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Abstract

The origin of spurious solutions in computational electromagnetics, which violate the divergence equations, is deeply rooted in a misconception about the first-order Maxwell's equations and in an incorrect derivation and use of the curl-curl equations. The divergence equations must be always included in the first-order Maxwell's equations to maintain the ellipticity of the system in the space domain and to guarantee the uniqueness of the solution and/or the accuracy of the numerical solutions. The div-curl method and the least-squares method provide rigorous derivation of the equivalent second-order Maxwell's equations and their boundary conditions. The node-based least-squares finite element method(LSFEM) is recommended for solving the first-order full Maxwell equations directly. Examples of the numerical solutions by LSFEM for time-harmonic problems are given to demonstrate that the LSFEM is free of spurious solutions.

1 Introduction

The occurrence of spurious solutions in computational electromagnetics has been known for more than two decades, and elimination of such non-physical solutions is still a subject of great interest. The noted feature of these fictitious solutions has been their violating the divergence-free conditions in cases where the physical solution is completely solenoidal. There is a vast body of reports about spurious solutions associated with the finite element method, see e.g., Cendes and Silvester [10], Bird [3], Ikeuchi et al. [22], Davies et al. [14], Rahman and Davies [52] [53], Winkler and Davies [68], Webb [66], Welt and Webb [67], Koshiba et al. [30] [31], Ise et al. [21], Rahman et al. [54] and Schroeder and Wolff [56]. The majority of spurious solutions has been found in the context of eigenvalue analysis. A spurious mode does not correspond to the physical modes which the waveguide or resonator under consideration actually supports. The spurious mode problem is severe and often renders the numerical solution useless. The spurious solutions have been also revealed in boundary-value problems, see, e.g., Crowley et al. [13], Pinchuk et al. [50], Wong and Cendes [69] [70] and Paulsen and Lynch [49].

The phenomenon of spurious solutions is not exclusive with the finite element method. This phenomenon has been also reported in the context of the finite difference method, see e.g., Corr and Davies [12], Mabaya et el. [36], Schwieg and Bridges [57] and Su [61], the boundary element method, see e.g., Ganguly and Spielman [17] and Swaminathan et al. [62], and the spectral method, see Farrar and Adams [15]. This fact itself undermines the common belief that the spurious solution is a result of numerical process. In our opinion, the trouble of spurious solutions in computational electromagnetics is deeply rooted in a misconception of the first-order Maxwell's equations and in an incorrect derivation and use of the second-order curl-curl equations. We agree with Mur [44] [45] that spurious solutions can only be avoided by a correct formulation of the problem to be solved.

In terms of the type of differential equations to be solved, conventional numerical methods in computational electromagnetics may be classified into four categories: (1) those based on the first-order curl equations; (2) those based on the second-order curl-curl equations; (3) those based on the Helmholtz equations; (4) those based on the potentials.

The most widely used numerical method for the solution of time-dependent electromagnetic problems has been the finite-difference time-domain (FD-TD) scheme developed by Yee [72] and extensively utilized and refined by Taflove and Umashankar [63] and Kunz and Luebbers [33], as well as others. In the Yee scheme, only the two Maxwell's curl equations are solved. Some other time-domain methods are also based on the two Maxwell's curl equations, such as the finite volume method developed by Shankar et al. [58], the finite difference and finite volume methods by Shang [59] and Shang and Gaitonde [60], and the finite element methods by Mei and his colleagues [8], Madsen and his colleagues [37] [34], Noack and Anderson [47] and Ambrosiano et al. [1]. In general, these approaches do not produce noticeable spurious solutions. This is attributed to the fact that by taking the divergence of the Faraday and Ampere laws, one finds that these divergence-free conditions will be satisfied for all time if they are satisfied initially. However, it is not so easy to satisfy them initially in these methods. In fact, in these papers the satisfaction of divergence-free conditions was not even considered except by Shang and Gaitonde [60] who seriously examined the value of divergence of the computed magnetic field.

In the original full Maxwell's equations, when the constitutive relations are specified, for three dimensional cases there are eight first-order equations but only six unknown vector components, and for two dimensional TE and TM cases four equations and three unknowns. That is, the number of equations is larger than the number of the unknown functions. For this reason, it is traditionally believed that the full first-order Maxwell's equations are "overdetermined" or "overspecified", and the two divergence equations are thus regarded as "auxiliary" or "dependent" and are often neglected in numerical computation.

The first-order full Maxwell's equations have a mathematical structure in which the fundamental ingredient is the div-curl system that looks "overdetermined". A similar situation exists in fluid dynamics, see Jiang et el. [27]. By introducing a dummy variable(Chang and Gunzburger [11]), however, it can be shown that the div-curl system is not "overdetermined". In this paper, we use this technique to study the full Maxwell's equations and show that they are properly determined, that is, the two divergence equations should not be ignored regardless in either the static or in the time-varying cases.

In electromagnetics, there are mainly two reasons why the second-order curl-curl equations are often employed. First, it is hard for conventional numerical methods to deal with the non-self-adjoint first-order derivatives. Second, in the curl-curl equations the electric field and the magnetic field are decoupled. The curl-curl equations are derived from the first-order Maxwell's curl equations by applying the curl operator. It seems that no one has addressed a very important issue: the curl-curl equations obtained by simple differentiation without additional equations and boundary conditions admit more solutions than do its progenitors. In order to derive an equivalent higher-order system from a system of vector partial differential equations, one should use the div-curl method that is based on the theorem: if a vector is divergence-free and curl-free in a domain, and its normal component or tangential components on the boundary is zero, then this vector is identically zero. In other words, the curl and the divergence operators must act together with appropriate boundary conditions to guarantee that there are no spurious solutions in the resulting higher-order equations. In this paper, this div-curl method originally developed by Jiang et al. [27] is employed to derive the second-order system of time-dependent Maxwell's equations and its boundary conditions, and to show that the divergence equations and additional boundary conditions must be supplemented to the curl-curl equations.

The common approach to removing spurious vector modes in the curl-curl equations is to modify the variational functional by penalizing the non-zero divergence.

The key to success with this so-called penalty method, first used by Hara et al [20] and Rahman and Davies [53], depends on the choice of the correct penalty factor – values too small or too large do not eliminate spurious solutions. Unfortunately, this is an ad hoc and problem-dependent treatment and there has been a lack of systematic study of the rationale for selecting this parameter for general problems.

Recently, the edge element method of Nedelec [46], see e.g., Bossavit and Verite [5], Hano [19], Mur and Hoop [43], Barton and Cendes [2], Bossavit [4], Bossavit and Mayergoyz [6], Monk [41], Jin [28], Volakis et al. [65] and the references therein, has been advocated, because it is believed to be a cure for many difficulties that are encountered when attempting to solve electromagnetic field problems by using conventional node-based finite elements. Apart from the fact that such an approach can only be used in the simple divergence-free case, edge elements violate the normal field continuity between adjacent elements in the homogeneous material domain, see Mur [45] for comments and an example. The accuracy of edge elements is lower than that of the nodal elements for the same number of unknowns, or the computational cost of edge elements is much higher than that of nodal elements for the same accuracy, see Mur [45] and Monk [42]. The edge element method also needs non-conventional meshing and postprocessing which are not normally available. Moreover, Ross et al. [55] reported that the edge element method broke down for large-scale computations due to the fact that edge elements cannot represent purely TE fields.

It is well known that the solution of the Helmholtz equations with proper boundary conditions is free of spurious modes, see Mayergoyz and D'Angelo [38]. The key issue in the Helmholtz method is how to specify proper boundary conditions. In this paper, we use the div-curl method and the least-squares method to derive the Helmholtz equations and their boundary conditions, and show that the divergence equations need to be enforced only on a part of boundary, and they will be implicitly satisfied in the domain. We also give a Galerkin variational formulation which corresponds to the Helmholtz equations. This theoretically justifies that the penalty parameter s in the penalty method should be equal to one.

The potential approach is widely used in computation of static fields and eddy currents. Although the potential approach, see e.g., Boyse et al. [7] for time-harmonic problems, does not give rise to spurious modes, it involves difficulties related to the appropriate gauging method and the loss of accuracy of the calculated field intensity from the potentials by the numerical differentiation.

This paper emphasizes that in any case the divergence equations must be included explicitly or implicitly as a part of the formulation for electromagnetic problems. However, it is not so easy to combine the divergence equations in conventional methods. Attempts to satisfy the divergence-free equations by using edge elements merely complicate the situation by introducing the need to impose an additional condition of normal field continuity.

This paper shows that the satisfaction of the divergence equations and the elimination of spurious solutions can be achieved easily by the application of the node-based least-squares finite element method (LSFEM). We believe that the LSFEM is the best choice among the available methods for numerical solution of many problems in electromagnetics, since it is simple, universal, optimal, robust and efficient. The LSFEM is based on the minimization of the residuals in first-order partial differential equations. The LSFEM has been successfully applied to various fluid dynamics problems, see e.g., Jiang et al. [24] [26], Tang and Tsang [64] and Lefebvre et al. [35]. The LSFEM is naturally suitable for the first-order full Maxwell's equations. The preliminary results of LSFEM for time-domain scattering wave problems can be found in Wu and Jiang [71]. The theory and the least-squares method for the div-curl system discussed in this paper can be employed to directly solve static electric or magnetic fields without introducing the potentials and gauging. In the last section of this paper we briefly discuss the general formulation of the LSFEM and apply it to time-harmonic problems. Numerical examples are given to demonstrate that the LSFEM is free of spurious solutions.

2 The Div-Curl System

In this section, we study the div-curl system. We shall show that the three dimensional div-curl system is not "overdetermined". We shall introduce the div-curl method to derive a second-order system equivalent to the div-curl system. We shall show why the least-squares method is the best method for the solution of the div-curl system. The technique and the procedure developed here will be applied to dealing with the Maxwell's equations. Since the static Maxwell's equations are typical div-curl systems, the least-squares method introduced in this section can be applied to the direct solution of static electric or magnetic fields.

2.1 Basic Theorems

First we introduce some notations which are common in functional analysis. These notations will help us to write the mathematical formulations more concisely. Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected, convex and open domain with a piecewise smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Either Γ_1 or Γ_2, not both, may be empty. Also Γ_1 and Γ_2 must have at least one commom point. $\mathbf{x} = (x, y, z)$ be a point in Ω, $\mathbf{n}$ be a unit outward normal vector and $\boldsymbol{\tau}$ be a tangential vector to Γ at a boundary point, respectively. $L_2(\Omega)$ denotes the space of square-integrable functions defined on Ω equipped with the inner product

$$(u, v) = \int_\Omega uv d\Omega$$

and the norm

$$\|u\|_{0,\Omega}^2 = (u, u).$$

$H^r(\Omega)$ denotes the Sobolev space of functions with square-integrable derivatives of order up to r. $\|\cdot\|_{r,\Omega}$ denotes the usual norm for $H^r(\Omega)$. For vector-valued functions $\mathbf{u}$ with m components, we have the product spaces

$$L_2(\Omega)^m, \ H^r(\Omega)^m$$

with the inner product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} d\Omega$$

and the corresponding norm

$$\|\mathbf{u}\|_{0,\Omega}^2 = \sum_{j=1}^m \|u_j\|_{0,\Omega}^2, \qquad \|\mathbf{u}\|_{r,\Omega}^2 = \sum_{j=1}^m \|u_j\|_{r,\Omega}^2.$$

Further we define

$$< u, v >_\Gamma = \int_\Gamma uv d\Gamma.$$

When there is no chance for confusion, we will often omit the measure Ω or Γ from the inner product and norm designation.

Throughout the paper C denotes a positive constant dependent on Ω with possibly different values in each appearance.

The following theorems are essential in this paper.

Theorem 1. If $\mathbf{u} \in H^1(\Omega)^3$, then $\mathbf{n} \times \mathbf{u} = \mathbf{0}$ on $\Gamma_2 \neq 0 \Leftrightarrow \mathbf{n} \cdot \nabla \times \mathbf{u} = 0$ on Γ_2.

Here the notation "$\Leftrightarrow$" stands for "leading to and vice versa". The proof of Theorem 1 is straightforward by using the Stokes theorem, see Pironneau [51] or Jiang et al. [27].

Theorem 2 (Friedrichs' Div-Curl Inequality). Every function $\mathbf{u}$ of $H^1(\Omega)^3$ with $\mathbf{n} \cdot \mathbf{u} = 0$ on Γ_1 and $\mathbf{n} \times \mathbf{u} = \mathbf{0}$ on Γ_2 satisfies:

$$\|\mathbf{u}\|_1^2 \leq C(\|\nabla \cdot \mathbf{u}\|_0^2 + \|\nabla \times \mathbf{u}\|_0^2), \tag{2.1}$$

where the constant $C > 0$ depends only on Ω.

The proof of Theorem 2 involves lots of mathematics. We refer to Girault and Raviart [18], Krizek and Neittaanmaki [32] and Jiang et al. [27]. This theorem implies that the div-curl norm appearing in the right-hand side of (2.1) is equivalent to the H^1 norm. This theorem plays a key role in the analysis of the least-squares method. From Theorem 2, we can immediately obtain the following theorem which is the basis of the div-curl method for deriving higher-order vector equations:

Theorem 3 (The Div-Curl Theorem). If $\mathbf{u} \in H^1(\Omega)^3$ satisfies

$$\nabla \cdot \mathbf{u} = 0 \quad in \quad \Omega,$$

$$\nabla \times \mathbf{u} = \mathbf{0} \quad in \quad \Omega,$$
$$\mathbf{n} \cdot \mathbf{u} = 0 \quad on \quad \Gamma_1,$$
$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad on \quad \Gamma_2,$$

then

$$\mathbf{u} \equiv \mathbf{0} \quad in \quad \Omega.$$

This theorem can also be proved easily by introducing the potential.

