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# STIFFNESS AND STRENGTH TAILORING IN UNIFORM SPACE-FILLING TRUSS STRUCTURES

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## SUMMARY

This paper presents a deterministic procedure for tailoring the continuum stiffness and strength of uniform space-filling truss structures through the appropriate selection of truss geometry and member sizes (i.e., flexural and axial stiffnesses and length). The trusses considered herein are generated by replication of a characteristic truss cell uniformly through space. The repeating cells are categorized by one of a set of possible geometric symmetry groups derived using the techniques of crystallography. The elastic symmetry associated with each geometric symmetry group is identified to aid in the selection of an appropriate truss geometry for a given application. Stiffness and strength tailoring of a given truss geometry is enabled through explicit expressions relating the continuum stiffnesses and failure stresses of the truss to the stiffnesses and failure loads of its members. These expressions are derived using an existing equivalent continuum analysis technique and a newly developed analytical failure theory for trusses. Several examples are presented to illustrate the application of these techniques, and to demonstrate the usefulness of the information gained from this analysis.

## INTRODUCTION

In the future, the primary structures of many large orbiting spacecraft will be lightweight trusses. Although numerous studies have been performed to determine the feasibility and structural characteristics of these trusses (e.g. refs. 1-3), little work has been done to establish deterministic procedures for their design. The selection of appropriate truss designs is influenced by both structural optimization and spacecraft operational considerations. Currently, structural optimization of these trusses is a predominantly heuristic process involving trial and error procedures. The purpose of this paper is to present a deterministic procedure for truss geometry selection and member design based on tailoring the continuum stiffness and strength characteristics of the truss. Analysis of the truss stiffness and strength characteristics is performed using an equivalent continuum analogy (ref. 4). This approach is preferred because it offers better insight into structural behavior than conventional numerical analysis techniques.

The trusses considered herein are generated by replication (rotational and/or translational) of a characteristic cell uniformly through space, as shown in figure 1, and are thus called uniform space-filling trusses. In most cases, the repeating truss cell and the resulting truss structure inherently possess some geometric symmetry. The presence of geometric symmetry implies elastic symmetry which reduces the number of independent equivalent elastic constants characterizing the truss. In this study, the techniques of crystallography are used to define the possible geometric symmetry groups associated with repeating cells which generate uniform trusses. In addition, the number of independent elastic constants associated with each geometric symmetry group is identified to aid in the selection of an appropriate truss geometry for a given application.

The independent elastic constants characterizing a truss can be tailored to specific values by selecting appropriate member stiffnesses. In the present study, this stiffness tailoring is accomplished using explicit relationships between the equivalent continuum stiffnesses of a truss and the axial stiffnesses of its members. Also, the continuum strength characteristics of a truss are tailored using a strength tensor which is written explicitly in terms of the local elastic buckling loads of the truss members. To illustrate the application of these techniques, a commonly used truss geometry is analyzed to determine member sizes which produce optimum isotropic and orthotropic (i.e., one direction of high stiffness and strength) designs.

All derivations presented herein were performed symbolically using a computerized mathematics routine (ref. 5), and results were converted into a numerical form when necessary. The advantage in using symbolic algebra is that explicit relationships can be determined between the design parameters and the continuum elastic behavior of the truss. This significantly enhances the utility of the stiffness and strength tailoring procedures presented herein.

## SYMBOLS

| | |
|---|---|
| $A$ | cross-sectional area of members in the regular Octahedral truss |
| $A_o$ | cross-sectional area of members in the octahedral lattice of the Warren truss |
| $A_c$ | cross-sectional area of members in cubic lattice of the Warren truss |
| $A_n$ | cross-sectional area of members in *n*th group |
| $c_{mn}$ | continuum elastic constants (matrix form) |
| $C_{ijkl}$ | continuum elastic constants (tensor form) |
| $C'_{mnop}$ | transformed continuum elastic constants |
| $(C'_{1111})_n$ | continuum unidirectional stiffness for *n*th group of parallel members |
| $E$ | Young's modulus of truss material |
| $E_{eq}$ | equivalent continuum Young's modulus |
| $(E_{eq})_z$ | equivalent *z*-direction Young's modulus |
| $(E_{eq})_{iso.}$ | equivalent Young's modulus of isotropic Warren truss |
| $G_{eq}$ | equivalent continuum shear modulus |
| $L$ | characteristic dimension of truss repeating cell |
| $l_n$ | length of members in *n*th group |
| $r_n$ | radius of gyration of members in *n*th group |
| $s_{mn}$ | continuum compliance constants (matrix form) |
| $S_{ijkl}$ | continuum compliance constants (tensor form) |
| $T_{ij}$ | coordinate transformation tensor |
| $v_n$ | volume fraction of *n*th group of parallel members |
| $x, y, z$ | Cartesian coordinates |
| $\beta$ | length ratio of repeating truss cell in *z* direction |
| $\delta_n$ | ratio of cross-sectional area of members in *n*th group to that of first group |
| $\varepsilon_{ij}$ | strain tensor |
| $(\varepsilon_{crit.})_n$ | critical axial strain for *n*th group of members |
| $\nu_{eq}$ | equivalent Poisson's ratio |
| $\phi_i$ | direction cosine with the *i*th coordinate axis |
| $\rho$ | density of truss material |

| | |
|---|---|
| $\rho_{eq}$ | equivalent continuum density |
| $\sigma_{ij}$ | stress tensor |
| $\sigma_{ult}$ | continuum compression strength |
| $(\sigma_{ult})_z$ | $z$-direction compression strength |
| $(\sigma_{ult})_{iso.}$ | compression strength of isotropic Warren truss |
| $[\Omega_{kl}]_n$ | strength tensor |
| $\theta,\ \varphi$ | spherical coordinates |

## TRUSS GEOMETRY SELECTION

The design of a truss is often governed by considerations other than the structural performance (e.g. ref. 6). For example, operational concerns such as the arrangement and integration of spacecraft subsystems onto a truss might dictate a particular geometry for the truss repeating cell. For applications in which operational concerns do not dominate, it is prudent to select a truss geometry by matching its inherent elastic behavior with the structural requirements of the spacecraft. Even in situations where operational concerns prevail, it is probable that enough latitude exists in the selection of a truss geometry that structural considerations can be incorporated. The purpose of this section is to categorize the elastic characteristics of most uniform space-filling truss structures by examining their geometric symmetry.

The uniform truss structures considered herein are similar to crystalline lattices since they both can be generated by replication of a characteristic repeating cell which typically possesses geometric symmetry. Of interest are symmetry with respect to specific rotations about one or more axes, and/or symmetry with respect to reflection about one or more planes. Symmetry in the truss geometry (i.e., lattice arrangement and member designs) implies symmetry in the elastic characteristics of the truss. This implied elastic symmetry reduces the number of independent equivalent elastic constants characterizing the continuum behavior of the truss, and thus simplifies the task of stiffness and strength tailoring.

