From the limits

$$\lim_{\lambda_1 \to 0} \frac{(e^{\lambda_1 t} - e^{-\lambda_1 t})}{2\lambda_1} = t, \quad \lim_{\lambda_1 \to 0} e^{\lambda_1 t} = 1,$$

we have

$$e^{\Lambda t} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix}, \quad \hat{M}(t)^{-1} = \frac{1}{\lambda^3} \begin{pmatrix} \lambda t & e^{-\lambda t} - 1 \\ e^{-\lambda t} - 1 & (1 - e^{-2\lambda t})/2 \end{pmatrix}, \tag{4.14}$$

$$e^{\Lambda t}\hat{M}(t)^{-1} = \frac{1}{\lambda^3} \begin{pmatrix} \lambda t & e^{-\lambda t} - 1 \\ 1 - e^{\lambda t} & (e^{\lambda t} - e^{-\lambda t})/2 \end{pmatrix}. \tag{4.15}$$

Therefore we end up with the exponential spline with basis functions given by linear combinations of $1, t, e^{2\gamma t}, e^{-2\gamma t}$. Later we shall further show that this is the classical exponential spline [9].

2. $\gamma^2 + \beta < 0$. There are two complex eigenvalues: $\lambda_1 = \gamma + i\omega$, $\lambda_2 = \gamma - i\omega$, where $\omega = \sqrt{-\gamma^2 - \beta}$.

2.a. $\gamma \neq 0$, $\beta < 0$. Evaluating (4.10) and (4.12), we have the exponential-trigonometric spline with basis functions given by linear combinations of $e^{\gamma t}\sin\omega t$, $e^{\gamma t}\cos\omega t$, $e^{-\gamma t}\sin\omega t$, $e^{-\gamma t}\cos\omega t$.

2.b. $\gamma = 0$, $\beta < 0$. Again this is a degenerated case where $\lambda_1 = \lambda_2 = i\omega = i\sqrt{-\beta}$. Therefore (4.10) is now

$$e^{\Lambda t} = \begin{pmatrix} \cos\omega t + i\sin\omega t & 0 \\ 0 & \cos\omega t + i\sin\omega t \end{pmatrix}. \tag{4.16}$$

Taking the limit $\gamma \to 0$ in (4.12), we then have

$$e^{\Lambda t}\hat{M}(t)^{-1} = \frac{-1}{4\omega^2}\begin{pmatrix} \sin\omega t/\omega & -t(\cos\omega t + i\sin\omega t) \\ -t(\cos\omega t - i\sin\omega t) & \sin\omega t/\omega \end{pmatrix}. \tag{4.17}$$

Hence, we have the polynomial-trigonometric spline with basis functions given by linear combinations of $\sin\sqrt{-\beta}t, \cos\sqrt{-\beta}t, t\sin\sqrt{-\beta}t, t\cos\sqrt{-\beta}t$.

3. $\gamma^2 + \beta = 0$. In this case $\lambda_1 = \lambda_2 = \gamma$.

3.a. $\gamma \neq 0$. We have non-degenerated Jordan form in this case,

$$\Lambda = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1/\gamma \\ \gamma & 0 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 0 & 1/\gamma \\ -\gamma & 1 \end{pmatrix}. \tag{4.18}$$

Therefore,

$$e^{\Lambda t} = \begin{pmatrix} e^{\gamma t} & te^{\gamma t} \\ 0 & e^{\gamma t} \end{pmatrix} = e^{\gamma t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \tag{4.19}$$

$$\begin{aligned} \hat{M}(t)^{-1} &= \int_0^t e^{-2\gamma s}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\gamma \\ 1 \end{pmatrix}(1/\gamma, 1)\begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} ds \\ &= \int_0^t e^{-2\gamma s}\begin{pmatrix} (1/\gamma - s)^2 & 1/\gamma - s \\ 1/\gamma - s & 1 \end{pmatrix} ds. \end{aligned} \tag{4.20}$$

After some more detailed manipulation (see the next section) we can show that this is the exponential spline with basis functions given by linear combinations of $e^{\gamma t}$, $te^{\gamma t}$, $e^{-\gamma t}$, $te^{-\gamma t}$, similar to the case 1.b.

3.b. $\gamma = 0$. In this case, A itself is a Jordan matrix. We compute directly the following quantities:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \tag{4.21}$$

$$M(t)^{-1} = \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1)\begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} ds = \begin{pmatrix} t^3/3 & -t^2/2 \\ -t^2/2 & t \end{pmatrix}. \tag{4.22}$$

$$e^{At}M(t)^{-1} = \begin{pmatrix} -t^3/6 & t^2/2 \\ -t^2/2 & t \end{pmatrix}. \tag{4.23}$$

Then we have the polynomial spline with basis functions given by linear combinations of $1, t, t^2, t^3$. In the next section, we shall further show that this is the well-known cubic spline [4].

From the above discussion, we see that we may encounter all kinds of splines by varying parameters β and γ. Two general cases are 1.a. and 2.a. where we have full sized exponential or exponential-trigonometric splines. Degeneration occurs when zero or multiple eigenvalues appear. The extremal is the case 3.b. when both eigenvalues are zero. It is this extremal case that draws most of the attention. This is evidenced by extensive investigation regarding the cubic spline in the literature. Case 1.c. also has been investigated from a different point of view. But we can hardly find any work regarding the other cases (except 1.c. and 3.b.) listed above.

The situation for $m > 2$ is similar. Let $\lambda_1, \cdots, \lambda_m$ be eigenvalues of A. When $\lambda_1, \cdots, \lambda_m$; $-\lambda_1, \cdots, -\lambda_m$ are all distinct, we have the exponential spline with the basis functions given by linear combinations of

$$e^{\lambda_1 t}, \quad e^{-\lambda_1 t}, \quad \cdots, \quad e^{\lambda_m t}, \quad e^{-\lambda_m t}.$$

See case 1.a. When complex eigenvalues appear, we get the exponential-trigonometric splines with basis functions $e^{\lambda_k t}\sin\omega t$, $e^{-\lambda_k t}\cos\omega t$ (see case 2.a.). If we have multiple eigenvalues, the terms like

$$te^{\lambda t}, \quad t\sin\omega t, \quad te^{-\lambda t}\cos\omega t, \quad t^2 e^{\lambda t}, \quad \cdots$$

will appear in basis functions (see cases 1.b., 2.b. and 3.a.). Finally, zero eigenvalues will introduce polynomials into basis functions (see case 1.c.) and the extremal situation is that all eigenvalues are zero in which case we recover polynomial splines of order $2m-1$ (see case 3.b.).

5. Examples of splines.

In this section, we shall work out in detail some classes of splines. We shall explicitly construct the nodal shape functions and the linear system needed to solve for the unknown parameters.