Theorem 4 (The Gradient Theorem). If $g \in H^1(\Omega)$ satisfies

$$\nabla g = \mathbf{0} \quad in \quad \Omega,$$
$$g = 0 \quad on \quad \Gamma_1 \neq 0 \ (or\ on \quad \Gamma_2 \neq 0),$$

then

$$g \equiv 0 \quad in \quad \Omega.$$

The validation of Theorem 4 is obvious. In fact, $g = 0$ needs to be specified only at any point in the domain or on the boundary. This theorem will be used to derive the higher-order equations which are equivalent to a scalar equation.

2.2 The Div-Curl System

Let us consider the following three-dimensional div-curl system:

$$\nabla \times \mathbf{u} = \boldsymbol{\omega} \ \ in\ \Omega, \tag{2.2a}$$

$$\nabla \cdot \mathbf{u} = \rho \ \ in\ \Omega, \tag{2.2b}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \ \ on\ \Gamma_1, \tag{2.2c}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \ \ on\ \Gamma_2, \tag{2.2d}$$

where the given vector function $\boldsymbol{\omega} \in L_2(\Omega)^3$ must satisfy the following compatibility conditions:

$$\nabla \cdot \boldsymbol{\omega} = 0 \ \ in\ \Omega, \tag{2.3a}$$

$$\mathbf{n} \cdot \boldsymbol{\omega} = 0 \ \ on\ \Gamma_2, \tag{2.3b}$$

$$\int_\Gamma \mathbf{n} \cdot \boldsymbol{\omega} ds = 0. \tag{2.3c}$$

If Γ_2 is empty, then the given scalar function $\rho \in L_2(\Omega)$ must satisfy the compatibility condition:

$$\int_\Omega \rho d\Omega = 0. \tag{2.3d}$$

At first glance, System (2.2) seems "overdetermined" or "overspecified", since there are four equations and three unknowns. For this reason, indeed, solving (2.2) is not trivial by conventional finite difference or finite element methods. However, after careful investigation we shall find that System (2.2) is properly determined and elliptic.

By introducing a dummy variable ϑ, System (2.2) can be written as

$$\nabla\vartheta + \nabla \times \mathbf{u} = \boldsymbol{\omega} \quad in\ \Omega, \tag{2.4a}$$

$$\nabla \cdot \mathbf{u} = \rho \quad in\ \Omega, \tag{2.4b}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad on\ \Gamma_1, \tag{2.4c}$$

$$\vartheta = 0 \quad on\ \Gamma_1, \tag{2.4d}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad on\ \Gamma_2. \tag{2.4e}$$

Notice that we impose $\vartheta = 0$ on Γ_1, and do not specify any boundary condition for the dummy variable ϑ on Γ_2.

By virtue of Theorem 3, Eq. (2.4a) is equivalent to the following equations and boundary conditions:

$$\nabla \times (\nabla\vartheta + \nabla \times \mathbf{u} - \boldsymbol{\omega}) = \mathbf{0} \quad in\ \ \Omega, \tag{2.5a}$$

$$\nabla \cdot (\nabla\vartheta + \nabla \times \mathbf{u} - \boldsymbol{\omega}) = 0 \quad in\ \ \Omega, \tag{2.5b}$$

$$\mathbf{n} \times (\nabla\vartheta + \nabla \times \mathbf{u} - \boldsymbol{\omega}) = \mathbf{0} \quad on\ \ \Gamma_1, \tag{2.5c}$$

$$\mathbf{n} \cdot (\nabla\vartheta + \nabla \times \mathbf{u} - \boldsymbol{\omega}) = 0 \quad on\ \ \Gamma_2. \tag{2.5d}$$

Taking into account the compatibility conditions (2.3a) and (2.3b), the boundary condition (2.4e) and Theorem 1, Eq. (2.5b), (2.4d) and (2.5d) lead to

$$\Delta\vartheta = 0 \quad in\ \ \Omega, \tag{2.6a}$$

$$\vartheta = 0 \quad on\ \ \Gamma_1, \tag{2.6b}$$

$$\frac{\partial\vartheta}{\partial n} = 0 \quad on\ \ \Gamma_2. \tag{2.6c}$$

From (2.6) we know that $\vartheta \equiv 0$ in Ω. That is, the introduction of ϑ into (2.2) does not change anything, and thus System (2.4) with four equations and four unknowns is indeed equivalent to System (2.2).

Now let us classify System (2.4). In Cartesian coordinates the equations in System (2.4) are given as

$$\frac{\partial\vartheta}{\partial x} + \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \omega_x,$$

$$\frac{\partial\vartheta}{\partial y} + \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \omega_y, \tag{2.7}$$

$$\frac{\partial \vartheta}{\partial z}+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega_z,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\rho.$$

We may write System (2.7) in the standard matrix form:

$$\mathbf{A}_1\frac{\partial \mathbf{u}}{\partial x}+\mathbf{A}_2\frac{\partial \mathbf{u}}{\partial y}+\mathbf{A}_3\frac{\partial \mathbf{u}}{\partial z}+\mathbf{A}_0\mathbf{u}=\mathbf{f}, \tag{2.8}$$

in which

$$\mathbf{A}_1=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \mathbf{A}_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$\mathbf{A}_3=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\ \mathbf{A}_0=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\mathbf{f}=\begin{pmatrix}\omega_x\\\omega_y\\\omega_z\\\rho\end{pmatrix},\qquad \mathbf{u}=\begin{pmatrix}u\\v\\w\\\vartheta\end{pmatrix}.$$

The characteristic polynomial associated with System (2.7) is

$$det(\mathbf{A}_1\xi+\mathbf{A}_2\eta+\mathbf{A}_3\zeta)=det\begin{pmatrix}0&-\zeta&\eta&\xi\\\zeta&0&-\xi&\eta\\-\eta&\xi&0&\zeta\\\xi&\eta&\zeta&0\end{pmatrix}=(\xi^2+\eta^2+\zeta^2)^2\neq 0$$

for all nonzero real triplets (ξ,η,ζ), System (2.4) is thus elliptic and properly determined.

The fist-order elliptic system (2.4) has four equations and four unknowns, so two boundary conditions on each boundary are needed to make System (2.4) well-posed. Here $\vartheta=0$ and $\mathbf{n}\cdot\mathbf{u}=0$ serve as two boundary conditions on Γ_1; while $\mathbf{n}\times\mathbf{u}=0$ implies that two tangential components of $\mathbf{u}$ are zero on Γ_2.

Since System (2.2) is equivalent to System (2.4), and System (2.4) is elliptic and properly determined, so is System (2.2).

Remark In fact, the compatibility conditions (2.3a,b) can be obtained by applying the div-curl method to the equation (2.2a).

2.3 The Div-Curl Method

Let us derive a higher-order system which is equivalent to the div-curl system (2.2). By virtue of Theorem 3, System (2.2) is equivalent to the following system:

$$\nabla \times (\nabla \times \mathbf{u} - \boldsymbol{\omega}) = \mathbf{0} \quad in \;\; \Omega, \tag{2.9a}$$

$$\nabla \cdot (\nabla \times \mathbf{u} - \boldsymbol{\omega}) = 0 \quad in \;\; \Omega, \tag{2.9b}$$

$$\mathbf{n} \times (\nabla \times \mathbf{u} - \boldsymbol{\omega}) = \mathbf{0} \quad on \;\; \Gamma_1, \tag{2.9c}$$

$$\mathbf{n} \cdot (\nabla \times \mathbf{u} - \boldsymbol{\omega}) = 0 \quad on \;\; \Gamma_2. \tag{2.9d}$$

$$\nabla \cdot \mathbf{u} = \rho \;\; in \; \Omega, \tag{2.9e}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \;\; on \; \Gamma_1, \tag{2.9f}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \;\; on \; \Gamma_2. \tag{2.9g}$$

Due to the compatibility conditions (2.3a,b), the boundary condition (2.9g) and Theorem 1, (2.9b) and (2.9d) are satisfied. Therefore, System (2.9) can be simplified as

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla \times \boldsymbol{\omega} \quad in \;\; \Omega, \tag{2.10a}$$

$$\nabla \cdot \mathbf{u} = \rho \;\; in \; \Omega, \tag{2.10b}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \;\; on \; \Gamma_1, \tag{2.10c}$$

$$\mathbf{n} \times (\nabla \times \mathbf{u}) = \mathbf{n} \times \boldsymbol{\omega} \quad on \;\; \Gamma_1, \tag{2.10d}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \;\; on \; \Gamma_2. \tag{2.10e}$$

Now at least one thing is made clear by the div-curl method. That is, the curl-curl equation (2.10a) cannot stand alone; it must go with the divergence equation (2.10b) and the additional Neumann boundary condition (2.10d).

System (2.10) can be further simplified. By virtue of Theorem 4, Eq. (2.10b) is equivalent to the following system of equations and boundary condition (assuming that $\Gamma_2 \neq 0$):

$$\nabla(\nabla \cdot \mathbf{u} - \rho) = \mathbf{0} \;\; in \; \Omega, \tag{2.11a}$$

$$\nabla \cdot \mathbf{u} = \rho \;\; on \; \Gamma_2. \tag{2.11b}$$

Taking into account (2.11) and the following vector identity:

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \Delta \mathbf{u}, \tag{2.12}$$

System (2.10) can be reduced as

$$\Delta \mathbf{u} = -\nabla \times \boldsymbol{\omega} + \nabla \rho \quad in \;\; \Omega, \tag{2.13a}$$

$$\nabla(\nabla \cdot \mathbf{u} - \rho) = \mathbf{0} \;\; in \; \Omega, \tag{2.13b}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad on\ \Gamma_1, \tag{2.13c}$$

$$\mathbf{n} \times (\nabla \times \mathbf{u}) = \mathbf{n} \times \boldsymbol{\omega} \quad on\ \ \Gamma_1, \tag{2.13d}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad on\ \Gamma_2, \tag{2.13e}$$

$$\nabla \cdot \mathbf{u} = \rho \quad on\ \Gamma_2. \tag{2.13f}$$

The solution of the derived second-order system (2.10) or (2.13) is completely identical to the solution of the original div-curl system (2.2), therefore no spurious solution will be produced by the system (2.10) or (2.13). Moreover, the divergence equation (2.13b) in System (2.13) can be deleted. That is, the divergence equation is implicitly satisfied by the equation (2.13a) and boundary conditions (2.13c-f). The rigorous proof of this statement will be given by using the least-squares method in the next section. Here we give a simple explanation adopted from Mayergoyz and D'Angelo [38]. Let us consider a slightly different problem:

$$\Delta \mathbf{u} = -\nabla \times \boldsymbol{\omega} + \nabla \rho \quad in\ \ \Omega, \tag{2.14a}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad on\ \Gamma_1, \tag{2.14b}$$

$$\mathbf{n} \times (\nabla \times \mathbf{u}) = \mathbf{n} \times \boldsymbol{\omega} \quad on\ \ \Gamma_1, \tag{2.14c}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad on\ \Gamma_2, \tag{2.14d}$$

$$\nabla \cdot \mathbf{u} - \rho = 0 \quad on\ \Gamma. \tag{2.14e}$$

That is, we let the divergence equation be satisfied on the whole boundary. Although this condition needs to be specified only on Γ_2, it is not wrong for it to be enforced on Γ. By taking the divergence of (2.14a) we obtain a Poisson equation of $\phi = \nabla \cdot \mathbf{u} - \rho$:

$$\Delta \phi = 0 \quad in\ \ \Omega. \tag{2.15}$$

Since $\phi = 0$ on the whole boundary, ϕ must be equal to zero in the whole domain, i.e., the divergence equation is implicitly satisfied in the system (2.14).

2.4 The Least-Squares Method

Let us introduce a more powerful and systematic method, the least-squares method, to solve System (2.2) and to derive a higher-order system without spurious solutions. We construct the following quadratic functional:

$$I : \mathcal{H} \longrightarrow \mathbb{R},$$

$$I(\mathbf{u}) = \|\nabla \times \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\nabla \cdot \mathbf{u} - \rho\|_0^2,$$

where $\mathcal{H} = \{\mathbf{u} \in H^1(\Omega)^3 | \mathbf{n} \cdot \mathbf{u} = 0 \ on\ \Gamma_1, \mathbf{n} \times \mathbf{u} = 0 \ on\ \Gamma_2\}$. We note that the introduction of a dummy variable ϑ in Section 2.2 is only for the verification of the determination, and it is not required in the least-squares functional I. Taking the variation of I with respect to $\mathbf{u}$, and letting $\delta\mathbf{u} = \mathbf{v}$ and $\delta I = 0$, we obtain a least-squares variational formulation of the following type: find $\mathbf{u} \in \mathcal{H}$ such that

$$B(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{H}, \tag{2.16}$$

where $B(\cdot,\cdot)$ is a bilinear form of the type

$$B(\mathbf{u},\mathbf{v}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}),$$

and $L(\cdot)$ is a linear form of the type

$$L(\mathbf{v}) = (\boldsymbol{\omega}, \nabla \times \mathbf{v}) + (\rho, \nabla \cdot \mathbf{v}).$$

Obviously the bilinear form $B(\mathbf{u},\mathbf{v})$ is symmetric and continuous. The coerciveness of $B(\mathbf{u},\mathbf{v})$ is due to Theorem 2. Therefore, we immediately have

$$\frac{1}{C}\|\mathbf{u}\|_1^2 \leq B(\mathbf{u},\mathbf{u}) = L(\mathbf{u}) \leq \|\mathbf{u}\|_1(\|\boldsymbol{\omega}\|_0 + \|\rho\|_0).$$

By virtue of the Lax-Milgram theorem, see e.g., Oden and Reddy [48] or Johnson [29], in fact we have proved the following theorem.

