IN-34
47942
P-11

NASA Contractor Report 195061

ICASE Report No. 95-19

ICASE

CLASSICAL CLOSURE THEORY AND LAM'S INTERPRETATION OF ϵ-RNG

Ye Zhou

Contract No. NAS1-19480
March 1995

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA 23681-0001

Operated by Universities Space Research Association

Classical closure theory and Lam's interpretation of ϵ-RNG[1]

Ye Zhou

Institute for Computer Applications in Science and Engineering

NASA Langley Research Center, Hampton, VA 23681-0001

Abstract

Lam's phenomenological ϵ-renormalization group (RNG) model is quite different from the other members of that group. It does not make use of the correspondence principle and the ϵ-expansion procedure. In this report, we demonstrate that Lam's ϵ-RNG model [*Phys. Fluids A*, **4**, 1007 (1992)] is essentially the physical space version of the classical closure theory [Leslie and Quarini, *J. Fluid Mech.*, **91**, 65 (1979)] in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

[1]This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-19480 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681-0001.

Introduction

In this note, we demonstrate that Lam's ϵ-RNG model[1] is essentially the physical space version of the classical closure theory[2] in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

Analysis

The incompressible N-S equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu_0 \nabla^2 \mathbf{v} \tag{1}$$

where ν_0 is the molecular viscosity, ρ is the density, and p is the pressure and can be determined from (1) using $\nabla \cdot \mathbf{v} = 0$. The external driving force that sustains the turbulence and which acts in the very small wavenumber region is not included in (1) since it plays no part in the energy cascade process in the inertial range[2].

As in both closure and RNG theories, the velocity field is filtered into two components

$$\mathbf{v} = \mathbf{v}^< + \mathbf{v}^>, \qquad p = p^< + p^> \tag{2}$$

where the Fourier-transformed fields

$$v_i^<(\mathbf{k}, t) = G(k) v_i(\mathbf{k}, t), \tag{3}$$

$$v_i^>(\mathbf{k}, t) = (1 - G(k)) v_i(\mathbf{k}, t). \tag{4}$$

The sharp cut-off filter of classical closure theory is exactly the same as the RNG technique of separating the subgrid from the resolvable scales at the cutoff wavenumber Λ

$$G(k) = \begin{cases} 0 & \text{if } k > \Lambda \ ; \\ 1 & \text{if } k < \Lambda. \end{cases} \tag{4}$$

In the classical closure theory of Leslie and Quarini (LQ)[2], the filtered N-S equation is

$$\Big(\frac{\partial}{\partial t} + [\nu_0 + \nu_E(k)] k^2\Big) v_\alpha^<(\mathbf{k}, t) = M_{\alpha\beta\gamma}(k) \int d\,\mathbf{p} d\,\mathbf{q} v_\beta^<(\mathbf{p}, t) v_\gamma^<(\mathbf{q}, t) + f_\alpha(\mathbf{k}, t), \tag{5}$$

where $M_{\alpha\beta\gamma}(k)$ is the standard nonlinear coupling coefficient[2,3]. For convenience we have added to *both* sides a wavenumber dependent turbulent eddy viscosity $\nu_E(k)$, which is at the moment unspecified. The term $f(k,t)$ accounts for the Reynolds stress[2,4],

$$R_{\beta\gamma} \equiv v^>_\beta(\mathbf{p},t)v^>_\gamma(\mathbf{q},t). \tag{6}$$

the cross stress[2,4],

$$C_{\beta\gamma} \equiv v^<_\beta(\mathbf{p},t)v^>_\gamma(\mathbf{q},t) + v^>_\beta(\mathbf{p},t)v^<_\gamma(\mathbf{q},t) \tag{7}$$

and the added eddy viscosity $\nu_E(k)$:

$$f_\alpha(\mathbf{k},t) \equiv \nu_E(k)k^2 v^<_\alpha(\mathbf{k},t) + M_{\alpha\beta\gamma}(k)\int d\mathbf{p}d\mathbf{q}\left[C_{\beta\gamma} + R_{\beta\gamma}\right]. \tag{8}$$

In (6)-(7), $|\mathbf{p}+\mathbf{q}| < \Lambda$. It is important to realize that no random force has been inserted here.