1. Our first example is the case 3.b. which turns out to be the classical cubic spline. We first construct the nodal shape functions. From Theorem 4.1 we need only to calculate the first row of matrix (4.4) and the first row of matrix (4.5). Let $t = h$ in (4.22), and we have

$$M(h) = \begin{pmatrix} h^3/3 & -h^2/2 \\ -h^2/2 & h \end{pmatrix}^{-1} = \frac{12}{h^4}\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}.$$

Thus from (4.21) and (4.23), we have

$$e^{At} - e^{At}M(t)^{-1}M(h) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -t^3/6 & t^2/2 \\ -t^2/2 & t \end{pmatrix}\frac{12}{h^4}\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -2t^3/h^3 + 3t^2/h^2 & t(-t^2/h^2 + 2t/h) \\ \cdot & \cdot \end{pmatrix}.$$

$$\begin{aligned} e^{At}M(t)^{-1}M(h)e^{-Ah} &= \begin{pmatrix} -2t^3/h^3 + 3t^2/h^2 & t(-t^2/h^2 + 2t/h) \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2t^3/h^3 + 3t^2/h^2 & t(t^2/h^2 - t/h) \\ \cdot & \cdot \end{pmatrix}. \end{aligned}$$

Hence

$$p_1(t) = 1 + 2(\frac{t}{h})^3 - 3(\frac{t}{h})^2, \quad p_2(t) = t[1 + (\frac{t}{h})^2 - 2(\frac{t}{h})]; \tag{5.1}$$

$$q_1(t) = -2(\frac{t}{h})^3 + 3(\frac{t}{h})^2, \quad q_2(t) = t(\frac{t}{h})[(\frac{t}{h}) - 1]. \tag{5.2}$$

They are precisely the nodal shape functions for the Hermit interpolation.

Next we set in (3.8), $r = 0$, $\vec{x}^k = (\alpha_k, \beta_k)^T$, and $h_k = h_{k+1} = h$ (by this, we are using equally spaced intervals). The equation (3.8) is now

$$-\vec{b}^T e^{-A^T h} M(h)\vec{x}^{k-1} + \vec{b}^T [e^{-A^T h} M(h) e^{-Ah} + M(h)]\vec{x}^k - \vec{b}^T M(h) e^{-Ah} \vec{x}^{k+1} = 0, \tag{5.3}$$

for $k = 1, \cdots, n-1$. Substituting

$$e^{-A^T h} M(h) = \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \frac{12}{h^4} \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix} = \frac{12}{h^3} \begin{pmatrix} 1 & h/2 \\ -h/2 & -h^2/6 \end{pmatrix},$$

$$M(h)e^{-Ah} = [e^{-A^T h} M(h)]^T = \frac{12}{h^3} \begin{pmatrix} 1 & -h/2 \\ h/2 & -h^2/6 \end{pmatrix},$$

$$e^{-A^T h} M(h) e^{-Ah} = \frac{12}{h^3} \begin{pmatrix} 1 & h/2 \\ -h/2 & -h^2/6 \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} = \frac{12}{h^3} \begin{pmatrix} 1 & -h/2 \\ -h/2 & h^2/3 \end{pmatrix},$$

into (5.3) yields,

$$(\frac{6}{h^2}, \frac{2}{h}) \begin{pmatrix} \alpha_{k-1} \\ \beta_{k-1} \end{pmatrix} + (0, \frac{8}{h}) \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} - (\frac{6}{h^2}, -\frac{2}{h}) \begin{pmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{pmatrix} = 0,$$

or

$$\beta_{k-1} + 4\beta_k + \beta_{k+1} = \frac{3}{h}(\alpha_{k+1} - \alpha_{k-1}), \quad k = 1, \cdots, n-1. \tag{5.4}$$

In the matrix form (5.4) is

$$\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{pmatrix} = \frac{3}{h} \begin{pmatrix} \alpha_2 - \alpha_0 \\ \alpha_3 - \alpha_1 \\ \alpha_4 - \alpha_2 \\ \vdots \\ \alpha_{n-1} - \alpha_{n-3} \\ \alpha_n - \alpha_{n-2} \end{pmatrix} - \begin{pmatrix} \beta_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \beta_n \end{pmatrix}. \tag{5.5}$$

This is precisely the same linear system as we construct the cubic spline. After solving β_k's from (5.5), the desired cubic spline can be expressed piecewisely on the subinterval $[t_{k-1}, t_k]$, $k = 1, \cdots, n$ as

$$y(t) = \alpha_{k-1} p_1(t - t_{k-1}) + \beta_{k-1} p_2(t - t_{k-1}) + \alpha_k q_1(t - t_{k-1}) + \beta_k q_2(t - t_{k-1}),$$

where p_1, p_2, q_1, q_2 are given by (5.1) and (5.2).

2. The second example is the case 1.c. which is the classical exponential spline [9]. The nodal shape functions are again derived from the first rows of matrices (4.4) and (4.5). We use the Jordan form. let $\lambda_1 = 0$, and $\lambda_2 = \lambda$ (4.9), and we have

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & \lambda \end{pmatrix}, \quad Q^{-1} = \frac{1}{\lambda} \begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix}.$$

Using (4.14) we can calculate (by setting $t = h$),

$$\hat{M}(h) = \lambda^3 \begin{pmatrix} \lambda t & e^{-\lambda t} - 1 \\ e^{-\lambda t} - 1 & (1 - e^{-2\lambda t})/2 \end{pmatrix}^{-1} = C(\lambda) \begin{pmatrix} (1 + e^{-\lambda h})/2 & 1 \\ 1 & \lambda h (1 - e^{-\lambda h})^{-1} \end{pmatrix},$$

where

$$C(\lambda) = \frac{2\lambda^3}{\lambda h (1 + e^{-\lambda h}) - 2(1 - e^{-\lambda h})}.$$

Recall (4.14) and (4.15), and we have

$$\begin{aligned} & Q e^{\Lambda t} [I - \hat{M}(t)^{-1} \hat{M}(h)] Q^{-1} \\ = & \begin{pmatrix} 1 & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix} [I - \frac{C(\lambda)}{\lambda^3} \begin{pmatrix} \lambda t & e^{-\lambda t} - 1 \\ 1 - e^{\lambda t} & (e^{\lambda t} - e^{-\lambda t})/2 \end{pmatrix} \cdot \\ & \begin{pmatrix} (1 + e^{-\lambda h})/2 & 1 \\ 1 & \lambda h (1 - e^{-\lambda h})^{-1} \end{pmatrix}] \frac{1}{\lambda} \begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} 1 & (e^{\lambda t} - 1)/\lambda \\ 0 & e^{\lambda t} \end{pmatrix} - \frac{C(\lambda)}{\lambda^4} \begin{pmatrix} 1 & 1 \\ 0 & \lambda t \end{pmatrix} \begin{pmatrix} \lambda t & e^{-\lambda t} - 1 \\ 1 - e^{\lambda t} & (e^{\lambda t} - e^{-\lambda t})/2 \end{pmatrix} \cdot \\ & \begin{pmatrix} (1 + e^{-\lambda h})/2 & 1 \\ 1 & \lambda h (1 - e^{-\lambda h})^{-1} \end{pmatrix} \begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} p_1(\lambda; t) & p_2(\lambda; t) \\ \cdot & \cdot \end{pmatrix}, \end{aligned}$$