Theorem 5. The solution of (2.16) uniquely exists and satisfies:

$$\|\mathbf{u}\|_1 \leq C(\|\boldsymbol{\omega}\|_0 + \|\rho\|_0). \tag{2.17}$$

The following theorem about the error estimate is also a direct consequence of the above results.

Theorem 6. The LSFEM based on (2.16) has an optimal rate of convergence and an optimal satisfaction of the divergence condition:

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{k+1}\|\mathbf{u}\|_2, \tag{2.18a}$$

$$\|\nabla \cdot (\mathbf{u}_h - \rho)\|_0 \leq Ch^k\|\mathbf{u}\|_2, \tag{2.18b}$$

where $\mathbf{u}_h$ is the finite element solution, k is the order of complete polynomials used in the finite element interpolation.

The error estimate (2.18a) is not totally new. The early results were obtained by Fix and Rose [16] for the case $\Gamma_2 = 0$ and by Krizek and Neittaanmaki [32] for the case $\Gamma_1 = 0$. The two-dimensional numerical results obtained by using the primitive variables insteady of the potential and a preliminary analysis can also be found in Jiang and Chai [23] and Carey and Jiang [9].

The advantages of the LSFEM over the potential method for solving the div-curl system is obvious: the trouble of selecting a proper gauging method is avoided; the electric or magnetic fields are obtained directly without numerical differentiation and thus have a higher accuracy; and the electric or magnetic fields are continuous across the element boundaries.

In order to further understand the least-squares method, we derive the Euler-Lagrange equations associated with the least-squares variational formulation (2.16) which can be rewritten as: find $\mathbf{u} \in \mathcal{H}$ such that

$$(\nabla \times \mathbf{u} - \boldsymbol{\omega}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u} - \rho, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{H}. \tag{2.19}$$

Suppose that $\mathbf{u}$, $\boldsymbol{\omega}$ and ρ are sufficiently smooth. By using Green's formulae, Eq. (2.19) can be written as

$$\begin{aligned}&(\nabla \times (\nabla \times \mathbf{u} - \boldsymbol{\omega}), \mathbf{v}) + < (\nabla \times \mathbf{u} - \boldsymbol{\omega}), \mathbf{n} \times \mathbf{v} >_\Gamma + \\ &(-\nabla(\nabla \cdot \mathbf{u} - \rho), \mathbf{v}) + < (\nabla \cdot \mathbf{u} - \rho), \mathbf{n} \cdot \mathbf{v} >_\Gamma = 0 \quad \forall \mathbf{v} \in \mathcal{H}.\end{aligned} \tag{2.20}$$

Taking into account (2.12) and that $\mathbf{v}$ satisfies $\mathbf{n} \cdot \mathbf{v} = 0$ on Γ_1 and $\mathbf{n} \times \mathbf{v} = \mathbf{0}$ on Γ_2, from (2.20) we obtain

$$\begin{aligned}&(-\Delta \mathbf{u} - \nabla \times \boldsymbol{\omega} + \nabla \rho, \mathbf{v}) \\ &- < \mathbf{n} \times (\nabla \times \mathbf{u} - \boldsymbol{\omega}), \mathbf{v} >_{\Gamma_1} + < (\nabla \cdot \mathbf{u} - \rho), \mathbf{n} \cdot \mathbf{v} >_{\Gamma_2} = 0\end{aligned} \tag{2.21}$$

for all admissible $\mathbf{v} \in \mathcal{H}$, hence we have the Euler-Lagrange equation and boundary conditions:

$$\Delta \mathbf{u} = -\nabla \times \boldsymbol{\omega} + \nabla \rho \quad in \quad \Omega, \tag{2.22a}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad on \quad \Gamma_1, \tag{2.22b}$$

$$\mathbf{n} \times (\nabla \times \mathbf{u}) = \mathbf{n} \times \boldsymbol{\omega} \quad on \quad \Gamma_1, \tag{2.22c}$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad on \quad \Gamma_2, \tag{2.22d}$$

$$\nabla \cdot \mathbf{u} = \rho \quad on \quad \Gamma_2. \tag{2.22e}$$

We note that in System (2.22) the divergence equation does not appear in the domain. In fact, we have rigorously proved that the solution of the Helmholtz-type system (2.22a) under additional boundary conditions (2.22c,e) automatically satisfies the divergence equation. We also remark that if Γ_2 is empty, the divergence equation does not even appear on the boundary. The attraction of using the higher-order system (2.22) now becomes apparent: one avoids dealing with the divergence condition (2.2b) which is implicitly satisfied; instead, one deals with three Poisson equations that everyone would rather solve. However, we should mention that if one chooses to use the finite difference method to solve (2.22a), the additional natural boundary conditions (2.22c,e) must be supplemented.

Now it is clear that the following four formulations are equivalent to each other: (1) the first-order div-curl system (2.2); (2) the least-squares variational formulation (2.16); (3) The Helmholtz-type system (2.22); and (4) the Galerkin formulation (2.21). It turns out that the least-squares method (2.16) for the div-curl equations

(2.2) corresponds to using the Galerkin method (2.21) to solve System (2.22) which consists of three independent second-order Poisson equations (2.22a) and three coupled boundary conditions on each boundary, where the original first-order equations (2.22c) and (2.22e) serve as the natural boundary conditions, and (2.12b) and (2.22d) as the essential boundary conditions.

Obviously, the least-squares problem is formally equivalent to a higher-order problem with additional natural boundary conditions provided by the original first-order differential equations. The least-squares method (2.16) is the simplest approach among these equivalent methods, because it does not need any additional boundary conditions. The trial function **u** and the test function **v** need to satisfy only the original essential boundary conditions. This is one of the reasons why we strongly recommend the least-squares method.

Now we have shown that the three-dimensional div-curl system can have three equivalent differential forms: (1) the first-order system (2.2); (2) the curl-curl equation (2.10a) which must be accompanied by the divergence equation (2.10b) and the additional Neumman boundary condition (2.10d); (3) three uncoupled Poisson equations (2.22a) with additional Neumman boundary conditions (2.10c) and (2.10e) provided by the original first-order system.

In the following sections, we will show that Maxwell's equations have similar equivalent forms.

3 The First-Order Maxwell's Equations

In this section we shall show that the first-order full Maxwell's Equations are not "overdetermined", and thus the divergence equations should not be ignored.

3.1 The Basic Equations

For general time-varying fields, the original first-order full Maxwell's equations can be written as

$$\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} = -\mathbf{K}^{imp} \qquad in\ \Omega, \qquad (Faraday's\ Law) \tag{3.1a}$$

$$\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E} = \mathbf{J}^{imp} \qquad in\ \Omega, \qquad (Maxwell-Ampere's\ Law) \tag{3.1b}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho^{imp} \qquad in\ \Omega, \qquad (Gauss's\ Law - Electric) \tag{3.1c}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \qquad in\ \Omega, \qquad (Gauss's\ Law - Magnetic) \tag{3.1d}$$

where **E** and **H** are the electric and magnetic field intensities respectively, ρ^{imp} is the imposed source of electric charge density, and $\mathbf{J}^{imp}$ and $\mathbf{K}^{imp}$ are imposed sources of electric and magnetic current density. All imposed sources are given functions of the space and time coordinates.

In System (3.1) we have already made use of the following constitutive relations:

$$\mathbf{D} = \varepsilon\mathbf{E},$$

$$\mathbf{B} = \mu\mathbf{H},$$

$$\mathbf{J} = \sigma\mathbf{E},$$

where $\mathbf{D}$ is the electric flux density, $\mathbf{B}$ is the magnetic flux density and $\mathbf{J}$ is the electric (eddy) current density; and the constitutive parameters ε, μ and σ denote, respectively, the permittivity, permeability and conductivity of the medium. These parameters are tensors for anisotropic media, and may be functions of position and time, and may depend on the field intensities. For simplicity, we consider isotropic and homogeneous media, therefore they are constant scalars.

The field equations are supplemented by the boundary conditions:

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad \mathit{on}\ \Gamma_1, \tag{3.1e}$$

$$\mathbf{n} \cdot (\mu\mathbf{H}) = 0 \qquad \mathit{on}\ \Gamma_1, \tag{3.1f}$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{0} \qquad \mathit{on}\ \Gamma_2, \tag{3.1g}$$

$$\mathbf{n} \cdot (\varepsilon\mathbf{E}) = 0 \qquad \mathit{on}\ \Gamma_2, \tag{3.1h}$$

where Γ_1 is an electric wall, and Γ_2 is a magnetic symmetry wall. Here we consider only homogeneous boundary conditions, since inhomogeneous boundary terms can always be converted into source terms.

For transient problems, the initial conditions on $\mathbf{E}$ and $\mathbf{H}$ should also be provided.

To allow System (3.1) to have a solution, the source terms must satisfy the following compatibility conditions:

$$\nabla \cdot \mathbf{K}^{imp} = 0 \qquad \mathit{in}\ \Omega, \tag{3.2a}$$

$$\mathbf{n} \cdot \mathbf{K}^{imp} = 0 \qquad \mathit{on}\ \Gamma_1, \tag{3.2b}$$

$$\int_\Gamma \mathbf{n} \cdot \mathbf{K}^{imp} d\Gamma = 0, \tag{3.2c}$$

$$\nabla \cdot \mathbf{J}^{imp} + \frac{\partial \rho^{imp}}{\partial t} + (\sigma/\varepsilon)\rho^{imp} = 0 \qquad \mathit{in}\ \Omega, \tag{3.2d}$$

$$\mathbf{n} \cdot \mathbf{J}^{imp} = 0 \qquad \mathit{on}\ \Gamma_2. \tag{3.2e}$$

We remark that the compatibility conditions (3.2a,b,d,e) can be obtained by applying the div-curl method to the Maxwell's curl equations (3.1a,b).

3.2 The Determination

Consider the following system augmented by the variables φ and χ:

$$\nabla\varphi + \nabla\times\mathbf{E} + \frac{\partial(\mu\mathbf{H})}{\partial t} = -\mathbf{K}^{imp} \qquad in\ \Omega, \tag{3.3a}$$

$$\nabla\chi + \nabla\times\mathbf{H} - \frac{\partial(\varepsilon\mathbf{E})}{\partial t} - \sigma\mathbf{E} = \mathbf{J}^{imp} \qquad in\ \Omega, \tag{3.3b}$$

$$\nabla\cdot(\varepsilon\mathbf{E}) = \rho^{imp} \qquad in\ \Omega, \tag{3.3c}$$

$$\nabla\cdot(\mu\mathbf{H}) = 0 \qquad in\ \Omega, \tag{3.3d}$$

$$\mathbf{n}\times\mathbf{E} = \mathbf{0} \qquad on\ \Gamma_1, \tag{3.3e}$$

$$\chi = 0, \qquad \mathbf{n}\cdot(\mu\mathbf{H}) = 0 \qquad on\ \Gamma_1, \tag{3.3f}$$

$$\mathbf{n}\times\mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{3.3g}$$

$$\varphi = 0, \qquad \mathbf{n}\cdot(\varepsilon\mathbf{E}) = 0 \qquad on\ \Gamma_2, \tag{3.3h}$$

We shall prove that φ and χ in (3.3) are dummy variables, i.e., System (3.3) is equivalent to System (3.1). In fact, by virtue of the div-curl theorem, Eq. (3.3a) and (3.3b) are equivalent to the following equations:

$$\nabla\times\{\nabla\varphi + \nabla\times\mathbf{E} + \frac{\partial(\mu\mathbf{H})}{\partial t} + \mathbf{K}^{imp}\} = \mathbf{0} \qquad in\ \Omega, \tag{3.4a}$$

$$\nabla\cdot\{\nabla\varphi + \nabla\times\mathbf{E} + \frac{\partial(\mu\mathbf{H})}{\partial t} + \mathbf{K}^{imp}\} = 0 \qquad in\ \Omega, \tag{3.4b}$$

$$\mathbf{n}\cdot\{\nabla\varphi + \nabla\times\mathbf{E} + \frac{\partial(\mu\mathbf{H})}{\partial t} + \mathbf{K}^{imp}\} = 0 \qquad on\ \Gamma_1, \tag{3.4c}$$

$$\mathbf{n}\times\{\nabla\varphi + \nabla\times\mathbf{E} + \frac{\partial(\mu\mathbf{H})}{\partial t} + \mathbf{K}^{imp}\} = \mathbf{0} \qquad on\ \Gamma_2, \tag{3.4d}$$

$$\nabla\times\{\nabla\chi + \nabla\times\mathbf{H} - \frac{\partial(\varepsilon\mathbf{E})}{\partial t} - \sigma\mathbf{E} - \mathbf{J}^{imp}\} = \mathbf{0} \qquad in\ \Omega, \tag{3.5a}$$

$$\nabla\cdot\{\nabla\chi + \nabla\times\mathbf{H} - \frac{\partial(\varepsilon\mathbf{E})}{\partial t} - \sigma\mathbf{E} - \mathbf{J}^{imp}\} = 0 \qquad in\ \Omega, \tag{3.5b}$$

$$\mathbf{n}\times\{\nabla\chi + \nabla\times\mathbf{H} - \frac{\partial(\varepsilon\mathbf{E})}{\partial t} - \sigma\mathbf{E} - \mathbf{J}^{imp}\} = \mathbf{0} \qquad on\ \Gamma_1, \tag{3.5c}$$

$$\mathbf{n}\cdot\{\nabla\chi + \nabla\times\mathbf{H} - \frac{\partial(\varepsilon\mathbf{E})}{\partial t} - \sigma\mathbf{E} - \mathbf{J}^{imp}\} = 0 \qquad on\ \Gamma_2, \tag{3.5d}$$

Taking into account the divergence-free condition (3.3d) and the compatibility condition (3.2a), from (3.4b) we find that

$$\Delta\varphi = 0, \qquad in\ \Omega. \tag{3.6a}$$

Taking into account the boundary condition (3.3e) and Theorem 1, the boundary condition (3.3f), and the compatibility condition (3.2b), from (3.4c) we obtain

$$\mathbf{n} \cdot \nabla \varphi = 0 \qquad on\ \Gamma_1. \tag{3.6b}$$

From (3.3h) we have

$$\varphi = 0 \qquad on\ \Gamma_2. \tag{3.6c}$$

System (3.6) implies that $\varphi \equiv 0$ in Ω. Similarly, we can show that $\chi \equiv 0$ in Ω. Therefore, φ and χ in (3.3) are really dummy variables, and thus System (3.3) is equivalent to System (3.1).