### Rotational Symmetry Groups

Studies in crystallography (refs. 7,8) have shown that the rotational and reflectional symmetries in reticulated, or discrete, structures are limited to a set of 32 possible combinations which are commonly called crystallographic symmetry groups. Love (ref. 9) determined that the elastic behavior of most crystallographic symmetry groups can be derived by considering only rotational symmetry. For brevity, the few cases in which reflectional symmetry is important are not considered herein. By neglecting reflectional symmetry, the 32 crystallographic symmetry groups reduce to the ten rotational symmetry groups shown in figure 2.

Each symmetry group in figure 2 is identified by a specific combination of axes about which there is rotational symmetry. The orientations of these axes are shown relative to a Cartesian coordinate system, and the order of rotational symmetry is given by one of four graphical symbols: a cusped oval, a triangle, a square, or a hexagon. These symmetry symbols are related to the order of symmetry in the accompanying key. The order of symmetry is defined as $n$-gonal where the rotation angle is $2\pi/n$ and $n$ is either $2, 3, 4$, or $6$. Notice that in symmetry groups $i$ and $j$, the trigonal symmetry axes lie along lines connecting the center of a cube with its corners, thus structures of these symmetry groups are often referred to as "cubic" structures.

Symmetry groups which possess more than one axis of rotational symmetry are called multiaxial. The three rotational symmetry axes presented for each of the multiaxial groups are not the only symmetry axes possessed by those groups. A complete set can be generated by applying the symmetry operation of each axis to the others. For example in symmetry group *d*, applying trigonal symmetry about the *z* axis identifies four additional digonal symmetry axes in the *x-y* plane separated by 60°.

Any truss structure which possesses axes of rotational symmetry can be categorized by one of the ter rotational symmetry groups in figure 2. This classification is accomplished by identifying all rotational symmetry axes within the structure, and then selecting a Cartesian coordinate system relative to these axes which matches one of the given symmetry groups. Once the symmetry group of the truss is identified, its inherent elastic behavior is determined using the methods which follow.

## Elastic Characteristics of Rotational Symmetry Groups

A uniform truss structure can be represented by an equivalent homogeneous anisotropic continuum characterized by 21 empirical elastic constants. These elastic constants appear as stiffnesses, $c_{mn}$ or $C_{ijkl}$, in the constitutive equations given in equation (1a) in matrix form and equation (1b) in tensor form.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \qquad (1a)$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \qquad (1b)$$

When the truss possesses geometric symmetry, elastic symmetry is implied which reduces the number of independent continuum elastic constants.

A continuum which possesses geometric symmetry with respect to a linear orthogonal transformation $T_{ij}$, such as a rotation or a reflection, also possesses symmetry in its elastic constants (see for example ref. 10). Therefore, the transformed stiffness tensor $C'_{ijkl}$ must be identical to the original tensor $C_{ijkl}$. Hence,

$$C'_{ijkl} = C_{mnop}\,T_{im}T_{jn}T_{ko}T_{lp} = C_{ijkl} \qquad (2)$$

The number of independent elastic constants associated with each symmetry group, presented in figure 2, is determined using equation (2). A transformation tensor $T_{ij}$ is determined for the specified rotation about each symmetry axis, and substituted into equation (2) to give 21 conditions on the stiffnesses $C_{ijkl}$. Some of these conditions are identically satisfied, whereas others can be satisfied only by the elimination or restriction of certain elastic constants. This process is repeated for all rotational symmetry axes in the given symmetry group, and the resulting reduced set of elastic constants defines the continuum elastic characteristics of any truss structure which is a member of that symmetry group.

For example, the independent elastic constants characterizing trusses of symmetry group *a* are determined by enforcing elastic symmetry with respect to a 180° rotation about the *z* axis. The transformation matrix for this rotation is

$$T_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

Substitution of equation (3) into equation (2) gives the following result

$$C_{ijkl} = \begin{cases} C_{ijkl} \ , & \text{if an even number (or none) of the indices are 3} \\ -C_{ijkl} \ , & \text{if an odd number of the indices are 3} \end{cases} \tag{4}$$

Satisfying the second condition in equation (4) requires the following to be true (note that, due to symmetry in $C_{ijkl}$, many possible permutations of the subscripts have been omitted).

$$\begin{gathered} C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = \\ C_{3313} = C_{2312} = C_{1312} = 0 \end{gathered} \tag{5}$$

Employing the usual conversion from tensor to matrix form (ref. 10), the following equivalent conditions exist for the components of the stiffness matrix.

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0 \tag{6}$$

Similar calculations can be made for the remaining symmetry groups in figure 2. Without presentation of the details, the conditions on continuum stiffnesses as well as the number of independent elastic constants for each symmetry group are presented in table I. A similar derivation shows that the conditions presented in table I must also be obeyed by the components of the continuum compliance tensor.

An obvious conclusion to be drawn from table I is that the presence of any symmetry in a truss lattice significantly reduces the number of independent elastic constants characterizing its continuum behavior. This result greatly simplifies the task of tailoring the stiffness and strength of most trusses. It should also be cautioned that the conditions on the elastic constants presented in table I are valid only for the coordinate axes presented in figure 2. For example, symmetry groups *b*, *f*, *g*, *h*, *i*, and *j* are indicated to have zero shear coupling stiffnesses (e.g., $c_{14}$, $c_{15}$, and $c_{16}$) in the given coordinate system, but they might have non-zero coupling stiffnesses in an alternate coordinate system. As explained by Rosen (ref. 11), and seen in table I, none of the permissible geometric symmetry groups possess sufficient symmetry to insure isotropic elastic behavior. However, it will be shown that isotropy can be obtained by tailoring the relative stiffnesses of different truss members.

The information in table I should help in the selection of appropriate truss geometries for particular truss applications, and in the determination of additional stiffness tailoring requirements for the selected truss geometry. For example, if the primary loads in a truss are expected to occur in only one direction, it is most efficient to consider geometries which have less symmetry and can easily be tailored to have significantly higher stiffnesses and strengths in that direction (i.e., an orthotropic design). However, for a structure which may have to sustain loads in multiple directions, or one for which the loading conditions are not well defined, it may be best to consider truss geometries which possess more symmetry and can be tailored to behave isotropically.

## STIFFNESS AND STRENGTH TAILORING

Once a truss geometry has been selected, its independent elastic constants are identified using table I. The values of these constants can be adjusted for a particular application by tailoring the relative axial stiffnesses of the members comprising the truss. Likewise, changing the relative elastic buckling loads of different members alters the equivalent continuum strengths of the truss. Changing truss dimensions and member stiffnesses which do not violate the geometric symmetry of the truss allows the truss to remain in the same rotational symmetry group, thus the conditions on its continuum stiffnesses given in table I remain valid. Alternatively, changing dimensions and member stiffnesses of a truss which violate its geometric symmetry changes its rotational symmetry group, thus altering the number of independent elastic constants characterizing its behavior. Stiffness and strength tailoring will be demonstrated later for a truss in which geometric symmetry is maintained and one in which geometric symmetry is altered.