where

$$p_1(\lambda; t) = 1 - \frac{\lambda t (1 + e^{-\lambda h}) - (1 - e^{-\lambda h}) - e^{\lambda (t-h)} + e^{-\lambda t}}{\lambda h (1 + e^{-\lambda h}) - 2(1 - e^{-\lambda h})}, \tag{5.6}$$

$$p_2(\lambda;t)=\frac{e^{\lambda t}-1}{\lambda}-\frac{h}{1+e^{-\lambda h}-2\frac{1-e^{-\lambda h}}{\lambda h}}[\frac{1-e^{-\lambda h}}{\lambda h}(\frac{t}{h}+\frac{1-e^{\lambda t}}{\lambda h})-\frac{1-\frac{\lambda h}{(1-e^{-\lambda h})}}{\lambda h}\cdot\frac{e^{\lambda t}-2+e^{-\lambda t}}{\lambda h}]. \tag{5.7}$$

$$\begin{aligned}
&Qe^{\Lambda t}\hat{M}(t)^{-1}\hat{M}(h)e^{-\Lambda h}Q^{-1}\\
=\;&\frac{C(\lambda)}{\lambda^4}\begin{pmatrix}1&1\\0&\lambda\end{pmatrix}\begin{pmatrix}1&0\\0&e^{\lambda t}\end{pmatrix}\begin{pmatrix}\lambda t&e^{-\lambda t}-1\\e^{-\lambda t}-1&(1-e^{-2\lambda t})/2\end{pmatrix}\cdot\\
&\begin{pmatrix}(1+e^{-\lambda h})/2&1\\1&\lambda h(1-e^{-\lambda h})^{-1}\end{pmatrix}\begin{pmatrix}1&0\\0&e^{-\lambda h}\end{pmatrix}\begin{pmatrix}\lambda&-1\\0&1\end{pmatrix}\\
=\;&\frac{C(\lambda)}{\lambda^4}\begin{pmatrix}1&1\\0&\lambda\end{pmatrix}\begin{pmatrix}\lambda t&e^{-\lambda t}-1\\1-e^{\lambda t}&(e^{\lambda t}-e^{-\lambda t})/2\end{pmatrix}\begin{pmatrix}(1+e^{-\lambda h})/2&e^{-\lambda h}\\1&\lambda h(e^{\lambda h}-1)^{-1}\end{pmatrix}\cdot\\
&\begin{pmatrix}\lambda&-1\\0&1\end{pmatrix}\\
=\;&\begin{pmatrix}q_1(\lambda;t)&q_2(\lambda;t)\\ \cdot&\cdot\end{pmatrix},
\end{aligned}$$

where

$$q_1(\lambda;t)=1-p_1(\lambda;t), \tag{5.8}$$

$$q_2(\lambda;t)=-\frac{h}{1+e^{-\lambda h}-2\frac{1-e^{-\lambda h}}{\lambda h}}[\frac{1-e^{-\lambda h}}{\lambda h}(\frac{t}{h}+\frac{1-e^{\lambda t}}{\lambda h})-\frac{\frac{\lambda h}{e^{\lambda h}-1}-1}{\lambda h}\cdot\frac{e^{\lambda t}-2+e^{-\lambda t}}{\lambda h}]. \tag{5.9}$$

(5.6) - (5.9) are nodal shape functions for the classical exponential spline.

Next we set in (3.12), $r=0$, $\vec{x}^k=(\alpha_k,\beta_k)^T$, and $h_k=h_{k+1}=h$. The equation (3.12) is now

$$\begin{aligned}
&-(Q^{-1}\vec{b})^T e^{-\Lambda^T h}\hat{M}(h)Q^{-1}\vec{x}^{k-1}+(Q^{-1}\vec{b})^T[e^{-\Lambda^T h}\hat{M}(h)e^{-\Lambda h}+\hat{M}(h)]Q^{-1}\vec{x}^k\\
&-(Q^{-1}\vec{b})^T\hat{M}(h)e^{-\Lambda h}Q^{-1}\vec{x}^{k+1}=0,\quad k=1,\cdots,n-1.
\end{aligned} \tag{5.10}$$

Substituting

$$(Q^{-1}\vec{b})^T=\frac{1}{\lambda}(-1,1),$$

$$\begin{aligned}
e^{-\Lambda h}\hat{M}(h)&=C(\lambda)\begin{pmatrix}1&0\\0&e^{-\lambda h}\end{pmatrix}\begin{pmatrix}(1+e^{-\lambda h})/2&1\\1&\lambda h(1-e^{-\lambda h})^{-1}\end{pmatrix}\\
&=C(\lambda)\begin{pmatrix}(1+e^{-\lambda h})/2&1\\e^{-\lambda h}&\lambda h(e^{\lambda h}-1)^{-1}\end{pmatrix},
\end{aligned}$$

$$\hat{M}(h)e^{-\Lambda h}=[e^{-\Lambda h}\hat{M}(h)]^T=C(\lambda)\begin{pmatrix}(1+e^{-\lambda h})/2&e^{-\lambda h}\\1&\lambda h(e^{\lambda h}-1)^{-1}\end{pmatrix},$$

$$
\begin{aligned}
e^{-\Lambda h}\hat{M}(h)e^{-\Lambda h} &= C(\lambda)\begin{pmatrix} 1 & 0 \\ 0 & e^{-\lambda h} \end{pmatrix}\begin{pmatrix} (1+e^{-\lambda h})/2 & e^{-\lambda h} \\ 1 & \lambda h(e^{\lambda h}-1)^{-1} \end{pmatrix} \\
&= C(\lambda)\begin{pmatrix} (1+e^{-\lambda h})/2 & e^{-\lambda h} \\ e^{-\lambda h} & \lambda h e^{-\lambda h}(e^{\lambda h}-1)^{-1} \end{pmatrix},
\end{aligned}
$$

into (5.10), canceling $C(\lambda)/\lambda^2$, we have

$$
\begin{aligned}
&-(-1,1)\begin{pmatrix} (1+e^{-\lambda h})/2 & 1 \\ e^{-\lambda h} & \lambda h(e^{\lambda h}-1)^{-1} \end{pmatrix}\begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_{k-1} \\ \beta_{k-1} \end{pmatrix} \\
&+(-1,1)\begin{pmatrix} 1+e^{-\lambda h} & 1+e^{-\lambda h} \\ 1+e^{-\lambda h} & \lambda h(1+e^{-2\lambda h}(1-e^{-\lambda h})^{-1} \end{pmatrix}\begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_{k} \\ \beta_{k} \end{pmatrix} \\
&-(-1,1)\begin{pmatrix} (1+e^{-\lambda h})/2 & e^{-\lambda h} \\ 1 & \lambda h(e^{\lambda h}-1)^{-1} \end{pmatrix}\begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{pmatrix} = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{1-e^{-2\lambda h}-2\lambda h e^{-\lambda h}}{2(1-e^{-\lambda h})}(\beta_{k-1}+\beta_{k+1}) + [\frac{\lambda h(1+e^{-2\lambda h})}{1-e^{-\lambda h}} - (1+e^{-\lambda h})]\beta_k \\
= {}& \lambda\frac{1-e^{-\lambda h}}{2}(\alpha_{k+1}-\alpha_{k-1}), \quad k=1,\cdots,n-1. \qquad (5.11)
\end{aligned}
$$