The first-order system (3.3) has eight equations, eight unknowns, and four boundary conditions on each part of the boundary, and thus is properly determined. It is valid for static, transient, and time-harmonic cases.

In static cases, the time-derivative terms in (3.3a) and (3.3b) disappear, and $\sigma\mathbf{E}$ is included into the given current density. The system (3.3) becomes two independent div-curl systems for the electric field and the magnetic field respectively. In Section 2.2, we have shown that each div-curl system is elliptic.

In time-harmonic cases, when the time factor $e^{j\omega t}$ is used and suppressed, the time-derivative terms become the zero-order terms, and the system (3.3) becomes two coupled div-curl systems. The coupling is through the zero-order terms. The principle part, i.e., the first-order derivative terms which classify the system, still have the div-curl structure, and thus the whole system is elliptic.

In transient cases, the whole system (3.3) is hyperbolic. However, in time-domain numerical methods, the time-derivative terms are discretized by explicit or implicit finite differences, hence the time-derivative terms become the zero-order terms in the space domain. For each time step, the time-discretized system is still elliptic.

In summary, in all cases, System (3.3) is properly determined, and is elliptic in the space domain. Since System (3.1) is equivalent to System (3.3), it is indeed properly determined and also elliptic in the space domain.

3.3 The Importance of Divergence Equations

It is commonly believed that the divergence equations (3.1c) and (3.1d) are "redundant" for transient and time-harmonic problems, and thus are neglected in computation. This misconception is the true origin of spurious or inaccurate solutions in computational electromagnetics due to the following reasons:

(1) The original first-order full Maxwell's equations reflect the general laws of physics. They govern all electromagnetic phenomena, no matter whether the problem is static, time-harmonic or transient. But the first-order curl equations (3.1a) and (3.1b) cannot govern static cases. For static problems, the divergence equations (3.1c) and (3.1b) must be explicitly included in the first-order Maxwell's equations (3.1).

(2) By taking the divergence of (3.1a), one can conclude only that $\partial(\nabla \cdot (\mu\mathbf{H}))/\partial t = 0$, that is, $\nabla \cdot (\mu\mathbf{H}) = F(\mathbf{x})$. In (3.1a) there is no information about this function.

Some literature asserts that, if the divergence of $\mu\mathbf{H}$ is zero at the beginning, it will be identically zero forever. The problem then is how one can set $\nabla\cdot(\mu\mathbf{H})=0$ initially. Let us examine the common practice: letting the initial field intensities be zero in the domain, and the boundary conditions be correctly given on the boundary. In this case, the divergence-free condition is significantly violated near the boundary at the first time step of the computation. If someone really can make the divergence be zero at the beginning, it is in fact equivalent to adding a divergence-free equation into the system. The discussion for the electric field runs along the same line.

(3) The neglect of the divergence equations destroys the ellipticity of Maxwell's equations in the space domain. In each curl system there are only three(odd) equations and three(odd) unknowns that cannot be elliptic in the ordinary sense. In general, the numerical methods based on a non-elliptic system without special treatment cannot be proved to possess an optimal rate of convergence. A related investigation can be found in Jiang and Povinelli [25].

(4) The time marching method is often an effective approach to solve steady-state non-linear problems where the material properties depend on the electromagnetic fields. The curl equations alone are not adequate for this approach. Neither are the curl equations appropriate for solving the scattering of waves excited by a pulse wave.

4 The Second-Order Maxwell's Equations

In this section we shall use the div-curl method to derive the second-order Maxwell's equations and their boundary conditions, and show that the curl-curl equations cannot stand alone. We shall give the Galerkin method corresponding to the correct second-order Maxwell's equations. We shall see that this Galerkin method is of the same form as the popular Galerkin/penalty method with the penalty parameter $s=1$. We shall also give a simple least-squares look-alike method to obtain a correct variational formulation which rigorously justifies that $s=1$ in the penalty method.

4.1 The Div-Curl Method

By virtue of the div-curl theorem, System (3.1) is equivalent to

$$\nabla\times\{\nabla\times\mathbf{E}+\frac{\partial(\mu\mathbf{H})}{\partial t}+\mathbf{K}^{imp}\}=\mathbf{0}\qquad in\ \Omega, \tag{4.1a}$$

$$\nabla\cdot\{\nabla\times\mathbf{E}+\frac{\partial(\mu\mathbf{H})}{\partial t}+\mathbf{K}^{imp}\}=0\qquad in\ \Omega, \tag{4.1b}$$

$$\mathbf{n}\cdot\{\nabla\times\mathbf{E}+\frac{\partial(\mu\mathbf{H})}{\partial t}+\mathbf{K}^{imp}\}=0\qquad on\ \Gamma_1, \tag{4.1c}$$

$$\mathbf{n}\times\{\nabla\times\mathbf{E}+\frac{\partial(\mu\mathbf{H})}{\partial t}+\mathbf{K}^{imp}\}=\mathbf{0}\qquad on\ \Gamma_2, \tag{4.1d}$$

$$\nabla \times \{\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E} - \mathbf{J}^{imp}\} = \mathbf{0} \qquad in\ \Omega, \tag{4.1e}$$

$$\nabla \cdot \{\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E} - \mathbf{J}^{imp}\} = 0 \qquad in\ \Omega, \tag{4.1f}$$

$$\mathbf{n} \times \{\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E} - \mathbf{J}^{imp}\} = \mathbf{0} \qquad on\ \Gamma_1, \tag{4.1g}$$

$$\mathbf{n} \cdot \{\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E} - \mathbf{J}^{imp}\} = 0 \qquad on\ \Gamma_2, \tag{4.1h}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho^{imp} \qquad in\ \Omega, \tag{4.1i}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \qquad in\ \Omega, \tag{4.1j}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad on\ \Gamma_1, \tag{4.1k}$$

$$\mathbf{n} \cdot (\mu \mathbf{H}) = 0 \qquad on\ \Gamma_1, \tag{4.1l}$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{4.1m}$$

$$\mathbf{n} \cdot (\varepsilon \mathbf{E}) = 0 \qquad on\ \Gamma_2, \tag{4.1n}$$

Due to the compatibility conditions (3.2), the divergence conditions $(4.1i, j)$ and the boundary conditions $(4.1k - n)$, we may eliminate Equations $(4.1b), (4.1c), (4.1f)$ and $(4.1h)$ and rewrite System (4.1) as

$$\nabla \times \{\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t}\} = -\nabla \times \mathbf{K}^{imp} \qquad in\ \Omega, \tag{4.2a}$$

$$\nabla \times \{\nabla \times \mathbf{H} - \frac{\partial(\varepsilon \mathbf{E})}{\partial t} - \sigma \mathbf{E}\} = \nabla \times \mathbf{J}^{imp} \qquad in\ \Omega, \tag{4.2b}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho^{imp} \qquad in\ \Omega, \tag{4.2c}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 \qquad in\ \Omega, \tag{4.2d}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad on\ \Gamma_1, \tag{4.2e}$$

$$\mathbf{n} \cdot (\mu \mathbf{H}) = 0 \qquad on\ \Gamma_1, \tag{4.2f}$$

$$\mathbf{n} \times (\nabla \times \mathbf{H}) = \mathbf{n} \times \mathbf{J}^{imp} \qquad on\ \Gamma_1, \tag{4.2g}$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{4.2h}$$

$$\mathbf{n} \cdot (\varepsilon \mathbf{E}) = 0 \qquad on\ \Gamma_2, \tag{4.2i}$$

$$\mathbf{n} \times (\nabla \times \mathbf{E}) = -\mathbf{n} \times \mathbf{K}^{imp} \qquad on\ \Gamma_2. \tag{4.2j}$$

System (4.2) is completely equivalent to System (3.1), the validation of (4.2) guarantees the validation of (3.1). Therefore, we can use the curl equations in (3.1) to decouple **E** and **H** in (4.2) as usual, then we obtain

$$\nabla \times (\nabla \times \mathbf{E}) + \mu \frac{\partial}{\partial t}(\frac{\partial(\varepsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E}) = -\nabla \times \mathbf{K}^{imp} - \mu \frac{\partial \mathbf{J}^{imp}}{\partial t} \qquad in\ \Omega, \tag{4.3a}$$

$$\nabla\cdot(\varepsilon\mathbf{E})=\rho^{imp}\qquad in\ \Omega, \tag{4.3b}$$

$$\mathbf{n}\times\mathbf{E}=\mathbf{0}\qquad on\ \Gamma_1, \tag{4.3c}$$

$$\mathbf{n}\cdot(\varepsilon\mathbf{E})=0\qquad on\ \Gamma_2, \tag{4.3d}$$

$$\mathbf{n}\times(\nabla\times\mathbf{E})=-\mathbf{n}\times\mathbf{K}^{imp}\qquad on\ \Gamma_2, \tag{4.3e}$$

and

$$\nabla\times(\nabla\times\mathbf{H})+(\varepsilon\frac{\partial}{\partial t}+\sigma)\frac{\partial(\mu\mathbf{H})}{\partial t}=-(\varepsilon\frac{\partial}{\partial t}+\sigma)\mathbf{K}^{imp}+\nabla\times\mathbf{J}^{imp}\qquad in\ \Omega, \tag{4.4a}$$

$$\nabla\cdot(\mu\mathbf{H})=0\qquad in\ \Omega, \tag{4.4b}$$

$$\mathbf{n}\cdot(\mu\mathbf{H})=0\qquad on\ \Gamma_1, \tag{4.4c}$$

$$\mathbf{n}\times(\nabla\times\mathbf{H})=\mathbf{n}\times\mathbf{J}^{imp}\qquad on\ \Gamma_1, \tag{4.4d}$$

$$\mathbf{n}\times\mathbf{H}=\mathbf{0}\qquad on\ \Gamma_2. \tag{4.4e}$$

We note that the curl-curl equations in (4.3) and (4.4) cannot stand alone; they must be supplemented by the divergence equations and the additional natural boundary conditions. In other words, the curl-curl equations admit more solutions than the first-order full system. This is the real reason that the numerical methods based on the curl-curl equations will give rise to spurious solutions.