## Equivalent Continuum Elastic Constants

Once a candidate truss for stiffness tailoring is selected, its continuum stiffnesses are calculated in terms of the axial stiffnesses of its members. The approach used in the recent study for calculating these stiffnesses was developed by Neyfeh and Hefzy (ref. 12) and can be thought of as a three-dimensional generalization of classical laminated plate theory (ref. 13) in which groups of parallel members within the truss are analogous to individual lamina. Since truss members carry only an axial load, each group of parallel members can be considered to form a unidirectional elastic continuum which has no transverse or shearing stiffnesses. The stiffnesses for the truss assemblage are obtained by summing the stiffnesses of each of the groups of parallel members. This superposition of stiffnesses implies that the continuum displacement field within a truss is single-valued which is consistent with the fact that truss members connected at a common point must have the same displacement at that point. Note that this is not the case for trusses with cross-laced members which can slide relative to one another, therefore it is cautioned that such designs should not be analyzed using the techniques of the present study.

Each group of parallel members is characterized by one non-zero equivalent stiffness which is in the local $x'$ direction (the member longitudinal direction). This equivalent unidirectional stiffness is determined in equation (7) for the $n$th group of members.

$$\left(C'_{1111}\right)_n = E v_n \tag{7}$$

where $E$ is the Young's modulus of the material in the members and $v_n$ is the volume fraction of the group of members (i.e., the ratio of the total volume of material in the members to the total volume of the truss).

The continuum stiffnesses for a truss are calculated by transforming the unidirectional stiffnesses for each of its groups of parallel members into a global coordinate system using equation (2), and summing the results as indicated by

$$C_{ijkl} = \sum_n \left(C'_{1111}\right)_n \left(T_{1i} T_{1j} T_{1k} T_{1l}\right)_n \tag{8}$$

Elements of the first row of the transformation tensor, $T_{1i}$, are simply the direction cosines between the longitudinal axis of the members and the $i$th coordinate axis. Therefore, equation (8) can be rewritten as

$$C_{ijkl} = \sum_n \left(C'_{1111}\right)_n \left(\phi_i\, \phi_j\, \phi_k\, \phi_l\right)_n \tag{9}$$

where $\phi_i$ are the direction cosines of the members. The continuum stiffnesses defined by equation (9) are explicit functions of the member extensional stiffnesses. This enables the translation of desired continuum stiffness characteristics into member axial stiffness tailoring rules.

Equation (9) produces additional restrictions on the continuum stiffnesses of uniform trusses which should be noted. Employing the usual conversion from the matrix form of the elastic constants to the tensor form (ref. 10), the values for the transverse and shear stiffnesses, $c_{12}$ and $c_{66}$ are found to be:

$$c_{12} = C_{1122} = \sum_n \left(C'_{1111}\right)_n \left(\phi_1^2 \phi_2^2\right)_n \tag{10}$$

$$c_{66} = C_{1212} = \sum_n \left(C'_{1111}\right)_n \left(\phi_1^2 \phi_2^2\right)_n \tag{11}$$

thus:

$$c_{12} = c_{66} \tag{12}$$

Similarly, it is found that

$$c_{13} = c_{55} \quad , \quad c_{23} = c_{44} \tag{13}$$

It is reiterated that these identities must hold for any uniform space-filling truss, regardless of its geometry, and therefore these identities should be added to those already presented in table I for all symmetry groups. Thus, under the assumptions above, a generally anisotropic space-filling truss structure has only 18 independent elastic constants rather than 21 as is normal for a generally anisotropic solid.

Trusses that are tailored to behave as isotropic continua can be characterized by two elastic constants, an equivalent Young's modulus, $E_{eq}$, and an equivalent Poisson's ratio, $\nu_{eq}$. Writing the stiffnesses in equation (12) in terms of these equivalent constants gives the following condition.

$$\frac{\nu_{eq} E_{eq}}{(1+\nu_{eq})(1-2\nu_{eq})} = \frac{E_{eq}}{2(1+\nu_{eq})} \tag{14}$$

Solving equation (14) for $\nu_{eq}$ gives the result that $\nu_{eq}$ is equal to 1/4. Therefore, any uniform three-dimensional space-filling truss structure which is globally isotropic must have an equivalent Poisson's ratio equal to 1/4, and thus, has only one remaining independent elastic constant, its equivalent Young's modulus. Using a similar procedure, it can be shown that two-dimensional space-filling trusses which behave isotropically must have an equivalent Poisson's ratio of 1/3.

## Equivalent Stiffness-to-Density Ratio

Stiffness-to-density ratios are commonly used as indicators of the efficiency of materials. Likewise, equivalent stiffness-to-density ratios are useful indicators of the efficiency of uniform trusses. Most equivalent truss stiffness-to-density ratios are dependent on the design of the truss. However, it will be shown that there exists an equivalent stiffness-to-density ratio which is only a function of the modulus-to-density ratio of the parent material.

In equation (15), a sum of equivalent continuum stiffnesses for a truss is shown to be equal to the sum of the uniaxial stiffnesses of its individual groups of members. Notice that the direction cosine terms drop out because the sum of the squares of the three direction cosines for any member is equal to one.

$$c_{11} + c_{22} + c_{33} + 2c_{23} + 2c_{13} + 2c_{12} = C_{1111} + C_{2222} + C_{3333} + 2C_{2233} + 2C_{1133} + 2C_{1122}$$

$$= \sum_n (C'_{1111})_n \left(\phi_1^4 + \phi_2^4 + \phi_3^4 + 2\phi_2^2\phi_3^2 + 2\phi_1^2\phi_3^2 + 2\phi_1^2\phi_2^2\right)_n$$

$$= \sum_n (C'_{1111})_n \left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)_n^2$$

$$= \sum_n (C'_{1111})_n \tag{15}$$

The equivalent density of a space-filling truss is determined by multiplying the density of the parent material, $\rho$, by the sum of the volume fractions of all groups of parallel members. Considering equation (7), this relationship can be written as

$$\rho_{eq} = \rho \sum_n v_n = \frac{\rho}{E} \sum_n (C'_{1111})_n \tag{16}$$

Dividing equation (15) by equation (16) gives the following equivalent stiffness-to-density ratio.

$$\frac{c_{11} + c_{22} + c_{33} + 2c_{23} + 2c_{13} + 2c_{12}}{\rho_{eq}} = \frac{E}{\rho} \tag{17}$$

Equation (17) is a unique relationship because it provides a direct correlation between an equivalent continuum stiffness-to-density ratio of the truss, and the modulus-to-density ratio of the parent material in the truss members. Once the parent material is defined for a truss, equation (17) provides a direct relationship between the equivalent anisotropic stiffness of a truss and its equivalent density. This relationship can be used in a number of ways. For example, changes in continuum stiffnesses due to stiffness tailoring of the truss members can be directly translated into a proportional change in equivalent density of the truss. Similarly, requiring the sum of the continuum stiffnesses in the numerator of equation (17) to be constant during stiffness tailoring results in the equivalent density remaining constant. This gives a convenient method for studying the effects of material redistribution within a truss lattice.