This is the linear system for the exponential spline, it can be written in the matrix form as

$$
\begin{pmatrix}
a(\lambda;h) & b(\lambda;h) & 0 & \cdots & 0 & 0 \\
b(\lambda;h) & a(\lambda;h) & b(\lambda;h) & \cdots & 0 & 0 \\
0 & b(\lambda;h) & a(\lambda;h) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a(\lambda;h) & b(\lambda;h) \\
0 & 0 & 0 & \cdots & b(\lambda;h) & a(\lambda;h)
\end{pmatrix}
\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{pmatrix}
= \frac{3}{h}\begin{pmatrix} \alpha_2-\alpha_0 \\ \alpha_3-\alpha_1 \\ \alpha_4-\alpha_2 \\ \vdots \\ \alpha_{n-1}-\alpha_{n-3} \\ \alpha_n-\alpha_{n-2} \end{pmatrix}
- \begin{pmatrix} b(\lambda;h)\beta_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b(\lambda;h)\beta_n \end{pmatrix}, \qquad (5.12)
$$

where

$$
a(\lambda;h) = 6\frac{\lambda h(1+e^{-2\lambda h})-(1-e^{-2\lambda h})}{\lambda h(1-e^{-\lambda h})^2}, \quad b(\lambda;h) = 3\frac{1-e^{-2\lambda h}-2\lambda h e^{-\lambda h}}{\lambda h(1-e^{-\lambda h})^2}.
$$

Again, after finding β_k's, the spline can be expressed piecewise by the nodal shape functions.

It is interesting to examine the limiting case for the exponential spline obtained above. Applying the L'Hospital's rule three times, we are able to verify that

$$\lim_{\lambda\to 0} a(\lambda;h)=4, \quad \lim_{\lambda\to 0} b(\lambda;h)=1. \tag{5.13}$$

We then recover (5.5), the tridiagonal systems for the cubic spline. Further we can verify that (by successively using the L'Hospital's rule)

$$\lim_{\lambda\to 0} q_1(\lambda;h)=\frac{3}{h}t-1-(\frac{t}{h}-1)^3-(\frac{t}{h})^3=-2(\frac{t}{h})^3+3(\frac{t}{h})^2=q_1(t),$$

$q_1(t)$ is one of the nodal shape functions for the cubic spline given by (5.2). The other three nodal shape functions can be verified similarly. So the cubic spline is the limiting case for the exponential spline 4.1.c. when $\lambda\to 0$. This is not a surprise from the following limit regarding matrix A of the system (2.1),

$$\lim_{\lambda\to 0}\begin{pmatrix} 0 & 1 \\ 0 & \lambda \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.14}$$

where the left hand side is the matrix A for the case 1.c., and the right hand side is the matrix A for the case 3.b. We can also examine the limit when $\lambda\to\infty$ where

$$\lim_{\lambda\to\infty} a(\lambda;h)=6, \quad \lim_{\lambda\to\infty} b(\lambda;h)=0, \tag{5.15}$$

$$\lim_{\lambda\to\infty} p_1(\lambda;t)=1-\frac{t}{h}, \quad \lim_{\lambda\to\infty} q_1(\lambda;t)=\frac{t}{h}, \tag{5.16}$$

$$\lim_{\lambda\to\infty} p_2(\lambda;t)=0=\lim_{\lambda\to\infty} q_2(\lambda;t). \tag{5.17}$$

Substituting (5.15) into (5.12), we then have $\beta_k=(\alpha_{k+1}-\alpha_{k-1})/2h$, $k=1,\cdots,n-1$, which is the central difference scheme. We see that (5.16) gives the nodal shape functions for the linear interpolation. In order to verify (5.17), it is convenient to rewrite

$$\begin{aligned} p_2(\lambda;t) &= -\frac{h}{\lambda h-2}\cdot\frac{e^{-\lambda h}}{1-e^{-\lambda h}}-\frac{(\lambda h+2)e^{-\lambda h}}{(\lambda h-2)[\lambda h(1+e^{-\lambda h})-2(1-e^{-\lambda h})]}(\frac{2}{\lambda}-\frac{h}{1-e^{-\lambda h}}) \\ &- \frac{1}{\lambda}+\frac{C(\lambda)}{2\lambda^3}[-t(1-e^{-\lambda h})+\frac{e^{-\lambda h}-2}{\lambda}-\frac{e^{\lambda(t-h)}}{\lambda}-\frac{1-e^{-\lambda t}}{\lambda}-\frac{h}{1-e^{-\lambda h}}(e^{-\lambda t}-2)], \\ q_2(\lambda;t) &= \frac{C(\lambda)}{2\lambda^3}[-t(1-e^{-\lambda h})+\frac{1-e^{-\lambda t}}{\lambda}+\frac{e^{-\lambda h}-e^{\lambda(t-h)}}{\lambda}+\frac{h(e^{\lambda(t-h)}-2e^{-\lambda h}+e^{-\lambda(t+h)})}{1-e^{-\lambda h}}]. \end{aligned}$$

So the linear spline (the piecewise linear interpolation) is the limiting case for the exponential spline 4.1.c. when $\lambda\to\infty$.

3. The third example is the case 2.b. when $\beta = -\omega^2$, $\gamma = 0$ and

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

Denote

$$C_1 = \omega^2 h^2 - \sin^2 \omega h, \quad C_2 = \frac{1}{2}\sin 2\omega h + \omega h, \quad C_3 = \sin^2 \omega h, \quad C_4 = \frac{1}{2}\sin 2\omega h - \omega h.$$

Following the same procedure as the first example, we have the nodal shape functions as following

$$\begin{aligned}
C_1 p_1(t) &= C_1 \cos \omega t + C_2(\sin \omega t - \omega t \cos \omega t) - C_3 \omega t \sin \omega t, \\
C_1 p_2(t) &= \omega h^2 \sin \omega t - C_3 t \cos \omega t + C_4 t \sin \omega t, \\
q_1(t) &= 1 - p_1(t), \\
C_1 q_2(t) &= h \sin \omega h(\sin \omega t - \omega t \cos \omega t) - t \sin \omega t(\sin \omega h - \omega h \cos \omega h).
\end{aligned}$$

When evenly spaced intervals are used, the tridiagonal system for unknowns β_k is given by

$$b(\omega;h)\beta_{k-1} + a(\omega;h)\beta_k + b(\omega;h)\beta_{k+1} = c(\omega;h)(\alpha_{k+1} - \alpha_{k-1}), \quad k = 1, \cdots, n, \tag{5.18}$$

where

$$a(\omega;h) = 2\omega h - \sin 2\omega h, \quad b(\omega;h) = \sin \omega h - \omega h \sin \omega h, \quad c(\omega;h) = \omega^2 h \sin \omega h.$$

It is easy to verify that,

$$\lim_{\omega \to 0} \frac{3a(\omega;h)}{(\omega h)^3} = 4, \quad \lim_{\omega \to 0} \frac{3b(\omega;h)}{(\omega h)^3} = 1, \quad \lim_{\omega \to 0} \frac{3c(\omega;h)}{(\omega h)^3} = \frac{3}{h},$$

which are the coefficients for the tridiagonal system of the cubic spline. We can also verify that at the limit $\omega \to 0$, the nodal shape functions $p_i(\omega;t)$ and $q_i(\omega;t)$, $i = 1, 2$, have the relative nodal shape functions of the cubic spline as their limit when $\omega \to 0$. Indeed, we recover the cubic spline from this trigonometric spline when $\omega \to 0$.