It is difficult to solve a second-order curl-curl equation (4.3a) with the explicit constraint of the first-order divergence equation (4.3b), since the problem has an overspecified number of partial differential equations and the first-order equation (4.3b) is hard to deal with numerically. We shall look for a simple way. By using Theorem 4 and the vector identity (2.12), System (4.3) and System (4.4) can be reduced to

$$-\Delta\mathbf{E}+\mu\frac{\partial}{\partial t}(\frac{\partial(\varepsilon\mathbf{E})}{\partial t}+\sigma\mathbf{E})=-\nabla\times\mathbf{K}^{imp}-\mu\frac{\partial\mathbf{J}^{imp}}{\partial t}-(\frac{1}{\varepsilon})\nabla\rho^{imp}\qquad in\ \Omega, \tag{4.5a}$$

$$\nabla(\nabla\cdot(\varepsilon\mathbf{E})-\rho^{imp})=\mathbf{0}\qquad in\ \Omega, \tag{4.5b}$$

$$\mathbf{n}\times\mathbf{E}=\mathbf{0}\qquad on\ \Gamma_1, \tag{4.5c}$$

$$\nabla\cdot(\varepsilon\mathbf{E})=\rho^{imp}\qquad on\ \Gamma_1, \tag{4.5d}$$

$$\mathbf{n}\cdot(\varepsilon\mathbf{E})=0\qquad on\ \Gamma_2, \tag{4.5e}$$

$$\mathbf{n}\times(\nabla\times\mathbf{E})=-\mathbf{n}\times\mathbf{K}^{imp}\qquad on\ \Gamma_2, \tag{4.5f}$$

and

$$-\Delta\mathbf{H}+(\varepsilon\frac{\partial}{\partial t}+\sigma)\frac{\partial(\mu\mathbf{H})}{\partial t}=-(\varepsilon\frac{\partial}{\partial t}+\sigma)\mathbf{K}^{imp}+\nabla\times\mathbf{J}^{imp}\qquad in\ \Omega, \tag{4.6a}$$

$$\nabla(\nabla\cdot(\mu\mathbf{H}))=\mathbf{0}\qquad in\ \Omega, \tag{4.6b}$$

$$\mathbf{n}\cdot(\mu\mathbf{H}) = 0 \qquad on\ \Gamma_1, \tag{4.6c}$$

$$\mathbf{n}\times(\nabla\times\mathbf{H}) = \mathbf{n}\times\mathbf{J}^{imp} \qquad on\ \Gamma_1, \tag{4.6d}$$

$$\mathbf{n}\times\mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{4.6e}$$

$$\nabla\cdot(\mu\mathbf{H}) = 0 \qquad on\ \Gamma_2. \tag{4.6f}$$

Due to the reasons pointed out in Section 2.3, Eq. (4.5b) and Eq. (4.6b) can be eliminated. That is, the divergence equations (4.5b) and (4.6b) are redundant, they are implicitly satisfied by the Helmholtz-type equations (4.5a) and (4.6a) and the boundary conditions. Therefore, System (4.5) and System (4.6) can be further simplified as

$$-\Delta\mathbf{E} + \mu\frac{\partial}{\partial t}(\frac{\partial(\varepsilon\mathbf{E})}{\partial t} + \sigma\mathbf{E}) = -\nabla\times\mathbf{K}^{imp} - \mu\frac{\partial\mathbf{J}^{imp}}{\partial t} - (\frac{1}{\varepsilon})\nabla\rho^{imp} \qquad in\ \Omega, \tag{4.7a}$$

$$\mathbf{n}\times\mathbf{E} = \mathbf{0} \qquad on\ \Gamma_1, \tag{4.7b}$$

$$\nabla\cdot(\varepsilon\mathbf{E}) = \rho^{imp} \qquad on\ \Gamma_1, \tag{4.7c}$$

$$\mathbf{n}\cdot(\varepsilon\mathbf{E}) = 0 \qquad on\ \Gamma_2, \tag{4.7d}$$

$$\mathbf{n}\times(\nabla\times\mathbf{E}) = -\mathbf{n}\times\mathbf{K}^{imp} \qquad on\ \Gamma_2, \tag{4.7e}$$

and

$$-\Delta\mathbf{H} + (\varepsilon\frac{\partial}{\partial t} + \sigma)\frac{\partial(\mu\mathbf{H})}{\partial t} = -(\varepsilon\frac{\partial}{\partial t} + \sigma)\mathbf{K}^{imp} + \nabla\times\mathbf{J}^{imp} \qquad in\ \Omega, \tag{4.8a}$$

$$\mathbf{n}\cdot(\mu\mathbf{H}) = 0 \qquad on\ \Gamma_1, \tag{4.8b}$$

$$\mathbf{n}\times(\nabla\times\mathbf{H}) = \mathbf{n}\times\mathbf{J}^{imp} \qquad on\ \Gamma_1, \tag{4.8c}$$

$$\mathbf{n}\times\mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{4.8d}$$

$$\nabla\cdot(\mu\mathbf{H}) = 0 \qquad on\ \Gamma_2. \tag{4.8e}$$

We note that the divergence conditions are required to be satisfied only on a part of boundary. We will rigorously prove this in Section 4.3 by using the least-squares method. As in Section 2.3 for the div-curl system, one may enforce the divergence conditions on the whole boundary Γ in (4.7) and (4.8) and show that the divergence conditions are satisfied in the domain Ω.

The Helmholtz-type equations (4.7a) and (4.8a) can be found in all text books on electromagnetics. However, it seems that all these books (except for Mayyergoyz and D'Angelo [38] who got the same conclusion as ours for a special case) claim that the Helmholtz equation must be solved with the divergence equation satisfied in the whole domain and do not mention that it needs additional boundary conditions. Our rigorous derivation using the div-curl method shows that the Helmholtz equation can stand alone, and the divergence equation should be satisfied only on a part of the boundary.

The advantages of using the Helmholtz equation over the curl-curl equation are obvious: one avoids the difficulty involving the explicit satisfaction of the divergence equations, instead one solves three decoupled second-order equations with coupled boundary conditions.

4.2 The Galerkin Method

One may elect to use, for example the finite difference method, to solve the Helmholtz-type systems (4.7) or (4.8). Usually, the finite difference method is based on rectangular structural grids. In this case, for example, the divergence condition (4.8e) can be simplified as the Neumann boundary condition (see Mayergoyz and D'Angelo [38]):

$$\frac{\partial}{\partial n} H_n = 0 \qquad on\ \Gamma_2.$$

For complex geometry it is not straightforward to implement the Neumann boundary condition in the finite difference method. By using the finite element method based on a variational principle, even the divergence conditions on the boundary do not appear. In the following we derive the variational formulation corresponding to (4.7).

By taking into account the vector identity (2.12), the Galerkin formulation associated with (4.7) is: find $\mathbf{E}$ satisfying (4.7b) and (4.7d) such that

$$\begin{aligned}
&(\nabla \times \{\nabla \times \mathbf{E} + \mathbf{K}^{imp}\}, \mathbf{E}^*) + < \nabla \times \mathbf{E} + \mathbf{K}^{imp}, \mathbf{n} \times \mathbf{E}^* >_{\Gamma_2} \\
&+(-\nabla\{\nabla \cdot \mathbf{E} - \rho^{imp}/\varepsilon\}, \mathbf{E}^*) + < \nabla \cdot \mathbf{E} - \rho^{imp}/\varepsilon, \mathbf{n} \cdot \mathbf{E}^* >_{\Gamma_1} \\
&(\mu\frac{\partial}{\partial t}\{\frac{\partial(\varepsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E}\}, \mathbf{E}^*) + (\mu\frac{\partial \mathbf{J}^{imp}}{\partial t}, \mathbf{E}^*) = 0
\end{aligned} \tag{4.9}$$

for all $\mathbf{E}^*$ satisfying (4.7b) and (4.7d). By virtue of Green's formula, the statement (4.9) can be simplified to a more symmetric form: find $\mathbf{E}$ satisfying (4.7b) and (4.7d) such that

$$\begin{aligned}
&(\nabla \times \mathbf{E}, \nabla \times \mathbf{E}^*) + (\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{E}^*) + (\mu\frac{\partial}{\partial t}\{\frac{\partial(\varepsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E}\}, \mathbf{E}^*) = \\
&-(\mathbf{K}^{imp}, \nabla \times \mathbf{E}^*) + (\rho^{imp}/\varepsilon, \nabla \cdot \mathbf{E}^*) - (\mu\frac{\partial}{\partial t}\mathbf{J}^{imp}, \mathbf{E}^*)
\end{aligned} \tag{4.10}$$

for all $\mathbf{E}^*$ satisfying (4.7b) and (4.7d).

For time-harmonic(eigenvalue) problems with $\sigma = 0$, the variational formulation takes the form

$$(\nabla \times \mathbf{E}, \nabla \times \mathbf{E}^*) + (\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{E}^*) - \omega^2 \mu\varepsilon(\mathbf{E}, \mathbf{E}^*) = 0, \tag{4.11}$$

where ω is the angular frequency. The formulations for the magnetic field are similar.

The variational formulations (4.10) and (4.11) are of the same structure as the most popular Galerkin/penalty formulations in the literature. However, in contrast to the commonly used penalty formulation, there is no free parameter in the Galerkin formulation (4.10) and (4.11). In other words, the penalty parameter $s = 1$ should be chosen in the penalty method in order for the penalty method to correspond to the Helmholtz-type equations (4.7).

4.3 The Least-Squares Look-Alike Method

In Section 4.1 the div-curl method is employed to derive the second-order (Helmholtz-type) Maxwell's equations and their boundary conditions that guarantee no spurious solutions. But there we cannot make sure that the divergence conditions should be specified only on a part of boundary. In this section we give a more powerful method to derive equivalent higher-order equations and rigorously prove the statement made in Section 4.1.

Consider the following div-curl system for the electric field:

$$\nabla \times \mathbf{E} = -\frac{\partial(\mu \mathbf{H})}{\partial t} - \mathbf{K}^{imp} \qquad in\ \Omega, \tag{4.12a}$$

$$\nabla \cdot \mathbf{E} = \rho^{imp}/\varepsilon \qquad in\ \Omega, \tag{4.12b}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad on\ \Gamma_1, \tag{4.12c}$$

$$\mathbf{n} \cdot (\varepsilon \mathbf{E}) = 0 \qquad on\ \Gamma_2, \tag{4.12d}$$

where $\mathbf{H}$ is assumed to be known and to satisfy Eq. (3.1b) and the boundary conditions (3.1f) and (3.1g), and the source terms satisfy the compatibility conditions (3.2a-e). In other words, when the magnetic field and the sources are given, the solution of (4.12) will give the corresponding electric field. Obviously, System (4.12) is a typical div-curl system that has been investigated in Section 2.

Following the steps in Section 2.4, we can derive the variational formulation which corresponds to System (4.7). We define the quadratic functional:

$$I(\mathbf{E}) = \|\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} + \mathbf{K}^{imp}\|^2 + \|\nabla \cdot \mathbf{E} - \rho^{imp}/\varepsilon\|^2,$$

in which $\mathbf{E}$ satisfies the boundary conditions (4.12c,d). The minimization of I leads to the variational formulation:

$$(\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} + \mathbf{K}^{imp}, \nabla \times \mathbf{E}^*) + (\nabla \cdot \mathbf{E} - \rho^{imp}/\varepsilon, \nabla \cdot \mathbf{E}^*) = 0, \tag{4.13}$$

where $\mathbf{E}^* = \delta \mathbf{E}$ and satisfies the same boundary conditions as $\mathbf{E}$. Since $\mathbf{H}$ satisfies (3.1b) and (3.1g), from (4.13) we have

$$(\nabla \times \mathbf{E}, \nabla \times \mathbf{E}^*) + (\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{E}^*) + (\mu \frac{\partial}{\partial t}\{\frac{\partial(\varepsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E}\}, \mathbf{E}^*) =$$

$$-(\mathbf{K}^{imp}, \nabla \times \mathbf{E}^*) + (\rho^{imp}/\varepsilon, \nabla \cdot \mathbf{E}^*) - (\mu \frac{\partial}{\partial t}\mathbf{J}^{imp}, \mathbf{E}^*), \tag{4.14}$$

which is exactly the same as (4.10). By using Green's formula, from (4.14) we can obtain the Euler-Lagrange equation (4.7a) and the natural boundary condition (4.7c) and (4.7e). That is, the correctness of (4.7) or (4.8) is completely proved.

Now we understand that the variational formulation (4.14), the Helmholtz-type equation (4.7a) and its boundary conditions, and the first-order system (4.12) are equivalent to each other. However, the finite element method based on (4.14) has superior advantages: the divergence condition (4.12b) is automatically satisfied, the test and trial functions are required to satisfy only the essential boundary conditions (4.12c,d).

We remark that the procedure to obtain the formulation (4.14) is not a true least-squares approach, because (1) we have assumed that **H** is given and satisfies (3.1b), and hence **H** is not subject to the variation; (2) the true least-squares method always leads to a symmetric bilinear form; here the σ related term is not symmetric. Even so, this procedure is mathematically justifiable. It is nothing but a rigorous method to derive the Galerkin variational formulation corresponding to the Helmholtz-type equations (4.7a) and their boundary conditions. All derivation provided in this section has rigorously proved that the penalty parameter in the Galerkin/penalty method should be equal to one.

5 The Least-Squares Method for First-Order Maxwell's Equations

In Section 2.4 we have introduced the least-squares method for the pure div-curl system governing static field problems, and in Section 4.3 the least-squares look alike method for the div-curl system describing time-dependent single(electric or magnetic) field problems, and demonstrated the power of the least-squares method. In this section we briefly give the formulations of the LSFEM for the general first-order partial differential equations, and apply it to the solution of the time-harmonic first-order Maxwell's equations.

5.1 The General Formulation

The least-squares method for the linear operator equation $\mathbf{A}\mathbf{u} = \mathbf{f}$ formally is equivalent to the solution of the higher-order equation $\mathbf{A}^*\mathbf{A}\mathbf{u} = \mathbf{A}^*\mathbf{f}$ with $\mathbf{A}\mathbf{u} = \mathbf{f}$ serving as an additional natural boundary condition, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$ in the inner product generated by the L_2 norm. When directly applied to second-order equations this approach requires the use of C^1 finite elements and leads to ill-conditioned discrete systems. In order to use simple C^0 elements and obtain a better-conditioned algebraic system, the least-squares method discussed here is based on the first-order system. The formulation of the least-squares finite element method for general first-order steady-state boundary-value problems can be found in Jiang and Povinelli [24]. This formulation can be directly applied to the solution of the first-order steady-state and time-harmonic Maxwell's equations. For time-dependent problems one always can use an appropriate finite difference method in the temporal domain, such as the backward Euler scheme or the Crank-Nicolson scheme, to discretize the time-derivative terms so that in each time-step the problems are converted into boundary-value problems. For completeness, we briefly derive the general least-squares formulation.