Equation (17) can be simplified for trusses that are tailored to be globally isotropic. Without presentation of the details, it can be shown that by writing the equivalent continuum stiffnesses in terms of an equivalent Young's modulus and Poisson's ratio (equal to *1/4*), equation (17) reduces to

$$\frac{E_{eq}}{\rho_{eq}} = \frac{1}{6}\frac{E}{\rho} \tag{18}$$

The significance of equation (18) is that all uniform space-filling trusses which are globally isotropic must have the same equivalent modulus-to-density ratio regardless of their geometries or member sizes. Furthermore, this modulus-to-density ratio must be exactly one sixth of the modulus-to-density ratio of the parent material.

### Equivalent Continuum Strength Tensor

The continuum strength of a truss structure is defined herein as the maximum continuum stress which it can sustain before any of its members buckle elastically. This failure mode is a local phenomenon within the truss lattice which would have one of two effects on the continuum behavior of the truss. f redundant members exist and load is redistributed, local buckling will cause a change in the continuum stiffnesses of the truss. However, if no load redistribution takes place, local buckling will precipitate a catastrophic failure of the truss lattice. These continuum effects are analogous, respectively, to yielding and ultimate failure in a material.

Since the local failure mode in trusses can be determined analytically it is possible to construct a purely analytical failure theory for trusses. In this section, a tensor which describes the strength of a truss will be constructed, and failure analysis using this strength tensor will be discussed. Having a tensor which represents the strength of a truss is advantageous, because it allows strength to be readily determined in alternate reference frames, or under multiaxial stress states. For materials it is known that strength is not a tensor quantity, and thus, analysis of failure in materials under multiaxial stress can be accomplished only with approximate, semi-empirical theories such as that proposed by von Mises (ref. 14).

The construction of a strength tensor for trusses is based on the assumption that applied stresses can be converted into strains using the compliance equations given in equation (19), and that these strains can be analyzed to determine if the axial compression strain in any truss member has exceeded its critical elastic buckling limit.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{19a}$$

$$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} \tag{19b}$$

Note that the compliance matrix in equation (19a) is simply the inverse of the stiffness matrix given in equation (1a). Therefore, the equivalent continuum compliances for a truss can be determined from the equivalent continuum stiffnesses derived previously.

The continuum strains, defined in tensor form in equation (19b), can be transformed into a new coordinate system described by the linear transformation tensor $T_{ij}$. The resulting transformed strains, $\varepsilon'_{ij}$, are

$$\varepsilon'_{ij} = T_{io}T_{jp}\varepsilon_{op} = T_{io}T_{jp}S_{opkl}\,\sigma_{kl} \tag{20}$$

The axial strain in any member of the truss is determined by defining an alternate coordinate system with one of its axes aligned along the longitudinal direction of the member, and evaluating the normal strain along that axis. Assuming that the $1$ axis of the alternate coordinate system is aligned in such a way, the axial strain in the member is given as

$$\varepsilon'_{11} = T_{1i}T_{1j}S_{ijkl}\,\sigma_{kl} = \phi_i\phi_j S_{ijkl}\,\sigma_{kl} \tag{21}$$

where, as defined before, $\phi_i$ is the $i$th direction cosine of the member.

Failure occurs in a member if its axial strain exceeds a critical value determined for elastic buckling. For the present study, it is assumed that the truss members are slender and therefore buckle as Euler columns (ref. 15), thus the critical strain for the $n$th group of members is defined as

$$(\varepsilon_{crit.})_n = -\pi^2\left(\frac{r_n}{l_n}\right)^2 \tag{22}$$

where $r_n$ is the radius of gyration and $l_n$ is the length of the members in the $n$th group. The minus sign in equation (22) indicates that the critical strain is compressive. A fail-safe criterion can be constructed from equations (21) and (22) by requiring the axial strains in all members to be less than the critical value. This fail-safe criterion can be written as

$$\left[\frac{(\phi_i\phi_j)_n S_{ijkl}}{-\pi^2\left(\frac{r_n}{l_n}\right)^2}\right]\sigma_{kl} = \left[\Omega_{kl}\right]_n \sigma_{kl} \leq 1 \tag{23}$$

The bracketed term in equation (23) can either be thought of as a third-order tensor representing the strength of the truss, or as a collection of second-order tensors, each representing the strength of a group of parallel members within the truss. The product of this strength tensor and the second-order applied stress tensor, $\sigma_{kl}$, is a vector of constants, one for each of the groups of parallel members. For elastic failure to occur, any one of these constants must be greater than or equal to one. Thus the critical stress at which failure occurs is that for which one or more of these constants is equal to one.

Equation (23) represents a purely analytical failure theory for space-filling trusses which can be used with equal ease to analyze strength under multiaxial or uniaxial loading. Similarly, strength in alternate coordinate systems can be readily handled by simply transforming the collection of second-order strength tensors, $\Omega_{kl}$, in the same way that a stress or strain tensor would be transformed.

Equation (23) can be used, as described, to determine the strength of a given truss design. Additionally, it is useful for tailoring the strength of a truss design because it is an explicit relationship between the strength of individual members (i.e., $r_n/l_n$) and the continuum strength of the truss. Strength tailoring is accomplished by varying the strength of individual members to effect a desired change in the continuum strength of the truss. It is important to note that since the continuum compliances of the truss appear in equation (23), strength tailoring is not independent of stiffness tailoring. Consequently, tailoring the continuum stiffnesses of a truss will also change its continuum strength characteristics.

In the remaining sections of this paper, examples of stiffness and strength tailoring of uniform trusses are presented. Truss geometries are selected for analytical simplicity, thus allowing emphasis to be placed on developing an understanding of the analysis techniques.

## EXAMPLES OF STIFFNESS AND STRENGTH TAILORING IN TRUSSES

Equations (9), (17), and (23) provide the basis for analysis of the continuum stiffness, density, and strength of uniform space-filling truss structures. By providing explicit relationships between these continuum quantities and truss design parameters, these equations are effective tools which enable efficient tailoring of the truss stiffness and strength characteristics. In this section, these equations are applied to the analysis of two commonly used truss geometries, and to the tailoring of designs which have continuum isotropic and orthotropic behaviors.

### Regular Octahedral Truss

The Octahedral truss (also known as the Tetrahedral truss, ref. 2, or the Octet truss) is a common geometry that derives its name from the fact that its members connect to form octahedrons and tetrahedrons. For the present study, a regular Octahedral truss is considered which has all identical members. A repeating cell from this truss is shown in figure 3. The cell contains a regular octahedron at its center (figure 3(a)) and tetrahedrons connected to each of the eight faces of the octahedron (figure 3(b)). Space is filled by translational replication of this cell in each of the three coordinate directions.

Since all members are identical, the octahedral truss has digonal symmetry axes along the lines $x=y$, $x=z$, and $y=z$; trigonal symmetry axes along the lines $x=y=z$, $-x=y=z$, $x=-y=z$, and $x=y=-z$; and quadragonal symmetry axes along the $x$, $y$, and $z$ axes. This combination of symmetry axes indicates that the regular Octahedral truss is a member of rotational symmetry group $j$.