In the Lam approach to ϵ -RNG[1], one works in physical space rather than wavenumber space. The exact resolvable scale Navier-Stokes equations can be written

$$[\frac{\partial}{\partial t} - (\nu_0 + \nu_T)\nabla^2]\mathbf{v}^< = -\frac{1}{\rho}\nabla p^< - \nabla\cdot(\mathbf{v}^<\mathbf{v}^<) + \mathbf{g}^{fast} \tag{9}$$

where $\mathbf{g}^{fast}$ is defined by

$$\mathbf{g}^{fast} = \nabla\cdot(\mathbf{v}^<\mathbf{v}^< - \mathbf{v}\mathbf{v}) - \nu_T\nabla^2\mathbf{v}^< = \nabla\cdot(2\mathbf{v}^>\mathbf{v}^< - \mathbf{v}^>\mathbf{v}^>) - \nu_T\nabla^2\mathbf{v}. \tag{10}$$

Note that Lam has introduced a *k-independent* turbulent eddy viscosity, ν_T,which remains to be chosen. g^{fast} is generated by the filtering process. The term $\mathbf{g}^{fast}$ in physical space corresponds to the term $\mathbf{f}(\mathbf{k},t)$ in wavenumber space, in Eq. (8).

The classical theory proceeds from this point by the use of certain "closure approximations"[2,3] An equation for the resolvable spectral energy, $\bar{E}(k,t)$, can readily be derived,

$$\left[\frac{\partial}{\partial t} + 2\nu_0 k^2\right]\bar{E}(k,t) = \bar{T}(k,t) + T^>(k,t), \tag{11}$$

where $\bar{T}(k,t)$ is the resolvable scale energy transfer and $T^>(k,t)$ is the energy transfer caused by the cross and Reynolds stresses[2] which can be put into the form[2,5]

$$T^>(k,t) \equiv -2\nu_d(k)k^2\bar{E}(k,t) + U(k). \tag{12}$$

$U(k)$, which represents the backscatter of energy from small to resolvable scales and is also the spectrum of the correlation function of $\mathbf{f}$, is given by

$$U(k) \equiv \int_\Delta dp\,dq\,B(k,p,q)E(p)E(q)G^2(k)[1 - G(p)G(q)]. \tag{13}$$

$\nu_d(k,t)$, the *drain eddy viscosity*, is given by

$$\nu_d(k) \equiv \int_\Delta dp\,dq\,A(k,p,q)E(q)[1 - G(p)G(q)]. \tag{14}$$

The integration domain is denoted by the expression Δ in which p and/or $q > \Lambda$. The explicit functional forms of A and B appearing in (13)-(14) are given in Leslie[3] and LQ[2].

Instead of trying to compute $\mathbf{g}^{fast}$ using closure approximations, Lam[1] simply tries to model its correlation function based on physical arguments. In his view, $\mathbf{f}$ is simply a guess of what $\mathbf{g}^{fast}$ should be for $k \approx \Lambda$ in the resolvable scale Navier-Stokes equation. He noted that in the absence of $\mathbf{f}$, the energy spectrum of the flow, computed from (5) driven by initial and/or boundary conditions, will have a Kolmogorov dissipation wavenumber substantially smaller than Λ. The primary role of $\mathbf{f}$ is to extend for the resolvable scale velocity field the inertial range with a guaranteed Kolmogorov scaling for $k \approx \Lambda$ and beyond.

The forcing function in classical closure theory arises from filtering at the small scales. In modeling the correlation function of $\mathbf{f}$, Lam[1] *assumes* the form

$$< f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega') >= \frac{2}{\Pi_3}\mathcal{E}\frac{1}{\Lambda^{4-\epsilon}}k^{-d+4-\epsilon}(2\pi)^{d+1}P_{ij}(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \tag{15}$$

where ω is frequency, $\mathcal{E}$ is the dissipation rate, d is the dimension of the physical space, Π_3 is a constant, and $P_{ij}(k) = \delta_{ij} - k_ik_j/k^2$. A multiplicative factor involving $\Lambda^{4-\epsilon}$ is introduced to maintain dimensional consistency for arbitrary ϵ. It is of some interest to compare Eq. (15) with the forcing correlation function introduced by Yakhot and Orszag (YO)[6]

$$< f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega') >= \frac{2}{\Theta}\mathcal{E} k^{-d+4-\epsilon}(2\pi)^{d+1} P_{ij}(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \tag{16}$$

where Θ is a known constant determined by $2D_0S_d/(2\pi)^{d+1} = 1.594\mathcal{E}$ (YO[6]) and S_d is the area of a d-dimensional unit sphere. This form[7] is assumed to arise from forcing at k $= 0$:

$$< ff >= \delta(k)\mathcal{E}\delta(\mathbf{k}+\mathbf{k}') \tag{17}$$

with the use of Gel'fand's δ-function representation in the limit of $\epsilon \to 4$ and $k \to 0$

$$\delta(k) = \lim_{\epsilon\to 4}(4-\epsilon)k^{1-\epsilon}. \ for \ k \to 0 \tag{18}$$

To recover (16), it appears that (18) needs to be applied for $k \neq 0$, without the $(4-\epsilon)$ factor.