We consider the linear boundary-value problem:

$$\mathbf{Au} = \mathbf{f} \qquad in \quad \Omega, \tag{5.1a}$$

$$\mathbf{Bu} = \mathbf{g} \qquad on \quad \Gamma, \tag{5.1b}$$

where $\mathbf{A}$ is a first-order partial differential operator:

$$\mathbf{Au} = \sum_{i=1}^{n_d} \mathbf{A}_i \frac{\partial \mathbf{u}}{\partial x_i} + \mathbf{A}_0 \mathbf{u}, \tag{5.2}$$

in which $\Omega \in \mathbf{R}^{n_d}$ is a bounded domain with a piecewise smooth boundary Γ, $n_d = 2$ *or* 3 represents the number of space dimensions, $\mathbf{u}^T = (u_1, u_2, ...u_m)$ is a vector of m unknown functions of $\mathbf{x} = (x_1, ..., x_{n_d})$, $\mathbf{A}_i$ and $\mathbf{A}_0$ are $n \times m$ matrices which depend on $\mathbf{x}$, $\mathbf{f}$ is a given vector-valued function, $\mathbf{B}$ is a boundary algebraic operator, and $\mathbf{g}$ is a given vector-valued function on the boundary. Without loss of generality we assume that the vector $\mathbf{g}$ is null. We should mention that the number of equations n in the system (5.1a) must be greater than or equal to the number of unknowns m.

Considering the boundary condition of the boundary-value problem, we also define the function space

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^m | \ \mathbf{Bv} = \mathbf{0} \quad on \ \Gamma\}. \tag{5.3}$$

Let us suppose that $\mathbf{f} \in \mathbf{L}_2(\Omega)$ and $\mathbf{A} : \mathbf{V} \rightarrow \mathbf{L}_2(\Omega)$. For an arbitrary trial function $\mathbf{v} \in \mathbf{V}$, we define the residual function:

$$\mathbf{R} = \mathbf{Av} - \mathbf{f} \qquad in \ \Omega. \tag{5.4}$$

In general the residual $\mathbf{R}$ is not equal to zero, except $\mathbf{v}$ is equal to the exact solution $\mathbf{u}$. The squared distance between $\mathbf{Av}$ and $\mathbf{f}$ will be nonnegative:

$$\|\mathbf{R}\|_0^2 = \int_\Omega (\mathbf{Av} - \mathbf{f})^2 d\Omega \geq 0. \tag{5.5}$$

A solution $\mathbf{u}$ to the problem (5.1) can thus be interpreted as a member of $\mathbf{V}$ that minimizes the squared distance between $\mathbf{Av}$ and $\mathbf{f}$:

$$0 = \|\mathbf{R}(\mathbf{u})\|_0^2 \leq \|\mathbf{R}(\mathbf{v})\|_0^2 \qquad \forall \mathbf{v} \in \mathbf{V}.$$

The least-squares method consists of seeking a minimizer of the squared distance $\|\mathbf{Av} - \mathbf{f}\|_0^2$ in $\mathbf{V}$. We write the quadratic functional in (5.5) as

$$I(\mathbf{v}) = \|\mathbf{Av} - \mathbf{f}\|_0^2 = (\mathbf{Av} - \mathbf{f}, \mathbf{Av} - \mathbf{f}). \tag{5.6}$$

A necessary condition that $\mathbf{u} \in \mathbf{V}$ be a minimizer of the functional I in (5.6) is that its first variation vanish at $\mathbf{u}$ for all admissible $\mathbf{v}$. That is,

$$\lim_{\tau \to 0} \frac{d}{d\tau} I(\mathbf{u} + \tau \mathbf{v}) \equiv 2 \int_\Omega (\mathbf{Av})^T (\mathbf{Au} - \mathbf{f}) d\Omega = \mathbf{0} \qquad \forall \mathbf{v} \in \mathbf{V}.$$

Thus, the least-squares method leads us to the variational boundary-value problem: Find $\mathbf{u} \in \mathbf{V}$ such that

$$B(\mathbf{u},\mathbf{v}) = F(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{5.7a}$$

where

$$B(\mathbf{u},\mathbf{v}) \equiv (\mathbf{Au},\mathbf{Av}), \tag{5.7b}$$

$$F(\mathbf{v}) \equiv (\mathbf{f},\mathbf{Av}). \tag{5.7c}$$

In the finite element analysis, we first subdivide the domain as a union of finite elements and then introduce an appropriate finite element basis. Let N_n denote the number of nodes for one element and ψ_j denote the element shape functions. If equal-order interpolations are employed, that is, for all unknown variables the same finite element is used, we can write the expansion in each element

$$\mathbf{u}_h^e(\mathbf{x}) = \sum_{j=1}^{N_n} \psi_j(\mathbf{x}) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}_j , \tag{5.8}$$

where $(u_1, u_2, ..., u_m)_j$ are the nodal values at the jth node, and h denotes the mesh parameter.

Introducing the finite element approximation defined in (5.8) into the variational statement (5.7), we have the linear algebraic equations

$$\mathbf{KU} = \mathbf{F}, \tag{5.9}$$

where $\mathbf{U}$ is the global vector of nodal values. The global matrix $\mathbf{K}$ is assembled from the element matrices

$$\mathbf{K}_e = \int_{\Omega_e} (\mathbf{A}\psi_1, \mathbf{A}\psi_2, ..., \mathbf{A}\psi_{N_n})^T (\mathbf{A}\psi_1, \mathbf{A}\psi_2, ..., \mathbf{A}\psi_{N_n}) d\Omega, \tag{5.10}$$

in which $\Omega_e \subset \Omega$ is the domain of the eth element, and T denotes the transpose, and the vector $\mathbf{F}$ is assembled from the element vectors

$$\mathbf{F}_e = \int_{\Omega_e} (\mathbf{A}\psi_1, \mathbf{A}\psi_2, ..., \mathbf{A}\psi_{N_n})^T \mathbf{f} d\Omega, \tag{5.11}$$

in (5.10) and (5.11)

$$\mathbf{A}\psi_j = \sum_{i=1}^{n_d} \frac{\partial \psi_j}{\partial x_i} \mathbf{A}_i + \psi_j \mathbf{A}_0. \tag{5.12}$$

From the above derivation we can immediately find out or further prove that:

(1) the least-squares method is universal for all types of partial differential equations, no matter whether they are elliptic, hyperbolic, parabolic or mixed; the only requirement is that they have a unique solution, see Mikhlin [40] and Marchuk [39];

(2) the LSFEM leads to a symmetric positive definite algebraic system which can be solved efficiently by matrix-free iterative methods, such as the element-by-element preconditioned conjugate gradient method, and thus the parallelization and large-scale 3D computation is made easy;

(3) the LSFEM formulation and its coding are general, therefore for a new problem one needs only to supply the coefficients, the load vector and the boundary conditions;

(4) the LSFEM is robust, no special treatments, such as upwinding, staggered grids, and operator splitting etc. are needed; the LSFEM leads to a minimization problem rather than a saddle-point problem, thus simple equal-order interpolations can be employed;

(5) the LSFEM can often be proved to have optimal numerical properties including an optimal rate of convergence;

(6) the LSFEM satisfies the divergence conditions in electromagnetics.

5.2 Time-Harmonic Fields

For three-dimensional time-harmonic fields, the first-order full Maxwell's equations can be written as

$$\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = -\mathbf{K}^{imp} \qquad in\ \Omega, \tag{5.13a}$$

$$\nabla \times \mathbf{H} - j\omega\varepsilon\mathbf{E} = \mathbf{J}^{imp} \qquad in\ \Omega, \tag{5.13b}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad in\ \Omega, \tag{5.13c}$$

$$\nabla \cdot \mathbf{H} = 0 \qquad in\ \Omega, \tag{5.13d}$$

where the time factor $e^{j\omega t}$ is used and suppressed, ω is the given angular frequency and not equal to the resonant frequencies of this problem, $\mathbf{E}$ and $\mathbf{H}$ are the complex electric and magnetic field intensities respectively, $\mathbf{J}^{imp}$ and $\mathbf{K}^{imp}$ are imposed harmonic sources of electric and magnetic current density respectively. All imposed sources are given functions of the space coordinates. For simplicity, we consider homogeneous isotropic media, i.e., ϵ and μ are constant scalars. The field equations are supplemented by the homogeneous boundary conditions:

$$\mathbf{n} \times \mathbf{E} = 0 \qquad on\ \Gamma_1, \tag{5.13e}$$

$$\mathbf{n} \cdot \mathbf{H} = 0 \qquad on\ \Gamma_1, \tag{5.13f}$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{0} \qquad on\ \Gamma_2, \tag{5.13g}$$

$$\mathbf{n} \cdot \mathbf{E} = 0 \qquad on\ \Gamma_2. \tag{5.13h}$$

where Γ_1 is an electric wall, and Γ_2 is a magnetic symmetry wall.

To allow System (5.13) to have a solution, the source terms cannot be arbitrary, they must satisfy the following compatibility conditions:

$$\nabla \cdot \mathbf{K}^{imp} = 0 \qquad in\ \Omega, \tag{5.14a}$$

$$\mathbf{n}\cdot\mathbf{K}^{imp}=0 \qquad on\ \Gamma_1, \tag{5.14b}$$

$$\int_\Gamma \mathbf{n}\cdot\mathbf{K}^{imp}d\Gamma=0, \tag{5.14c}$$

$$\nabla\cdot\mathbf{J}^{imp}=0 \qquad in\ \Omega, \tag{5.14d}$$

$$\mathbf{n}\cdot\mathbf{J}^{imp}=0 \qquad on\ \Gamma_2, \tag{5.14e}$$

$$\int_\Gamma \mathbf{n}\cdot\mathbf{J}^{imp}d\Gamma=0. \tag{5.14f}$$

As in the time-varying cases, the compatibility conditions (5.14a,b) and (5.14d,e) can be derived by applying the div-curl method to the curl equations (5.13a,b). The compatibility conditions (5.14c) and (5.14f) can be obtained by applying the Gauss divergence theorem to (5.14a) and (5.14d), respectively.

Separating the real and imaginary parts in (5.13) leads to

$$\nabla\times\mathbf{E}_r-\omega\mu\mathbf{H}_i=-\mathbf{K}_r^{imp} \qquad in\ \Omega, \tag{5.15a}$$

$$\nabla\times\mathbf{E}_i+\omega\mu\mathbf{H}_r=-\mathbf{K}_i^{imp} \qquad in\ \Omega, \tag{5.15b}$$

$$\nabla\times\mathbf{H}_r+\omega\varepsilon\mathbf{E}_i=\mathbf{J}_r^{imp} \qquad in\ \Omega, \tag{5.15c}$$

$$\nabla\times\mathbf{H}_i-\omega\varepsilon\mathbf{E}_r=\mathbf{J}_i^{imp} \qquad in\ \Omega, \tag{5.15d}$$

$$\nabla\cdot\mathbf{E}_r=0 \qquad in\ \Omega, \tag{5.15e}$$

$$\nabla\cdot\mathbf{E}_i=0 \qquad in\ \Omega, \tag{5.15f}$$

$$\nabla\cdot\mathbf{H}_r=0 \qquad in\ \Omega, \tag{5.15g}$$

$$\nabla\cdot\mathbf{H}_i=0 \qquad in\ \Omega. \tag{5.15h}$$

Obviously, System (5.15) is elliptic, since its principle part consists of four div-curl systems. For the solution of (5.15) the least-squares variational formulation is: find $\mathbf{u}=(\mathbf{E}_r,\mathbf{E}_i,\mathbf{H}_r,\mathbf{H}_i)\in\mathcal{H}$ such that

$$B(\mathbf{u},\mathbf{v})=L(\mathbf{v}) \quad \forall\mathbf{v}=(\mathbf{E}_r^*,\mathbf{E}_i^*,\mathbf{H}_r^*,\mathbf{H}_i^*)\in\mathcal{H}, \tag{5.16}$$

where $\mathcal{H}=\{\mathbf{u}\in H^1(\Omega)^3\times H^1(\Omega)^3\times H^1(\Omega)^3\times H^1(\Omega)^3|\mathbf{n}\times\mathbf{E}=\mathbf{0}\ on\ \Gamma_1,\mathbf{n}\cdot\mathbf{H}=0\ on\ \Gamma_1,\mathbf{n}\times\mathbf{H}=\mathbf{0}\ on\ \Gamma_2,\mathbf{n}\cdot\mathbf{E}=0\ on\ \Gamma_2\}$, and $B(\cdot,\cdot)$ is the bilinear form

$$\begin{aligned}B(\mathbf{u},\mathbf{v})=&(\nabla\times\mathbf{E}_r-\omega\mu\mathbf{H}_i,\nabla\times\mathbf{E}_r^*-\omega\mu\mathbf{H}_i^*)\\&+(\nabla\times\mathbf{E}_i+\omega\mu\mathbf{H}_r,\nabla\times\mathbf{E}_i^*+\omega\mu\mathbf{H}_r^*)\\&+(\nabla\times\mathbf{H}_r+\omega\varepsilon\mathbf{E}_i,\nabla\times\mathbf{H}_r^*+\omega\varepsilon\mathbf{E}_i^*)\\&+(\nabla\times\mathbf{H}_i-\omega\varepsilon\mathbf{E}_r,\nabla\times\mathbf{H}_i^*-\omega\varepsilon\mathbf{E}_r^*)\\&+(\nabla\cdot\mathbf{E}_r,\nabla\cdot\mathbf{E}_r^*)+(\nabla\cdot\mathbf{E}_i,\nabla\cdot\mathbf{E}_i^*)\\&+(\nabla\cdot\mathbf{E}_r,\nabla\cdot\mathbf{E}_r^*)+(\nabla\cdot\mathbf{E}_i,\nabla\cdot\mathbf{E}_i^*),\end{aligned} \tag{5.17a}$$

and $L(\cdot)$ is the linear form

$$
\begin{aligned}
L(\mathbf{v}) = &(-K_r^{imp}, \nabla \times \mathbf{E}_r^* - \omega\mu\mathbf{H}_i^*) \\
&+(-K_i^{imp}, \nabla \times \mathbf{E}_i^* + \omega\mu\mathbf{H}_r^*) \\
&+(J_r^{imp}, \nabla \times \mathbf{H}_r^* + \omega\varepsilon\mathbf{E}_i^*) \\
&+(J_i^{imp}, \nabla \times \mathbf{H}_i^* - \omega\varepsilon\mathbf{E}_r^*).
\end{aligned} \tag{5.17b}
$$

Obviously, the bilinear form $B(\mathbf{u},\mathbf{v})$ in (5.17a) is symmetric and continuous, and the linear form $L(\mathbf{v})$ in (5.17b) is continuous. One may prove that if the frequency of the exciting source is not equal to the resonant frequencies of this electromagnetic system, then the bilinear form $B(\mathbf{u},\mathbf{u})$ is coercive. By virtue of the Lax-Milgram theorem, the least-squares solution uniquely exists and the corresponding finite element solution is of an optimal rate of convergence. In fact, the following statement is the consequence of the coerciveness of the bilinear form $B(\mathbf{u},\mathbf{u})$. We will prove it in our future reports.