4. The fourth example is the case 3.a. where $\beta = -\gamma^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ -\gamma^2 & 2\gamma \end{pmatrix}.$$

Denote

$$C = 1 - 2e^{-2\gamma h} + e^{-4\gamma h} - 4\gamma^2 h^2 e^{-2\gamma h}.$$

Computing the first rows of matrices $e^{At} - e^{At}M(t)^{-1}M(h)$ and $e^{At}M(t)^{-1}M(h)e^{-Ah}$, we have the nodal shape functions

$$\begin{aligned}
Cp_1(t) &= e^{-\gamma(2h+t)}(2\gamma h - 2\gamma^2h^2 - \gamma t + 2\gamma^2 ht - 1) + e^{-\gamma t}(1+\gamma t) \\
&\quad + e^{-\gamma(2h-t)}(\gamma t + 2\gamma^2 ht - 2\gamma h + 2\gamma^2 h^2 - 1) + e^{-\gamma(4h-t)}(1-\gamma t), \\
Cp_2(t) &= t(e^{-\gamma(4h-t)} + e^{-\gamma t}) + e^{-\gamma(2h-t)}(2\gamma ht - 2\gamma h^2 - t) + e^{-\gamma(2h+t)}(2\gamma h^2 - t - 2\gamma ht), \\
q_1(t) &= 1 - p_1(t), \\
Cq_2(t) &= e^{-\gamma(3h-t)}(h - t - 2\gamma ht) + (e^{-\gamma(3h+t)} + e^{-\gamma(h-t)})(t-h) + e^{-\gamma(h+t)}(2\gamma ht + h - t).
\end{aligned}$$

Evaluating (5.3) in the current case, we get the tridiagonal system (5.18) with

$$\begin{aligned}
a(\omega;h) = a(\gamma;h) &= e^{\gamma h}(1 - e^{-4\gamma h} - 4\gamma h e^{-2\gamma h}), \\
b(\omega;h) = b(\gamma;h) &= e^{-2\gamma h}(1+\gamma h) - (1-\gamma h), \\
c(\omega;h) = c(\gamma;h) &= \gamma^2 h(1 - e^{-2\gamma h}).
\end{aligned}$$

Again we can verify that the cubic spline is the limiting case for this exponential spline when $\gamma \to 0$.

5. Our last example is a case for $m = 3$ when

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.19}$$

The system (2.1) with A given by (5.19) produces the quintic spline. In this case,

$$e^{At} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.20}$$

$$M(t)^{-1} = \int_0^t e^{-As}\vec{b}\vec{b}^T e^{-A^T s} ds = \begin{pmatrix} t^5/(20) & -t^4/8 & t^3/6 \\ -t^4/8 & t^3/3 & -t^2/2 \\ t^3/6 & -t^2/2 & t \end{pmatrix}, \tag{5.21}$$

$$M(h) = [M(h)^{-1}]^{-1} = 3\begin{pmatrix} 240/h^5 & 120/h^4 & 20/h^3 \\ 120/h^4 & 64/h^3 & 12/h^2 \\ 20/h^3 & 12/h^2 & 3/h \end{pmatrix}. \tag{5.22}$$

Using (5.20) - (5.22) to compute the first rows of matrices $e^{At} - e^{At}M(t)^{-1}M(h)$ and $e^{At}M(t)^{-1}M(h)e^{-Ah}$, we then have the nodal shape functions for the quintic interpolation:

$$p_1(t) = \frac{(h-t)^3}{h^5}[h^2 + 3ht + 6t^2], \quad q_1(t) = \frac{t^3}{h^5}[h^2 - 3ht + 6t^2],$$

$$
\begin{aligned}
p_2(t) &= \frac{(h-t)^3}{h^4}[ht+3t^2], \quad q_2(t) = \frac{t^3}{h^4}[h(t-h)-3t^2], \\
p_3(t) &= \frac{(h-t)^3t^2}{2h^3}, \quad q_3(t) = \frac{t^3(t-h)^2}{2h^3}.
\end{aligned}
$$

In order to compute the parameters for the optimal control, we set in (3.8) $r = 0, 1$ $\vec{x}^k = (\alpha_k, \beta_k, \gamma_k)^T$, and $h_k = h_{k+1} = h$. Then except (5.3), we also have

$$
-(A\vec{b})^T e^{-A^T h} M(h) \vec{x}^{k-1} + (A\vec{b})^T [e^{-A^T h} M(h) e^{-Ah} + M(h)] \vec{x}^k - (A\vec{b})^T M(h) e^{-Ah} \vec{x}^{k+1} = 0, \tag{5.23}
$$

for $k = 1, \cdots, n-1$. Substituting (5.20) - (5.22) into (5.3) and (5.23), after some tedious symbolic manipulation, we have the following linear system,

$$
8(\beta_{k+1} - \beta_{k-1}) + h(-\gamma_{k-1} + 6\gamma_i - \gamma_{k+1}) = \frac{20}{h}(f_{k-1} - 2f_k + f_{k+1}); \tag{5.24}
$$

$$
7\beta_{k-1} + 16\beta_k + 7\beta_{k+1} + h(\gamma_{k-1} - \gamma_{k+1}) = \frac{15}{h}(f_{k+1} - f_{k-1}), \tag{5.25}
$$

for $k = 1, \cdots, n-1$. If we arrange the unknowns as $(\beta_1, h\gamma_1, \cdots, \beta_{n-1}, h\gamma_{n-1})$, we will get a linear system with a banded 6-diagonal coefficient matrix; if we arrange the unknowns as $(\beta_1, \cdots, \beta_{n-1}, h\gamma_1, \cdots, h\gamma_{n-1}) = (\vec{\beta}^T, h\vec{\gamma}^T)$, we will have a linear system with a block tridiagonal coefficient matrix.