Lam pointed out that the forcing correlation function, Eq. (15), should peak around Λ; that its magnitude should be small for small k by an appropriate choice of ν_T; and that its behavior for $k \gg \Lambda$ is unimportant and irrelevant for the evolution of the resolved modes. Most importantly, the correlation function now depends on Λ, while in ϵ-RNG[6-7], the correlation function is assumed to be "scale invariant". The dimensionless parameter ϵ in the correlation function is now available as a freely adjustable parameter, and Lam used it to make the "predicted value" of Kolmogorov constant acceptable. He showed that either $\epsilon = 0$ or $\epsilon = 0.923$ yield good results.

The stochastic backscatter $\mathbf{f}$, for isotropic homogeneous turbulence in three dimensions, has a k^4 spectrum to lowest order in wavenumber k (e.g., Ref. 5). Specifically,

$$U(k) = \frac{14}{15}k^4 \int_\Lambda^\infty dp\theta_{k,p,q}(t)\frac{[E(p)]^2}{p^2} \quad for \quad k \to 0. \tag{19}$$

where $\theta_{k,p,q}(t) = 1/[\mu_{k,p,q}(t) + \nu_0(k^2+p^2+q^2)]$ and $\mu_{k,p,q}(t)$ is an "eddy-damping rate" of the third-order moments associated with the wavevectors $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$.

Thus, Lam's postulate (which was based on intuitive physical arguments) that $U(k)$ is small for small k is consistent with classical closure theory.

The advantage of the classical theory is that the energy equation is always satisfied and no restriction on the magnitude of Λ is imposed—so long as Λ is in the inertial range. On integrating (11) with respect to k for $0 < k < \Lambda$, we obtain:

$$\frac{\partial K}{\partial t} = \bar{\Pi} - \mathcal{E}. \tag{20}$$

where K is the integral of $\bar{E}(k)$ over the resolved wavenumbers, and $\mathcal{E}$ is defined by:

$$\mathcal{E} \equiv \int_0^\Lambda T^{>}(k)dk = \int_0^\Lambda 2k^2\nu_n(k)\bar{E}(k)dk. \tag{21}$$

and $\bar{\Pi}$, the resolved energy transfer term, is given by:

$$\bar{\Pi} \equiv \int_0^\Lambda \bar{T}(k)dk.$$

The *net eddy viscosity*, $\nu_n(k,t)$, is defined[2,5,8–9] as

$$\nu_n(k) \equiv \nu_d(k) - \nu_b(k). \tag{22}$$

and $\nu_b(k,t)$, the *back-scatter viscosity*, is given by

$$\nu_b(k) \equiv U(k)/(2k^2\bar{E}(k)). \tag{23}$$

From (14) and (23), one can show[10] that for k in the inertial range and $k \ll \Lambda$, the ratio of $\nu_b(k)$ to $\nu_d(k)$ is equal to $\frac{14}{15}(k/\Lambda)^{11/3}$. Spectral large-eddy simulations (LES) of Lesieur and Rogallo[5,11] was based on the resolvable scale Navier-Stokes equation

$$\Big(\frac{\partial}{\partial t} + [\nu_0 + \nu_n(k)]k^2\Big)v_\alpha^<(\mathbf{k},t) = M_{\alpha\beta\gamma}(k)\int\int d\,\mathbf{p}d\,\mathbf{q}v_\beta^<(\mathbf{p},t)v_\gamma^<(\mathbf{q},t). \tag{24}$$

Lam emphasized that $\mathcal{E}$, the energy dissipation rate of the turbulent flow in question, must be related to the parameters of the turbulent eddies by an *ad hoc* postulate under his formulation. Lam's choice[1] is

$$\mathcal{E}_L = \lim_{\Lambda\to\infty} 2\nu_T(\Lambda)\int_0^\Lambda k^2 E(k)dk. \tag{25}$$

The large Λ limiting process in (25) is needed to ensure that the dissipation rate can be adequately evaluated using information available from the resolved modes alone. In Lam's approach, the value of Λ must be sufficiently large such that the dissipation function $\mathcal{E}_L$ as given by (25) is independent of Λ. In physical variables, $\mathcal{E}_L$ is defined by:

$$\mathcal{E}_L \equiv \nu_T(\Lambda)\left(\frac{\partial u_i^<}{\partial x_k}\right)^2. \tag{26}$$