The LSFEM based on (5.16) has an optimal rate of convergence and an optimal satisfaction of divergence-free conditions:

$$\|\mathbf{E}_r - \mathbf{E}_{rh}\|_0 \le Ch^{k+1}, \quad \|\nabla \cdot \mathbf{E}_{rh}\|_0 \le Ch^k, \tag{5.18a}$$

$$\|\mathbf{E}_i - \mathbf{E}_{ih}\|_0 \le Ch^{k+1}, \quad \|\nabla \cdot \mathbf{E}_{ih}\|_0 \le Ch^k, \tag{5.18b}$$

$$\|\mathbf{H}_r - \mathbf{H}_{rh}\|_0 \le Ch^{k+1}, \quad \|\nabla \cdot \mathbf{H}_{rh}\|_0 \le Ch^k, \tag{5.18c}$$

$$\|\mathbf{H}_i - \mathbf{H}_{ih}\|_0 \le Ch^{k+1}, \quad \|\nabla \cdot \mathbf{H}_{ih}\|_0 \le Ch^k, \tag{5.18d}$$

where $\mathbf{E}_{rh}$, $\mathbf{E}_{ih}$, $\mathbf{H}_{rh}$, $\mathbf{H}_{ih}$ are the finite element solutions, k is the order of complete polynomials in the equal-order finite element interpolation.

5.3 Time-Harmonic TE Waves

For time-harmonic TE waves the first-order Maxwell's equations are

$$j\omega\mu H_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad in\ \Omega, \tag{5.19a}$$

$$j\omega\varepsilon^* E_x - \frac{\partial H_z}{\partial y} = 0 \qquad in\ \Omega, \tag{5.19b}$$

$$j\omega\varepsilon^* E_y + \frac{\partial H_z}{\partial x} = 0 \qquad in\ \Omega, \tag{5.19c}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad in\ \Omega, \tag{5.19d}$$

in which $\varepsilon^* = \varepsilon_r + j\varepsilon_i = \varepsilon + j\sigma/\omega$ is the complex permittivity where the subscripts r and i indicate the real part and the imaginary part, respectively. For a complete description of TE wave problems appropriate boundary conditions should be included. One may consider, for example

$$H_z = constant \qquad on\ \Gamma, \tag{5.19e}$$

$$E_x n_x + E_y n_y = 0 \qquad on\ \Gamma, \tag{5.19f}$$

where $\mathbf{n} = (n_x, n_y)$ is the unit vector normal to the boundary Γ. The condition (5.19e) is an inhomogeneous version corresponding to (5.13g), and (5.19f) is a 2D version of (5.13h). We also remark that the boundary conditions (5.19e,f) satisfy the boundary compatibility condition

$$j\omega\varepsilon^*(E_x n_x + E_y n_y) = \frac{\partial H_z}{\partial y} n_x - \frac{\partial H_z}{\partial x} n_y \qquad on\ \Gamma, \tag{5.20}$$

which is obtained by taking the operation $\mathbf{n}\cdot$ to Eq. (5.19b) and (5.19c).

In System (5.19) there are three unknowns and four equations, and thus the divergence-free equation (5.19d) seems redundant. By introducing a dummy variable ϑ into System (5.19) we have

$$j\omega\mu H_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad in\ \Omega, \tag{5.21a}$$

$$j\omega\varepsilon^* E_x + \frac{\partial \vartheta}{\partial x} - \frac{\partial H_z}{\partial y} = 0 \qquad in\ \Omega, \tag{5.21b}$$

$$j\omega\varepsilon^* E_y + \frac{\partial \vartheta}{\partial y} + \frac{\partial H_z}{\partial x} = 0 \qquad in\ \Omega, \tag{5.21c}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad in\ \Omega. \tag{5.21d}$$

By taking the operation $\partial/\partial x$ to Eq. (5.21b) and the operation $\partial/\partial y$ to Eq. (5.21c), and by adding the results together we obtain the Laplace equation for ϑ:

$$\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} = 0 \qquad in\ \Omega. \tag{5.22a}$$

By taking the operation $\mathbf{n}\cdot$ to the equations (5.21b) and (5.21c) and using the boundary compatibility condition (5.20) we obtain

$$\frac{\partial \vartheta}{\partial \mathbf{n}} = 0 \qquad on\ \Gamma. \tag{5.22b}$$

From (5.22) we know that $\vartheta = constant$, that is, System (5.19) is completely equivalent to the augmented system (5.21) with four unknowns and four equations. Since

System (5.21) consists of two two-dimensional div-curl systems, and thus is elliptic. Therefore, System (5.19) is not 'overdetermined', but is indeed properly determined and elliptic.

For numerical calculation separating the real and imaginary parts in (5.19a-d) leads to

$$-\omega\mu H_{zi} + \frac{\partial E_{yr}}{\partial x} - \frac{\partial E_{xr}}{\partial y} = 0 \qquad in\ \Omega, \tag{5.23a}$$

$$-\omega(\varepsilon_r E_{xi} + \varepsilon_i E_{xr}) - \frac{\partial H_{zr}}{\partial y} = 0 \qquad in\ \Omega, \tag{5.23b}$$

$$-\omega(\varepsilon_r E_{yi} + \varepsilon_i E_{yr}) + \frac{\partial H_{zr}}{\partial x} = 0, \qquad in\ \Omega \tag{5.23c}$$

$$\frac{\partial E_{xr}}{\partial x} + \frac{\partial E_{yr}}{\partial y} = 0 \qquad in\ \Omega, \tag{5.23d}$$

$$\omega\mu H_{zr} + \frac{\partial E_{yi}}{\partial x} - \frac{\partial E_{xi}}{\partial y} = 0 \qquad in\ \Omega, \tag{5.23e}$$

$$\omega(\varepsilon_r E_{xr} - \varepsilon_i E_{xi}) - \frac{\partial H_{zi}}{\partial y} = 0 \qquad in\ \Omega, \tag{5.23f}$$

$$\omega(\varepsilon_r E_{yr} - \varepsilon_i E_{yi}) + \frac{\partial H_{zi}}{\partial x} = 0 \qquad in\ \Omega, \tag{5.23g}$$

$$\frac{\partial E_{xi}}{\partial x} + \frac{\partial E_{yi}}{\partial y} = 0 \qquad in\ \Omega. \tag{5.23h}$$

Of course, in (5.23) the medium properties are different for different medium regions.

We may write System (5.23) in the standard matrix form:

$$\mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial x} + \mathbf{A}_2 \frac{\partial \mathbf{u}}{\partial y} + \mathbf{A}_0 \mathbf{u} = \mathbf{f}, \tag{5.24}$$

in which

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.25a}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.25b}$$

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & 0 & -\omega\mu & 0 & 0 \\ 0 & -\omega\epsilon_i & 0 & 0 & -\omega\epsilon_r & 0 \\ 0 & 0 & -\omega\epsilon_i & 0 & 0 & -\omega\epsilon_r \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega\epsilon_r & 0 & 0 & -\omega\epsilon_i & 0 \\ 0 & 0 & \omega\epsilon_r & 0 & 0 & -\omega\epsilon_i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.25c}$$

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} H_{zr} \\ E_{xr} \\ E_{yr} \\ H_{zi} \\ E_{xi} \\ E_{yi} \end{pmatrix}. \tag{5.25d}$$

At the interface Γ_{int} between two contiguous media (+) and (-) the following general conditions should be satisfied:

$$\mathbf{n} \times \mathbf{E}^+ = \mathbf{n} \times \mathbf{E}^- \qquad on\ \Gamma_{int}, \tag{5.26a}$$

$$\mathbf{n} \times \mathbf{H}^+ = \mathbf{n} \times \mathbf{H}^- \qquad on\ \Gamma_{int}, \tag{5.26b}$$

$$\mathbf{n} \cdot (\epsilon^{*+}\mathbf{E}^+) = \mathbf{n} \cdot (\epsilon^{*-}\mathbf{E}^-) \qquad on\ \Gamma_{int}, \tag{5.26c}$$

$$\mathbf{n} \cdot (\mu^+\mathbf{H}^+) = \mathbf{n} \cdot (\mu^-\mathbf{H}^-) \qquad on\ \Gamma_{int}. \tag{5.26d}$$

For two-dimensional TE waves the interface conditions (5.26a) and (5.26c) become

$$n_x E_{yr}^+ - n_y E_{xr}^+ = n_x E_{yr}^- - n_y E_{xr}^-, \tag{5.27a}$$

$$n_x E_{yi}^+ - n_y E_{xi}^+ = n_x E_{yi}^- - n_y E_{xi}^-, \tag{5.27b}$$

$$\begin{aligned} n_x(\epsilon_r^+ E_{xr}^+ - \epsilon_i^+ E_{xi}^+) + n_y(\epsilon_r^+ E_{yr}^+ - \epsilon_i^+ E_{yi}^+) = \\ n_x(\epsilon_r^- E_{xr}^- - \epsilon_i^- E_{xi}^-) + n_y(\epsilon_r^- E_{yr}^- - \epsilon_i^- E_{yi}^-), \end{aligned} \tag{5.27c}$$

$$n_x(\epsilon_i^+ E_{xr}^+ + \epsilon_r^+ E_{xi}^+) + n_y(\epsilon_i^+ E_{yr}^+ + \epsilon_r^+ E_{yi}^+) =$$

$$n_x(\epsilon_i^- E_{xr}^- + \epsilon_r^- E_{xi}^-) + n_y(\epsilon_i^- E_{yr}^- + \epsilon_r^- E_{yi}^-), \tag{5.27d}$$

The interface condition (5.26d) is automatically satisfied, and (5.26b) becomes

$$H_z^+ = H_z^-. \tag{5.27e}$$

In the LSFEM the treatment of the interface conditions is not difficult. As in other node-based finite element methods, the nodes on the interface should be double-numbered. If a direct solver is employed for solving the discretized system, two approaches are available: a simple way is to include the interface conditions into the least-squares functional; a better way is to use the conditions (5.27a-e) to modify the global stiffness matrix in the discretized system. If the conjugate gradient method is used, one just simply chooses the unknowns related to the medium (+) (or (−)) as the true unknowns and keeps the conditions (5.27a-e) satisfied for each solution vector.

Since the general formulation of the LSFEM has been given in Section 5.3, it is not necessary to write down the special one for the problem discussed in this section. One only needs to substitute the coefficients of (5.24) and the boundary conditions into a general-purpose LSFEM code.

We consider two test problems that are taken from Paulsen and Lynch [49] in which the spurious solutions given by the curl-curl formulation as well as the correct solutions are illustrated.

The first example is a cylinder (R=25) which is split into two regions having different complex permittivity. For the top region, $\varepsilon_r^+ = 3.0$, $\varepsilon_i^+ = -5.0$ and $\mu^+ = 1.0$; for the bottom region, $\varepsilon_r^- = 1.0$, $\varepsilon_i^- = 0.0$ and $\mu^- = 1.0$. This cylinder is excited by a uniform $H_z|_\Gamma = (1, 0)$ (with $\omega = 0.05$) imposed on the outer boundary. This problem is discretized by 932 bi-linear elements with 1016 nodes shown in Fig. 1(a). The contours of the computed real and imaginary magnetic field intensity are shown in Fig. 1(b) and (c), respectively. The vector plots of the real and imaginary electric field intensity are illustrated in Fig. 1(d) and (e), respectively.

In the second example, a smaller off-center cylinder (R=0.1) is embedded in a larger cylinder (R=0.25). The material properties for the outer region are $\varepsilon_r^+ = 0.0981$, $\varepsilon_i^+ = -0.0196$ and $\mu^+ = 1.0$; for the inner region $\varepsilon_r^- = 1.0$, $\varepsilon_i^- = 0.0$ and $\mu^- = 1.0$. A uniform $H_z|_\Gamma = (1, -0.15)$ (with $\omega = 44.7$) is imposed on the outer boundary. Fig. 2(a) shows the mesh with 2027 bi-linear elements and 2165. The contours of the computed real and imaginary magnetic field intensity are shown in Fig. 2(b,c), respectively. The vector plots of the real and imaginary electric field intensity are illustrated in Fig. 1(d,e), respectively.