***Calculation of Continuum Stiffness and Density*** From table I, it can be seen that the behavior of the regular Octahedral truss is characterized by the three independent elastic constants $c_{11}$, $c_{12}$, and $c_{66}$. Equation (12) further reduces this number to two. However, these constants lack the relationship $c_{66}=(c_{11}-c_{12})/2$; thus the regular Octahedral truss is not globally isotropic. Values for the elastic constants can be determined from equations (7) and (9). There are six different groups of parallel members in the Octahedral truss, and all members are identical and assumed to have a cross-sectional area of $A$. With the half-height of the regular octahedron defined to be $L$, as shown in figure 3, the length of each of the members is $\sqrt{2}L$. Then, the equivalent unidirectional stiffness for each of the six groups of parallel members is

$$\left(C'_{1111}\right)_n = \frac{EA}{\sqrt{2}\,L^2} \tag{24}$$

where $E$ is the Young's modulus of the material in the members. Substituting equation (24) into equation (9) along with the appropriate direction cosines for the different member groups, gives the result presented in equation (25) for the equivalent continuum stiffness matrix of the Octahedral truss.

$$[c_{mn}] = \frac{EA}{2\sqrt{2}\,L^2}\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Notice that the continuum stiffnesses obey the restrictions in table I and equation (12).

Because all members in the regular Octahedral truss are identical, the relative magnitudes of the continuum stiffnesses for the octahedral truss are constrained by the proportions given in the matrix of equation (25). Therefore, changing the axial stiffness of the truss members can only uniformly change all continuum stiffnesses.

The equivalent density of the Octahedral truss can be calculated by substituting the stiffnesses from equation (25) into equation (17). Rearranging and simplifying gives

$$\rho_{eq} = \frac{3\sqrt{2}\rho A}{L^2} \tag{26}$$

***Calculation of Continuum Strength*** Before applying equation (23) to calculate the continuum strength of the Octahedral truss, it is necessary to determine the tensor form of the continuum compliances from the stiffness matrix given in equation (25). This is done by inverting the stiffness matrix to get the compliance matrix, and then employing the usual conversion from matrix form to tensor form on the individual compliances (ref. 10). The only remaining unknown truss parameter is the radius of gyration of its members.

Suppose that the strength of the Octahedral truss under a continuum uniaxial compression is required. Assuming this stress to have magnitude $\sigma_{ult}$ and to be applied along a vector given by the spherical coordinates $\theta$ and $\varphi$ (as shown in figure 4), the applied continuum stress tensor can be written as

$$[\sigma_{kl}] = -\sigma_{ult} \begin{bmatrix} (\sin^2\theta \cos^2\varphi) & (\sin^2\theta \sin\varphi \cos\varphi) & (\sin\theta \cos\theta \cos\varphi) \\ (\sin^2\theta \sin\varphi \cos\varphi) & (\sin^2\theta \sin^2\varphi) & (\sin\theta \cos\theta \sin\varphi) \\ (\sin\theta \cos\theta \cos\varphi) & (\sin\theta \cos\theta \sin\varphi) & (\cos^2\theta) \end{bmatrix} \tag{27}$$

The compression strength, $\sigma_{ult}$, can be determined for any set of $\theta$ and $\varphi$ by substituting equation (27) into equation (23), along with the values for the continuum compliances, the member radius of gyration ($r$), the member length ($l = \sqrt{2}L$), and the appropriate direction cosines. After simplification, equation (23) reduces to a set of six scalar equations ($n = 1$ to 6), one for each group of parallel members in the truss. Each of these equations can be solved for the value of $\sigma_{ult}$ that is necessary to cause Euler buckling in the corresponding member. The minimum value determined from these six equations is the lowest uniaxial compression stress at which local buckling occurs within the truss lattice. The lowest value of compression stress found is then defined as the uniaxial compression strength for the given set of $\theta$ and $\varphi$.

A three dimensional plot of the uniaxial compression strength of the Octahedral truss is presented in figure 4 for a range of $\theta$ and $\varphi$ from 0° to 90°. Due to symmetry, the strength in all other quadrants is identical. There is a factor of two variation in the compression strength of the lattice and, not surprisingly, the directions of minimum strength are coincident with the directions of the members of the truss. Maximum strength occurs for loading along the three coordinate axes and along the line $x=y=z$. The value of the minimum strength is

$$\sigma_{ult} = \frac{EA\pi^2 r^2}{2\sqrt{2}\, L^4} \tag{28}$$

Since all members are identical, changing the strength of the members would change the vertical scale of the strength plot given in figure 4, but would not change its shape. Introducing member-specific properties will alter the equivalent continuum stiffness and strength; however, this would destroy the geometric symmetry of the lattice and introduce additional independent stiffnesses. In the following section, a truss based on the octahedral lattice is designed for isotropic stiffness, and nearly isotropic strength.

## Isotropic Warren Truss

The lattice of the regular Octahedral truss is modified by adding members that connect all six vertices of each octahedron to the geometric center of the octahedron as shown in figure 5(a). The resulting arrangement of new members forms a cubic lattice within the octahedral lattice, with the edges of the cube lying parallel to the three coordinate axes and each cube containing a regular tetrahedron as shown in figure 5(b). The members of the cubic lattice are of length $L$ whereas the members of the original octahedral lattice are of length $\sqrt{2}\, L$. This truss geometry is often referred to as the Warren truss because its lattice arrangement is similar to that of a common two-dimensional truss of the same name. Similar to the regular Octahedral truss, the Warren truss is a member of symmetry group $j$, and has two independent elastic constants, $c_{11}$ and $c_{12}$. However, unlike the Octahedral truss, the Warren truss has two different members whose relative stiffnesses and strengths can be tailored to affect the continuum behavior of the truss without violating its geometric and elastic symmetry. In this section, it is demonstrated that by redistributing material within the truss lattice the continuum strength and stiffness properties of the lattice can be tailored. In this case, material is transferred from the octahedral lattice members to the cubic lattice members so that the continuum stiffnesses become isotropic. Also the relative strengths of the members are tailored to reduce variations in continuum compression strength.