The Smagorinsky result for ν_T is recovered if $\mathcal{E}_L$ is eliminated between (26) and $\nu_T(\Lambda) = C_\nu \mathcal{E}_L^{1/3}\Lambda^{-4/3}$. In LES, the Lam requirement that Λ must be large enough is equivalent to requiring that (26), computed using data only from resolved modes, be "grid size" independent. In Lam's view, an LES calculation must exhibit a Kolmogorov spectrum using the resolved modes such that the limiting process in (25) is respected. If it does not, then the calculation would have no theoretical standing. Physically, if Λ is sufficiently large (so that $\mathcal{E}_L$ is independent of Λ), the contribution of back scattering to the dissipation would be negligible. The random force $\mathbf{f}$, the *surrogate* of the $\mathbf{g}^{fast}$, does not appear explicitly in the final LES model of Lam and one needs only to provide a profile of $<\mathbf{ff}>$ so as to introduce the adjustable parameter ϵ used in computing ν_T.

Conclusion

Thus, we find that Lam's formulation of ϵ-RNG[1] is essentially the physical space version of the spectral classical closure theory[2] with $\nu_n(k)$ being replaced by a phenomenological k-independent ν_T, but which now depends on arbitrary parameter ϵ.

Acknowledgments

The author gratefully acknowledges stimulating discussions with Professors S.H. Lam and G. Vahala.

References

1. S.H. Lam, On the RNG theory of turbulence, *Phys. Fluids A*, **4**, 1007 (1992).
2. D.C. Leslie and G.L. Quarini, The application of turbulence theory to the formulation of subgrid modelling procedure, *J. Fluid Mech.*, **91**, 65 (1979)
3. D.C. Leslie, Developments in the theory of turbulence (Clarendon, Oxford, 1972)
4. Y. Zhou, Eddy damping, backscatter, and subgrid stresses in subgrid modeling of turbulence, *Phys. Rev. A*, **43**, 7049 (1991)
5. M. Lesieur, *Turbulence in Fluids*, (Kluwer, Dordrecht, 1990)
6. V. Yakhot and S. Orszag, Renormalization group analysis of turbulence I. theory, *J. Sci.Compt.*, **1**, 3 (1986).
7. V. Yakhot, Presentation at ICASE panel discussion on RNG methods in turbulence modeling, June 29, 1993. [See Zhou and Speziale, An overview of RNG methods in turbulence modeling: Panel discussion summary. in *Transition, turbulence, and combustion*, Volume II, page 179-196, edited by M.Y. Hussaini et al. (Kluwer, Dordrecht, 1994)].
8. R.H. Kraichnan, Eddy viscosity in two and three dimensions, *J. Atmos. Sci.*, **33**, 1521 (1976).
9. J.P. Chollet and M. Lesieur, Parameterization of small scales of three-dimensional isotropic turbulence utilizing spectral closures, *J. Atmos. Sci.*, **38**, 2747 (1981)

10. J. R. Chasnov, Simulation of the Kolmogorov inertial range using an improved subgrid model, *Phys. Fluids A*, **3**, 188, 1991.

11. M. Lesieur and R.S. Rogallo, Large-eddy simulation of passive scalar diffusion in isotropic turbulence, *Phys. Fluids A*, **1**, 718 (1989).

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
March 1995

3. REPORT TYPE AND DATES COVERED
Contractor Report

4. TITLE AND SUBTITLE
CLASSICAL CLOSURE THEORY AND LAM'S INTERPRETATION OF ϵ-RNG

5. FUNDING NUMBERS
C NAS1-19480
WU 505-90-52-01

6. AUTHOR(S)
Ye Zhou

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Institute for Computer Applications in Science
and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23681-0001

8. PERFORMING ORGANIZATION REPORT NUMBER
ICASE Report No. 95-19

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA CR-195061
ICASE Report No. 95-19

11. SUPPLEMENTARY NOTES
Langley Technical Monitor: Dennis M. Bushnell
Final Report
To appear in Physical Review E

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified–Unlimited

Subject Category 34

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

Lam's phenomenological ϵ-renormalization group (RNG) model is quite different from the other members of that group. It does not make use of the correspondence principle and the ϵ-expansion procedure. In this report, we demonstrate that Lam's ϵ-RNG model [*Phys. Fluids A*, **4**, 1007 (1992)] is essentially the physical space version of the classical closure theory [Leslie and Quarini, *J. Fluid Mech.*, **91**, 65 (1979)] in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

14. SUBJECT TERMS
Turbulence; Renormalization Group; Classical closure theory

15. NUMBER OF PAGES
10

16. PRICE CODE
A02

17. SECURITY CLASSIFICATION OF REPORT
Unclassified

18. SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

19. SECURITY CLASSIFICATION OF ABSTRACT

20. LIMITATION OF ABSTRACT

NSN 7540-01-280-5500

Standard Form 298(Rev. 2-89)
Prescribed by ANSI Std Z39-18
298-102