As expected all computed results by the LSFEM are free of spurious modes.

6 Conclusions

(1) The system of the first-order full Maxwell's equations seems "overspecified" because it has more equations than unknowns. By taking into account of its div-

curl structure and introducing the dummy variables it proves to be properly determined and elliptic in the space domain. The information provided by the divergence equations is not completely contained in the curl equations. Therefore, the divergence equations must be explicitly included in the first-order system to assure the uniqueness of the solution in steady-state cases and to guarantee the accuracy of the numerical solution for time-varying cases.

(2) The least-squares method and the div-curl method are mathematically rigorous and useful tools for the derivation of correct second-order Maxwell's equations and their boundary conditions. The curl-curl equations cannot stand alone, they must be supplemented by the divergence equations and additional natural boundary conditions to eliminate the spurious solutions.

(3) The Helmholtz-type equations with appropriate natural boundary conditions, derived by the div-curl method or the least-squares method, can guarantee the implicit satisfaction of the divergence equations. For the solution of the Helmholtz-type equations the divergence conditions of the electric field and the magnetic field need to be enforced only on the electric wall and the magnetic symmetry wall, respectively.

(4) The variational formulation corresponding to the Helmholtz-type equations can be derived by using the least-squares look-alike method. This formulation theoretically justifies that the penalty parameter in the Galerkin/penalty method should be taken as one. The advantage of this formulation is that the trial and test functions need only to satisfy the conditions related to the essential boundary conditions.

(5) The node-based least-squares finite element method(LSFEM) can be used to solve both static and time-varying first-order Maxwell's equations directly and efficiently with the divergence equations satisfied easily. The introduction of potentials and the gauging method, the edge element method, the staggered grid and upwinding, etc. all turn out to be unnecessary.

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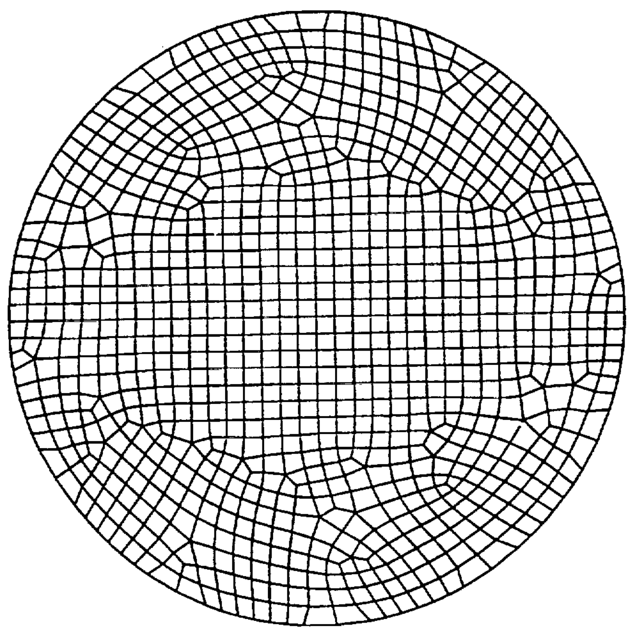

Fig. 1(a) The split cylinder and the mesh.

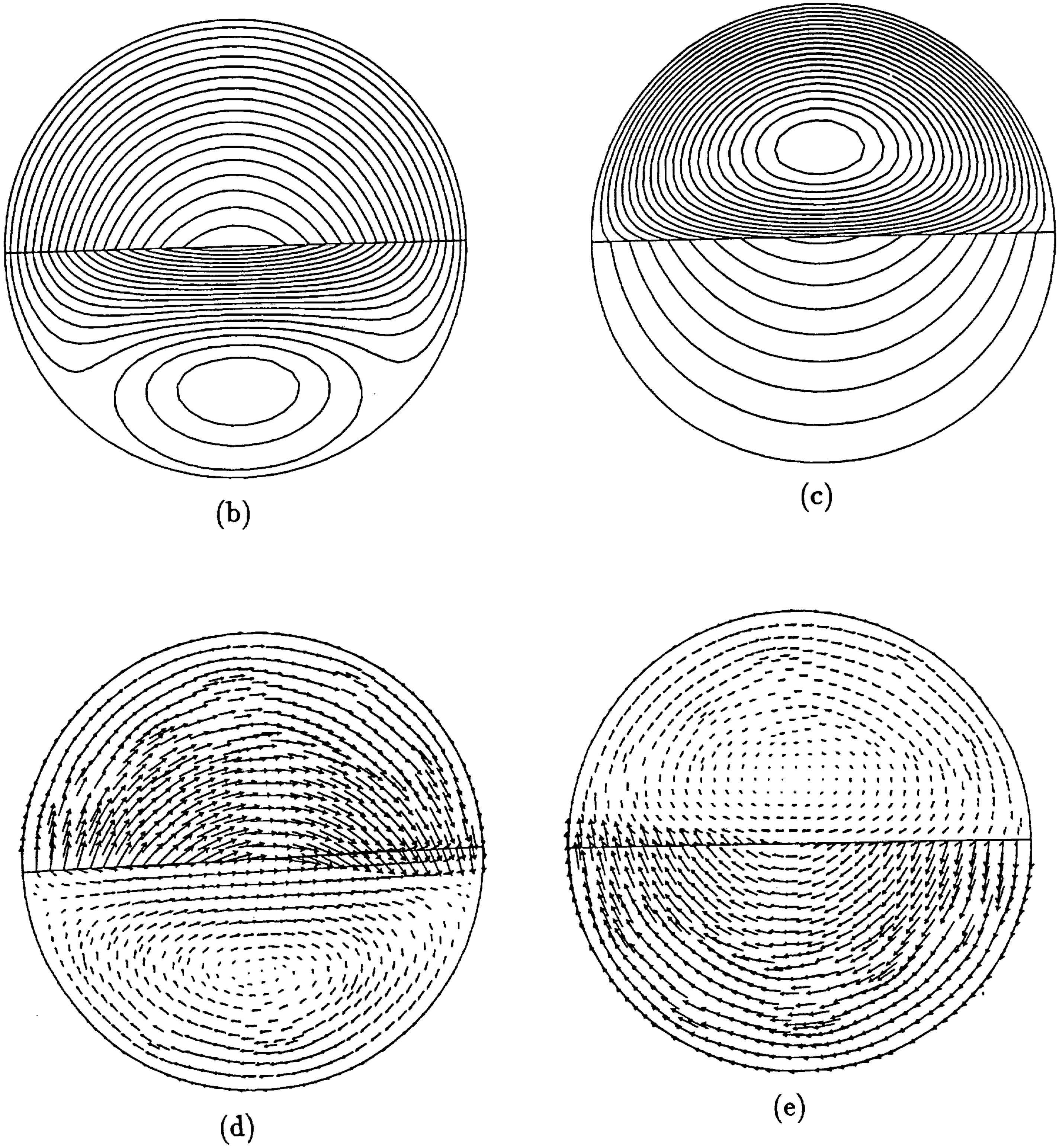

Fig. 1(b) Contours of constant magnetic field intensity H_r.
Fig. 1(c) Contours of constant magnetic field intensity H_i.
Fig. 1(d) Vectors of the computed electric field intensity $\mathbf{E}_r$.
Fig. 1(e) Vectors of the computed electric field intensity $\mathbf{E}_i$.

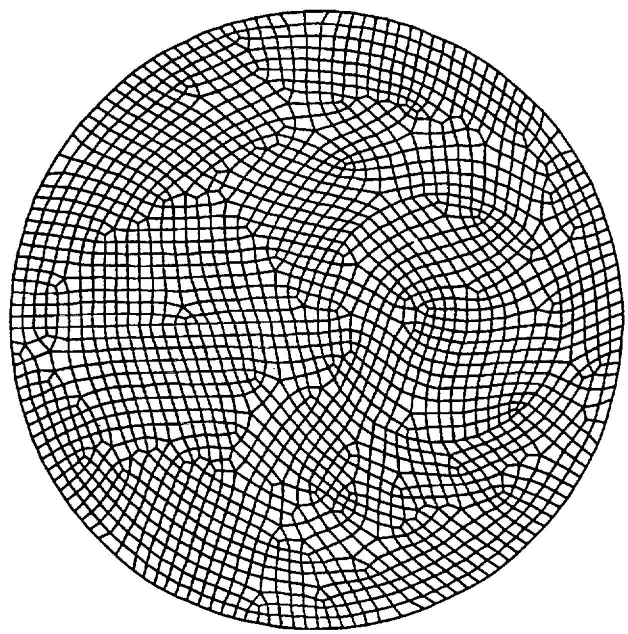

Fig. 2(a) The off-center cylinder.

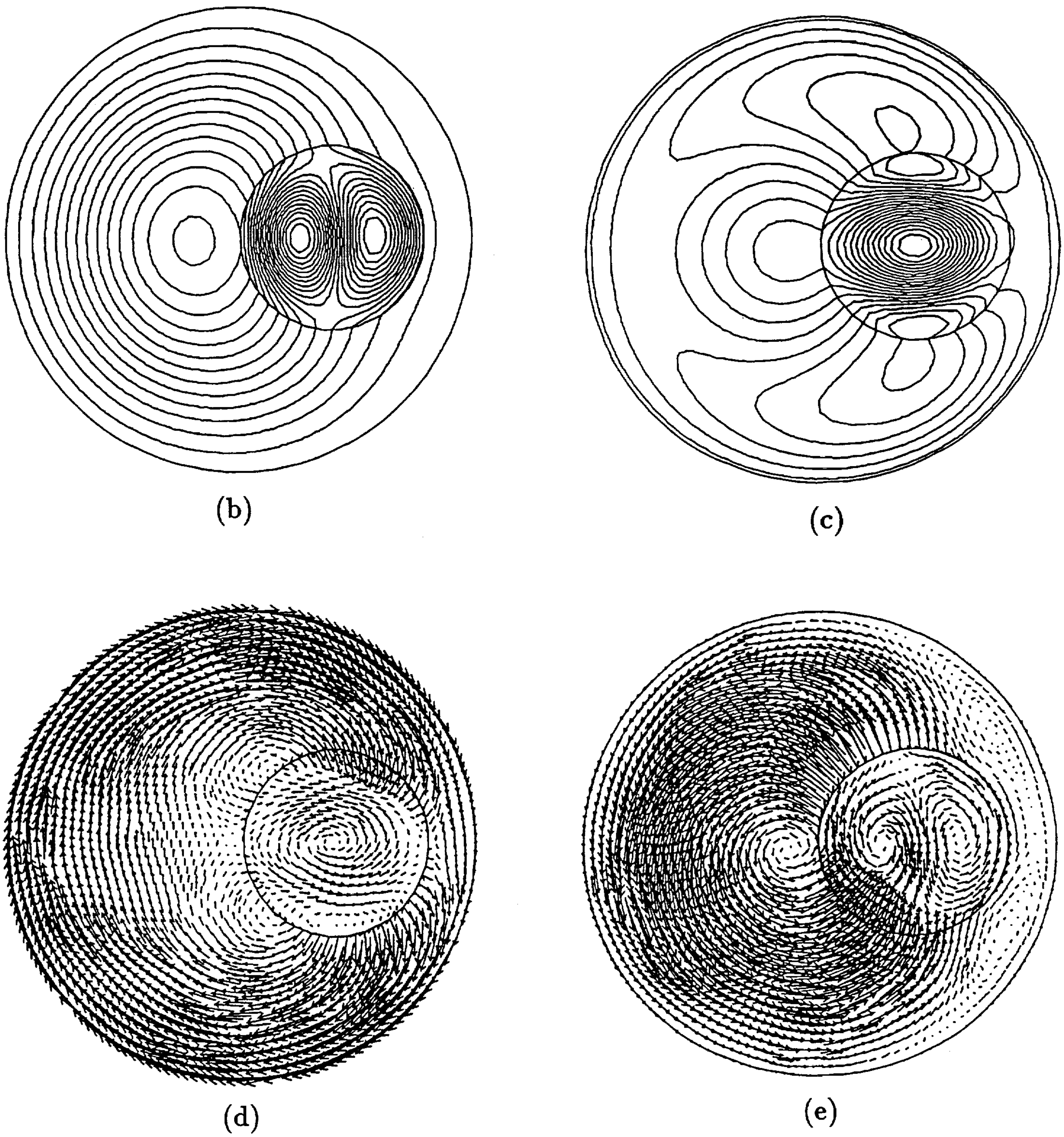

Fig. 2(b) Contours of constant magnetic field intensity H_r.
Fig. 2(c) Contours of constant magnetic field intensity H_i.
Fig. 2(d) Vectors of the computed electric field intensity $\mathbf{E}_r$.
Fig. 2(e) Vectors of the computed electric field intensity $\mathbf{E}_i$.

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13. ABSTRACT ***(Maximum 200 words)***

The origin of spurious solutions in computational electromagnetics, which violate the divergence equations, is deeply rooted in a misconception about the first-order Maxwell's equations and in an incorrect derivation and use of the curl-curl equations. The divergence equations must be always included in the first-order Maxwell's equations to maintain the ellipticity of the system in the space domain and to guarantee the uniqueness of the solution and/or the accuracy of the numerical solutions. The div-curl method and the least-squares method provide rigorous derivation of the equivalent second-order Maxwell's equations and their boundary conditions. The node-based least-squares finite element method (LSFEM) is recommended for solving the first-order full Maxwell equations directly. Examples of the numerical solutions by LSFEM for time-harmonic problems are given to demonstrate that the LSFEM is free of spurious solutions.

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