***Continuum Stiffness Tailoring*** The Warren truss is composed of nine different groups of parallel members. Three groups correspond to the cubic lattice, and six groups correspond to the octahedral lattice. The continuum stiffnesses for the Warren truss can be determined by adding the contributions due to the cubic lattice members to the result presented in equation (25) for the octahedral lattice. The cross-sectional areas of the members in the cubic lattice and the octahedral lattice are defined to be $A_c$ and $A_o$, respectively. Thus, the equivalent uniaxial stiffnesses of the three groups of parallel cubic lattice members are given by

$$\left(C'_{1111}\right)_n = \frac{EA_c}{L^2} \tag{29}$$

Substituting equation (29) into equation (9) along with the appropriate direction cosines, and adding the result to that presented in equation (25), gives

$$\left[c_{mn}\right] = \frac{EA_o}{2\sqrt{2}\,L^2}\begin{bmatrix} 2+2\sqrt{2}\delta_c & 1 & 1 & 0 & 0 & 0 \\ 1 & 2+2\sqrt{2}\delta_c & 1 & 0 & 0 & 0 \\ 1 & 1 & 2+2\sqrt{2}\delta_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

where $\delta_c$ is defined as $A_c/A_o$. It can be seen that if $\delta_c$ is equal to zero, the cross-sectional area of the cubic lattice members is zero and equation (30) is identical to equation (25). As before, an equivalent density can be calculated using equation (17) and the stiffnesses presented in equation (30). The result is

$$\rho_{eq} = \frac{\left(3\sqrt{2} + 3\delta_c\right)\rho A_o}{L^2} \tag{31}$$

To study the effects of redistribution of material within the truss, it is necessary to insure that the total amount of material remains constant. For convenience, the density of the Warren truss is required to be the same as that of the regular Octahedral truss by setting equation (26) equal to equation (31). The result is

$$A_o = \frac{A}{\left(1 + \delta_c/\sqrt{2}\right)} \tag{32}$$

where $A$ is the cross-sectional area of the members in the regular Octahedral truss analyzed previously. Equation (32) defines the relation between the cross-sectional areas of the cubic and octahedral lattice members within the Warren truss which must hold to keep the equivalent density of the Warren truss equal to that of the regular Octahedral truss. Substituting equation (32) into equation (30) gives explicit equations for the continuum stiffnesses of the Warren truss in terms of the member area ratio, $\delta_c$. To better understand the effects of redistribution of material, the stiffness components in equation (30) are translated into equivalent Young's modulus, Poisson's ratio, and shear modulus as follows.

$$E_{eq} = \frac{(c_{11} + 2c_{12})(c_{11} - c_{12})}{c_{11} + c_{12}} = \frac{4EA(1 + 2\sqrt{2}\delta_c)}{2\sqrt{2}\,L^2\,(3 + 2\sqrt{2}\delta_c)} \tag{33}$$

$$\nu_{eq} = \frac{c_{12}}{c_{11} + c_{12}} = \frac{1}{3 + 2\sqrt{2}\,\delta_c} \tag{34}$$

$$G_{eq} = c_{66} = \frac{EA}{2\sqrt{2}\,L^2\,(1 + \delta_c/\sqrt{2})} \tag{35}$$

These stiffness components are plotted in figure 6 as functions of the area ratio, $\delta_c$. For $\delta_c$ equal to zero, no material has been redistributed from the octahedral lattice to the cubic lattice, and the stiffnesses represent those of the Octahedral truss. As $\delta_c$ is increased, material is moved from the octahedral lattice to the cubic lattice, and this is accompanied by an increase in the equivalent Young's modulus and decreases in the equivalent Poisson's ratio and the equivalent shear modulus. From equations (34) and (35) it can be seen that as $\delta_c$ becomes large, both the Poisson's ratio and the shear modulus approach zero. This effect is consistent with the fact that the cubic lattice of members is not a kinematically stable truss by itself. Because of this, it is unreasonable to consider designs having very large values of $\delta_c$.

For the Warren truss to be globally isotropic, it is necessary for its stiffnesses to satisfy the following condition.

$$G_{eq} = \frac{E_{eq}}{2\,(1 + \nu_{eq})} \tag{36}$$

Substituting the expressions from equations (33) - (35) into equation (36) shows that $\delta_c$ must be equal to $1/(2\sqrt{2})$ for isotropy. Substituting this value of $\delta_c$ into equation (32) gives a value of $4A/5$ for the cross-sectional area of the members in the octahedral lattice, and consequently a value of $\sqrt{2}A/5$ for the cross-sectional area of the members in the cubic lattice. Thus, if one fifth of the material that was originally in the members of the octahedral truss is redistributed into the members of the cubic lattice, the resulting truss behaves isotropically. The isotropic values for the equivalent Young's modulus, Poisson's ratio, and shear modulus are

$$(E_{eq})_{iso.} = \frac{EA}{\sqrt{2}\,L^2} \quad , \quad (\nu_{eq})_{iso.} = \frac{1}{4} \quad , \quad (G_{eq})_{iso.} = \frac{\sqrt{2}EA}{5\,L^2} \tag{37}$$

Notice that the equivalent isotropic Poisson's ratio is 1/4 which is the value that was predicted earlier for globally isotropic trusses. Also, calculating the ratio of the equivalent isotropic Young's modulus (equation (37)) to the equivalent density (equation (26)) gives the result predicted in equation (18) for globally isotropic trusses.

***Continuum Strength Tailoring*** Applying the same procedure used for the Octahedral truss, it is possible to determine the continuum strength of the isotropic Warren truss and to evaluate the effects on continuum strength of varying the strength of the truss members. For the purpose of comparison, the same continuum stress tensor given in equation (27) is also applied to the Warren truss. Two cases are analyzed. In the first, it is assumed that all members in the truss have the same radius of gyration, and in the second it is assumed that all members have the same buckling load. The first case is representative of a truss with thin-walled members of equal cross-sectional diameter. The second case illustrates the effects of tailoring individual member buckling strengths on the continuum strength of the truss.

For the first case, the radius of gyration of all members is equal to $r$, and the lengths of the members are $L$ for the cubic lattice, and $\sqrt{2}L$ for the octahedral lattice. These values are substituted into equation (23) along with the continuum compliances determined from equation (30) and the appropriate direction cosines. The result is a set of nine scalar equations, one for each group of parallel members in the truss, from which the minimum value of $\sigma_{ult}$ is determined for the given set of $\theta$ and $\varphi$.

A three dimensional plot of the uniaxial compression strength of the isotropic Warren truss is presented in figure 7 for the same range of $\theta$ and $\varphi$ as before. The shape of the strength plot is very similar to that of the Octahedral truss, and despite the redistribution of material from the octahedral lattice, the values and the directions of the minimum and maximum strength are the same as those for the Octahedral truss. The directions and maximum strength are coincident with the directions of the cubic lattice members, and the directions of minimum strength are coincident with the directions of the octahedral lattice members. Selecting all members to have the same radius of gyration causes the cubic lattice members to have twice the buckling load of the octahedral lattice members, because of the difference in their lengths. This effect causes the factor of two variations in the continuum strength.

Variation in iruss strength might not be a concern for many design applications, however if it is desirable to have a truss which behaves isotropically in stiffness, it is likely that it is also desirable for it to be isotropic in strength. By tailoring the buckling loads of the cubic lattice members to be the same as those of the octahedral lattice, the variations in continuum strength can be significantly reduced. For this case, the radius of gyration of the cubic lattice members is reduced to $r/\sqrt{2}$ so that the buckling loads of all members are the same. A plot of the resulting continuum compression strength is presented in figure 8. Although some variation still exists in the continuum strength, the magnitude of the variation has been significantly reduced.