$$
\begin{bmatrix} S & E^T \\ 8E & H \end{bmatrix} \begin{bmatrix} \vec{\beta} \\ h\vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}, \tag{5.26}
$$

where

$$
S = \begin{bmatrix} 16 & 7 & 0 & \cdots & 0 & 0 \\ 7 & 16 & 7 & \cdots & 0 & 0 \\ 0 & 7 & 16 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 16 & 7 \\ 0 & 0 & 0 & \cdots & 7 & 16 \end{bmatrix}, \quad H = \begin{bmatrix} 6 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 6 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 6 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 6 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 6 \end{bmatrix},
$$

$$
E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},
$$

$$\vec{f} = \begin{bmatrix} 15(f_2 - f_0)/h - 7\beta_0 - h\gamma_0 \\ 15(f_3 - f_1)/h \\ 15(f_4 - f_2)/h \\ \vdots \\ 15(f_{n-1} - f_{n-3})/h \\ 15(f_n - f_{n-2})/h - 7\beta_n + h\gamma_n \end{bmatrix}, \quad \vec{g} = \begin{bmatrix} 20(f_0 - 2f_1 + f_2)/h + 8\beta_0 + h\gamma_0 \\ 20(f_1 - 2f_2 + f_3)/h \\ 20(f_2 - 2f_3 + f_4)/h \\ \vdots \\ 20(f_{n-3} - 2f_{n-2} + f_{n-1})/h \\ 20(f_{n-2} - 2f_{n-1} + f_n)/h - 8\beta_n + h\gamma_n \end{bmatrix}.$$

In general, if A is a $m \times m$ nilpotent matrix with 1's on the super diagonal and 0's elsewhere, we shall recover all odd degree polynomial splines (with degree $2m-1$).

6. Numerical experiments.

In this section, we test the behaviors of different splines numerically. Equally spaced intervals are used for all computations.

Example 1. Comparison of the cubic spline with the quintic spline.

Test function 1.

$$f(t) = \begin{cases} 0 & -1 \le t < 0 \\ 1/2 & t = 0 \\ 1 & 0 < t \le 1 \end{cases}$$

For the cubic spline, we pose, in (5.5), the boundary conditions:

$$\beta_0 = 0 = \beta_n;$$

and for the quintic spline, we pose, in (5.25), the boundary condition:

$$\beta_0 = 0 = \beta_n, \quad \gamma_0 = 0 = \gamma_n.$$

Recall that β_i and γ_j are the coefficients for the first and the second derivatives, respectively. The spline functions are then constructed for $h = .2$, $h = .1$, $h = .05$, and $h = .025$. Graphs are plotted in Figure 1(a), 1(b). We see that the qualitative behavior of the two splines are almost same, but the quintic spline has a little better accuracy.

One interesting phenomenon is that the mesh refinement does not effect the maximum overshoot of the spline approximation. Since this is very similar to the Gibbs phenomenon for the Fourier series, we term it as "Gibbs phenomenon" of splines. In fact, all spline functions have this property.

Test function 2.

$$g(t) = e^{-10t^3}, \quad -1 \le t \le 0.$$

For the cubic spline, we pose the boundary conditions:

$$\beta_0 = -30e^{10}, \quad \beta_n = 0;$$

and for the quintic spline, we pose the boundary conditions:

$$\beta_0 = -30e^{10}, \quad \beta_n = 0, \quad \gamma_0 = 960e^{10}, \quad \gamma_n = 0.$$

We use the mesh size $h = .2$, and plot the graphs in Figure 1(c), 1(d). We see that the quintic spline gives much better approximation in the neighborhood of $x = -1$ since it has the correct concavity information at $x = -1$ which the cubic spline does not have.

Example 2. Properties of the classical exponential spline case 1.c.

The test function is the same $f(t)$ as in Example 1. We have observed that for small parameter λ, the behavior is much like the cubic spline. This is not surprise from (5.14). The interesting fact is: For the moderate λ, the graph is very much the same as the cubic spline (Figure 2(a)). If we fix the parameter λ and refine the mesh, we observe Gibbs phenomenon as in the cubic and the quintic splines. But if we fix the mesh (here we choose $h = .1$) and increase the parameter λ, we see that the approximation converges to the piecewise linear function (Figure 2(b), 2(c), 2(d)). This confirms our theoretical analysis made in Section 5.

Example 3. Properties of the exponential spline case 3.a.

We use the same test function $f(t)$ as in the examples 1, 2.

For small parameter γ, the approximating feature of this spline is also like the cubic spline including the Gibbs phenomenon. But when we fix the mesh (here $h = .1$) and increase the parameter γ, an unexpected wiggling appears at $t = 0$ (Figure 3(a)-3(d)).

Example 4. Properties of the exponential spline case 1.a.

Here we choose

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix},$$

and we have $\lambda_1 = -1$, $\lambda_2 = -2$. Again, the testing function is $f(t)$ as in the previous examples.

We plot the approximation for $h = .1$, $h = .05$, $h = .025$ in Figure 4(a), 4(b), 4(c), respectively. again, we observe the similar behavior as that of the cubic spline.

Conclusions

1. Gibbs phenomenon exists for all splines.
2. The quintic spline is recommended if the concavity is important.

3. From the approximation point of view, the classical exponential spline is preferred when the function has points of discontinuity.

A final remark.

For the discussion purpose, we constructed spline approximation in this paper by introducing the nodal shape functions which is not necessary in practical computation. From the framework we have established based on the control theory in Section 3, all we need to do is: providing the matrix A, the vector $\vec{b}$ to the linear system (3.8), solving (3.8) numerically to obtain $\vec{x}^k$'s, and hence the control law $u(t)$ (see (2.9)). After we have the control $u(t)$, the expected spline function is given by the first component of $\vec{x}(t)$ defined by (3.13). Based on our analysis, we are able to choose different splines by simply selecting entries of the matrix A.

The significance of this investigation is two fold: first, it exposes the relationship between two important fields - control theory and spline approximations. This enables us to discover new spline functions and to investigate, systematically, the properties of the spline approximations. Secondly, it provides a practical way to construct different splines from a same simple framework. From our experience, we feel that this construction is more natural and easier than the traditional approach.

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Figure 1.