The use of three dimensional strength plots is particularly helpful for developing strength tailoring rules, because they provide visualization of the correlation between member orientations and continuum strength variations. Without this correlation it would be difficult to develop strength tailoring rationale for the members. The example presented is fairly simple due to the isotropic stiffness behavior and geometric symmetry of the Warren truss. Therefore, the correlation between variations in continuum strength and the orientation of members is fairly obvious. However, for trusses with less geometric symmetry or more complex applied stress tensors, this correlation might not be apparent without the use of a three dimensional strength plot.

## Orthotropic Warren Truss

Many applications exist for large truss structures with orthotropic, rather than isotropic, continuum properties. For orthotropic applications the requirements on continuum stiffness and strength are much higher in one direction than others. For example, many applications involve beam-like trusses which primarily carry bending and/or torsional loads. In these cases, the longitudinal (along the beam's length) stiffness and strength requirements are much higher than the transverse stiffness and strength requirements. Therefore, it is probably more efficient to use a truss with orthotropic continuum properties than one with isotropic properties.

From table I it can be seen that trusses of symmetry groups $i$ and $j$ are not candidates for orthotropic design because their stiffnesses (and strengths) must be the same in all three coordinate directions. Trusses of all other symmetry groups are candidates for orthotropic tailoring because their properties in the $z$ direction can differ from those in either the $x$ or the $y$ directions. The truss presented in figure 9 is a variation of the Warren truss design which is a member of symmetry group $f$, and is thus a likely candidate for orthotropic design. The lattice arrangement of this truss is identical to that of the Warren truss except the length of the repeating cell in the $z$ direction differs from that in either the $x$ or the $y$ directions by the proportion $\beta$. The purpose of this section is to apply stiffness and strength tailoring techniques to generate orthotropic designs which have high stiffnesses and strengths in the $z$ direction but have the same equivalent density as that of the isotropic Warren truss.

***Calculation of Continuum Stiffnesses*** The orthotropic Warren truss shown in figure 9 has four different members. The cross-sectional areas for members of groups 1 and 2 are defined as $\delta_1 A$ and $\delta_2 A$, respectively, where $\delta_1$ and $\delta_2$ are variable area ratios, and $A$ is the cross-sectional area assumed earlier for the members in the Octahedral truss. The equivalent uniaxial stiffnesses for groups of these members are determined using equation (7) and the results are given in equations (38) and (39).

$$\left(C'_{1111}\right)_1 = \frac{\delta_1 EA}{L^2} \tag{38}$$

$$\left(C'_{1111}\right)_2 = \frac{\delta_2 EA\left(1+\beta^2\right)^{1/2}}{2\beta L^2} \tag{39}$$

For simplicity, it is assumed that members of groups 3 and 4 are the same as those in the isotropic Warren truss. Therefore, the cross-sectional area of members of group 3 is $\sqrt{2}A/5$, and the cross-sectional area of members of group 4 is $4A/5$. The equivalent uniaxial stiffnesses are the same for member groups 1 and 2 and the value of this stiffness is given in equation (40).

$$\left(C'_{1111}\right)_3 = \left(C'_{1111}\right)_4 = \frac{\sqrt{2}EA}{5\beta L^2} \tag{40}$$

Substituting these uniaxial stiffnesses and the appropriate transformation tensors into equation (9) and simplifying gives the following values for the non-zero continuum stiffnesses.

$$c_{11} = c_{22} = \frac{EA}{\beta L^2}\left[\frac{2\sqrt{2}}{5} + \frac{\delta_2}{\left(1+\beta^2\right)^{3/2}}\right] \tag{41}$$

$$c_{12} = c_{66} = \frac{\sqrt{2}EA}{5\beta L^2} \tag{42}$$

$$c_{13} = c_{23} = c_{44} = c_{55} = \frac{EA}{\beta L^2}\left[\frac{\beta^2\delta_2}{\left(1+\beta^2\right)^{3/2}}\right] \tag{43}$$

$$c_{33} = \frac{EA}{\beta L^2}\left[\delta_1\beta + \frac{2\beta^4\delta_2}{\left(1+\beta^2\right)^{3/2}}\right] \tag{44}$$

Notice that these stiffnesses obey the conditions presented in table I and equations (12) and (13) for trusses of symmetry group *f*. Equations (41) through (44) are explicit functions of the three remaining design parameters, $\beta$, $\delta_1$, and $\delta_2$. Therefore, these equations can be used directly to determine how variations in the design parameters affect the orthotropic characteristics of the truss.

An equivalent density can be calculated for the orthotropic Warren truss by substituting the stiffnesses from equations (41) through (44) into equation (17). The result is

$$\rho_{eq} = \frac{\rho A}{\beta L^2}\left[\frac{6\sqrt{2}}{5} + \delta_1\beta + 2\left(1+\beta^2\right)^{1/2}\delta_2\right] \tag{45}$$

Setting equation (45) equal to equation (26) insures that the equivalent density of the orthotropic Warren truss is the same as that of the regular Octahedral truss and the isotropic Warren truss. The resulting expression can be rearranged to give the following condition on the area ratio $\delta_2$.

$$\delta_2 = \frac{\left(3\sqrt{2} - \delta_1\right)\beta - 6\sqrt{2}/5}{2\left(1 + \beta^2\right)^{1/2}} \tag{46}$$

Equation (46) reduces the set of independent design parameters to the repeating cell length ratio, $\beta$, and the cross-sectional area ratio, $\delta_1$.

An equivalent $z$-direction Young's modulus can be determined for the orthotropic Warren truss by inverting the $s_{33}$ component of the compliance matrix as follows.

$$\left(E_{eq}\right)_z = \frac{1}{s_{33}} \tag{47}$$

Performing this calculation gives the result

$$\left(E_{eq}\right)_z = \frac{\sqrt{2}EA\left[15\delta_1/\sqrt{2} + 18\beta^3 - 5\left(\delta_1/\sqrt{2} - 6\beta/5\right)^2\right]}{L^2\left[15 - 5\delta_1/\sqrt{2} + 12\beta + 6\beta^3\right]} \tag{48}$$

To determine the improvement in stiffness in the $z$ direction, the modulus given in equation (48) is divided by the Young's modulus of the isotropic Warren truss given in equation (37). The resulting normalized $z$-direction Young's modulus is

$$\frac{\left(E_{eq}\right)_z}{\left(E_{eq}\right)_{iso.}} = \frac{30\delta_1/\sqrt{2} + 36\beta^3 - 10\left(\delta_1/\sqrt{2} - 6\beta/5\right)^2}{15 - 5\delta_1/\sqrt{2} + 12\beta + 6\beta^3} \tag{49}$$

A three-dimensional plot of the normalized $z$-direction Young's modulus is presented in figure 10 for ranges of the length ratio, $\beta$, and the cross-sectional area ratio, $\delta_1$. The isotropic Warren truss is characterized by $\delta_1 = \sqrt{2}/5$ and $\beta = 1$, this point on the plot corresponds to a normalized $z$ modulus equal to 1. As $\delta_1$ is increased, for a fixed value of $\beta$, material is transferred from members of group 2 to members of group 1 (see figure 9). This causes an increase in the $z$ modulus because the group 1 members are oriented parallel to the $z$ direction. As $\beta$ is increased, for a fixed value of $\delta_1$, the number of group 3 and group 4 members in a given volume decreases. Thus, to maintain constant density, material is redistributed among group 1 and group 2 members also causing an increase in the $z$ modulus.