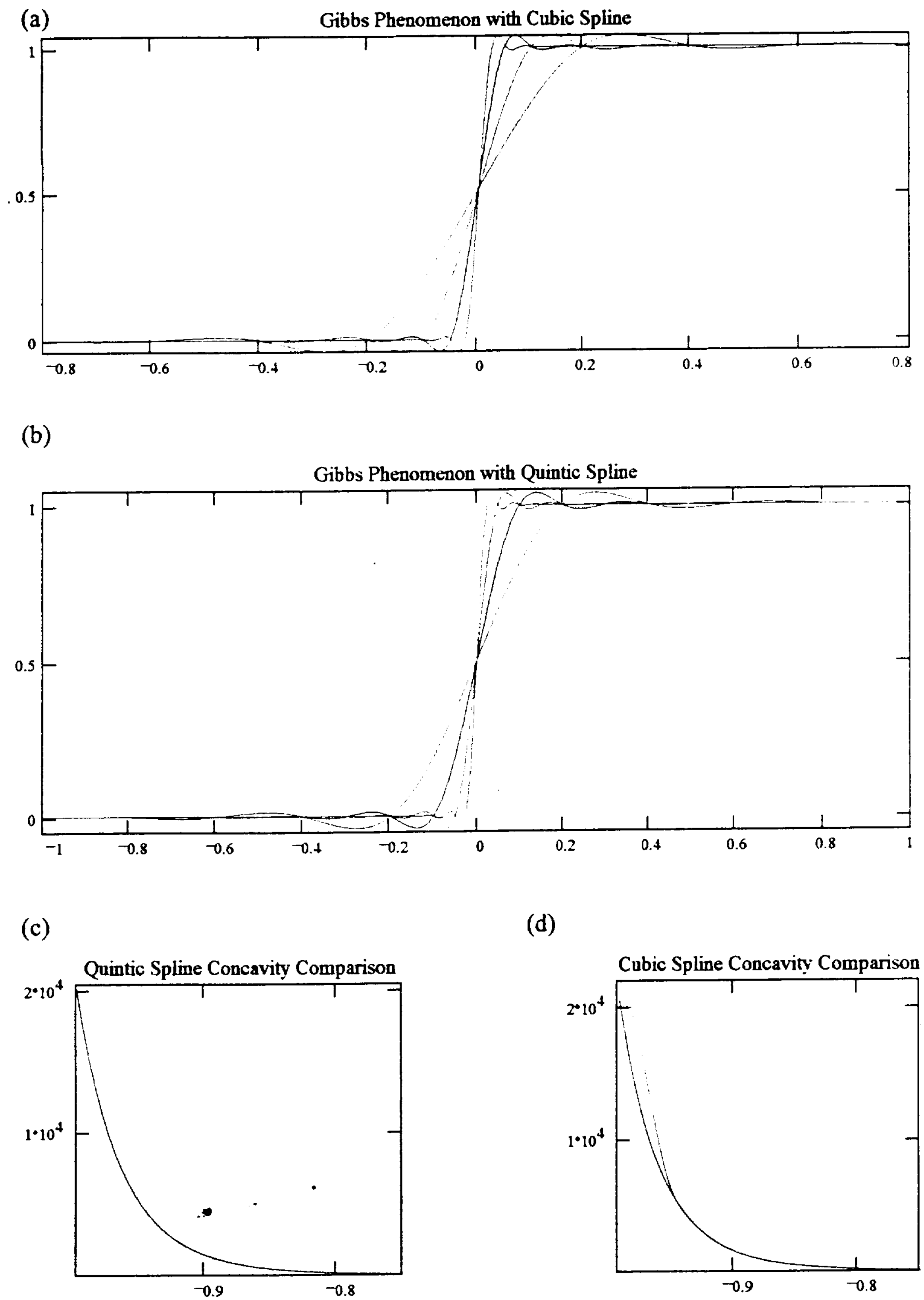

(a)
Gibbs Phenomenon with Cubic Spline
1
0.5
0
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
(b)
Gibbs Phenomenon with Quintic Spline
1
0.5
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
(c)
Quintic Spline Concavity Comparison
2•10^4
1•10^4
−0.9
−0.8
(d)
Cubic Spline Concavity Comparison
2•10^4
1•10^4
−0.9
−0.8

Figure 2.

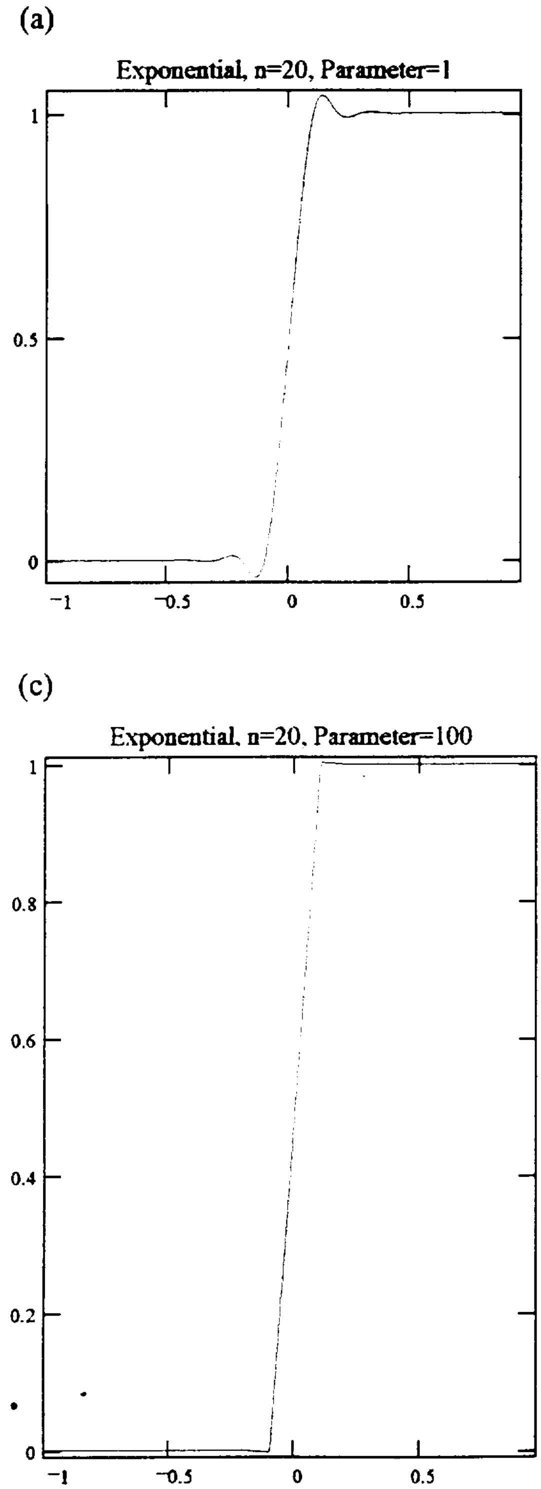

(a)
Exponential, n=20, Parameter=1
(b)
Exponential, n=20, Parameter=10
(c)
Exponential, n=20, Parameter=100
(d)
Exponential, n=20, Parameter=1000

Figure 3.

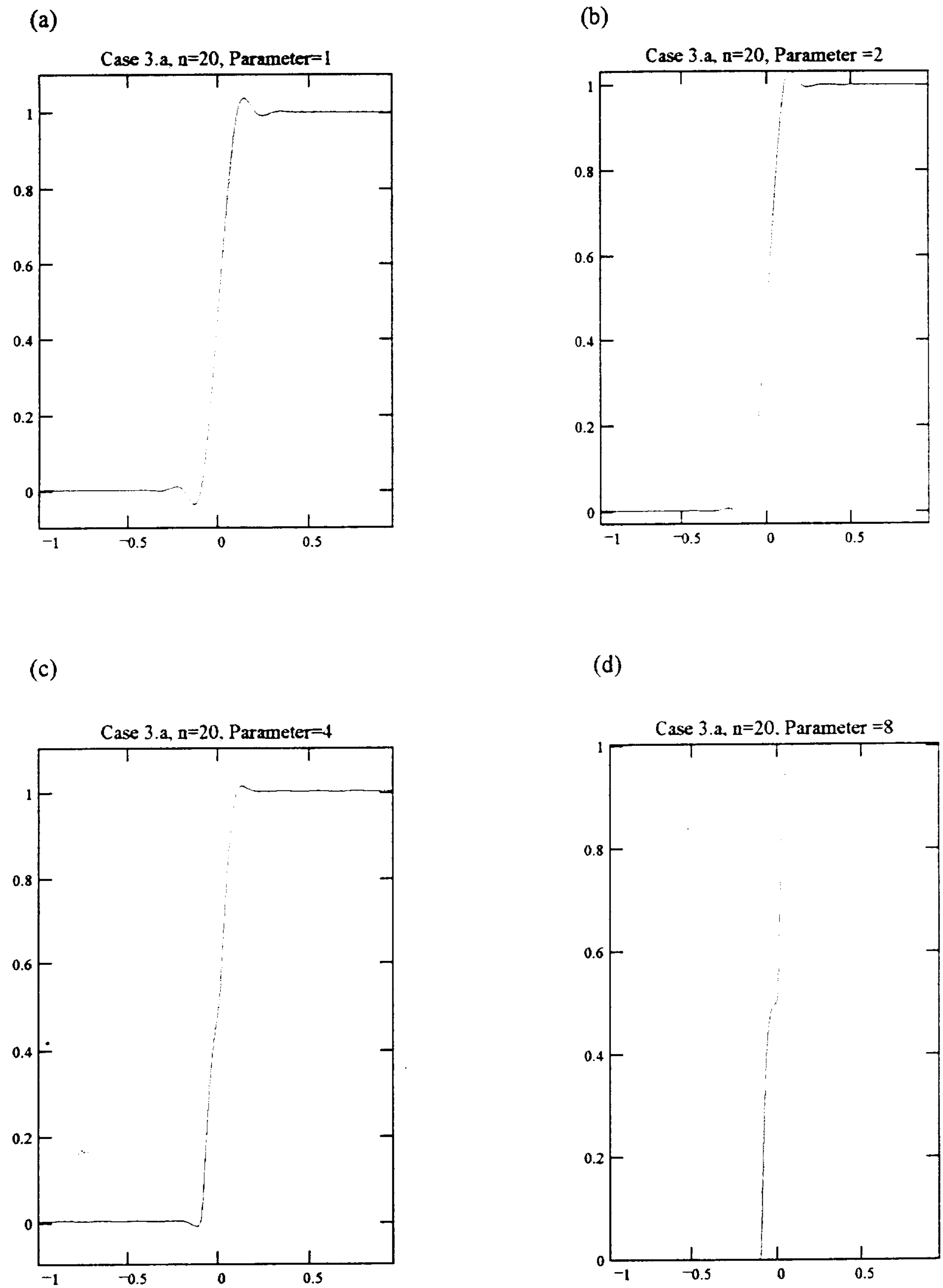
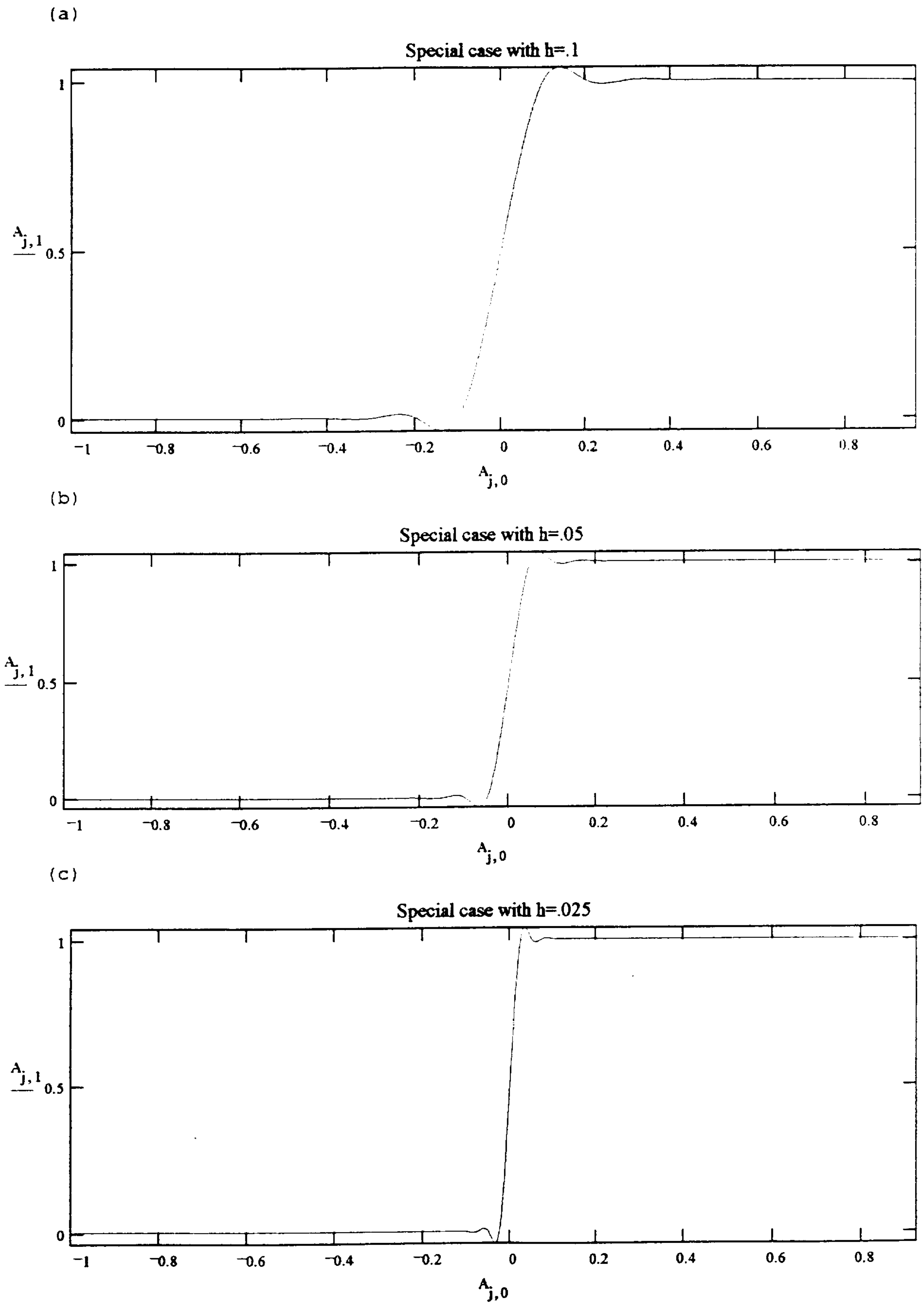

(a)
Case 3.a, n=20, Parameter=1
(b)
Case 3.a, n=20, Parameter =2
(c)
Case 3.a, n=20, Parameter=4
(d)
Case 3.a, n=20, Parameter =8

Figure 4.


(a)
Special case with h=.1
1
0.5
0
$A_{j,1}$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
$A_{j,0}$
(b)
Special case with h=.05
1
0.5
0
$A_{j,1}$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
$A_{j,0}$
(c)
Special case with h=.025
1
0.5
0
$A_{j,1}$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
$A_{j,0}$