***Calculation of Continuum z-Direction Strength*** The strength of the orthotropic Warren truss is calculated for a uniform continuum compression applied in the z-direction. This applied stress tensor is given in equation (50), and is substituted into equation (23).

$$\left[\sigma_{kl}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\left(\sigma_{ult}\right)_z \end{bmatrix} \tag{50}$$

Due to their alignment parallel to the $z$ direction, members in group 1 buckle at lower continuum stresses than the remaining members in the truss (this result was verified through additional analysis not presented herein). Thus, considering only buckling in group 1 members, equation (23) can be reduced to equation (51), where $r_1$ and $l_1$ are the radius of gyration and length of members in group 1.

$$(\sigma_{ult})_z = \frac{\pi^2 r_1^2}{l_1^2 s_{33}} \tag{51}$$

Defining the radius of gyration of these members to be $r$ and their length to be $\beta L$ (see figure 9), and substituting the result from equation (47) gives the following expression for the $z$-direction compression strength of the orthotropic Warren truss.

$$(\sigma_{ult})_z = \frac{\pi^2 r^2}{\beta^2 L^2}(E_{eq})_z \tag{52}$$

The $z$-direction compression strength of the isotropic Warren truss can be determined from figure 7 ($\theta$ = 0°), and this value can be used to normalize equation (52). The result is

$$\frac{(\sigma_{ult})_z}{(\sigma_{ult})_{iso.}} = \frac{(E_{eq})_z}{\beta^2 (E_{eq})_{iso.}} \tag{53}$$

Unlike the $z$ modulus, the factor of $\beta^2$ in the denominator of equation (53) causes the $z$-direction strength to decrease with increasing $\beta$. However, it should be apparent that both modulus and strength have the same variation with the cross-sectional area ratio, $\delta_l$. A three-dimensional plot of the normalized $z$-direction compression strength is presented in figure 11 for comparison to the modulus plot in figure 10. Since both modulus and strength increase as $\delta_l$ is increased, it is best to select the largest practical value for $\delta_l$. As an example, if the cross-sectional areas of all members within the truss are constrained such that they differ by no more than a factor of five, the maximum allowable value for $\delta_l$ would be $\sqrt{2}$. Assuming this value for $\delta_l$ gives the following for all of the member cross-sectional areas.

$$A_1 = \sqrt{2}A \quad , \quad A_2 = \frac{(10\beta - 6)A}{5(2 + 2\beta^2)^{1/2}} \quad , \quad A_3 = \sqrt{2}A/5 \quad , \quad A_4 = 4A/5 \tag{54}$$

A plot of the normalized $z$-direction strength and modulus is presented in figure 12 assuming $\delta_l$ is equal to $\sqrt{2}$. As already explained, extending the length of the Warren truss cell in the z direction (increasing $\beta$) increases the stiffness while decreasing the strength of the truss. Therefore, the optimum length for the truss cell depends on the relative importance of continuum strength and continuum stiffness in the design.

## CONCLUDING REMARKS

A deterministic procedure has been presented for tailoring the continuum stiffness and strength of uniform space-filling truss structures through the appropriate selection of truss geometry and member sizes (i.e., flexural and axial stiffnesses and length). A key aspect of this procedure is symbolic manipulation of the equivalent continuum constitutive equations to produce explicit relationships between truss member sizes and continuum strength and stiffness. To aid in the selection of an appropriate truss geometry for a given application, a finite set of possible geometric symmetry groups which can be possessed by uniform trusses was presented, and the implied elastic symmetry associated with each geometric symmetry group was identified.

Equivalent continuum stiffness were determined using an existing technique which assumes that the displacement field within a truss is single-valued, and the members within a truss carry only axial load.

Based on these assumptions, it was shown that generally anisotropic trusses are characterized by 18 independent elastic constants rather than 21 as is normal for a generally anisotropic solid. It was also shown that this result guarantees that all three-dimensional trusses which behave isotropically, in a continuum sense, must have an equivalent Poisson's ratio of 1/4. Furthermore, a direct relationship was derived between an anisotropic stiffness-to-density ratio of a truss and the stiffness-to-density ratio of its parent material. Using this relationship it was shown that the equivalent Young's modulus-to-density ratio of any isotropic three-dimensional truss is exactly 1/6 times the modulus-to-density ratio of the parent material of the truss.

A purely analytical failure theory was developed for trusses by defining failure to be elastic buckling of any member within the truss lattice. This theory allows the construction of a strength tensor which simplifies failure analysis under multiaxial stress and alternate coordinate systems.

To illustrate the application of these analysis techniques, truss designs were developed which behave isotropically and orthotropically under continuum loading. In these examples, stiffness tailoring was accomplished through redistribution of material among the truss members, and strength tailoring was accomplished by varying the relative buckling strengths of the members. This deterministic approach to the analysis and tailoring of truss behavior can significantly enhance the understanding of relationships between the design parameters and the continuum elastic behavior of trusses. Ultimately, this improved understanding should enable the creation of more efficient truss designs.

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TABLE I.- ELASTIC CHARACTERISTICS OF ROTATIONAL SYMMETRY GROUPS

| Symmetry Group[a] | Conditions on stiffnesses | Independent Constants |
|---|---|---|
| no symmetry | none | 21 |
| $a$ | $c_{14}, c_{15}, c_{24}, c_{25}, c_{34}, c_{35}, c_{46}, c_{56} = 0$ | 13 |
| $b$ | same as $a$ along with $c_{16}, c_{26}, c_{36}, c_{45} = 0$ | 9 |
| $c$ | $c_{16}, c_{26}, c_{34}, c_{35}, c_{36}, c_{45} = 0$ , $c_{11} = c_{22}$ , $c_{44} = c_{55}$ , $c_{13} = c_{23}$ , $c_{14} = -c_{24} = c_{56}$ , $c_{15} = -c_{25} = -c_{46}$ , $c_{66} = (c_{11} - c_{12})/2$ | 7 |
| $d$ | same as $c$ along with $c_{15}, c_{25}, c_{46} = 0$ | 6 |
| $e$ | same as $a$ along with $c_{36}, c_{45} = 0$, $c_{11} = c_{22}$ , $c_{44} = c_{55}$ , $c_{13} = c_{23}$ , $c_{16} = - c_{26}$ | 7 |
| $f$ | same as $e$ along with $c_{16}, c_{26} = 0$ | 6 |
| $g$ | same as $c$ along with $c_{14}, c_{15}, c_{24}, c_{25}, c_{46}, c_{56} = 0$ | 5 |
| $h$ | same as $g$ | 5 |
| $i$ | same as $b$ along with $c_{11} = c_{22} = c_{33}$ , $c_{12} = c_{13} = c_{23}$ , $c_{44} = c_{55} = c_{66}$ | 3 |
| $j$ | same as $i$ | 3 |

[a]See figure 2.

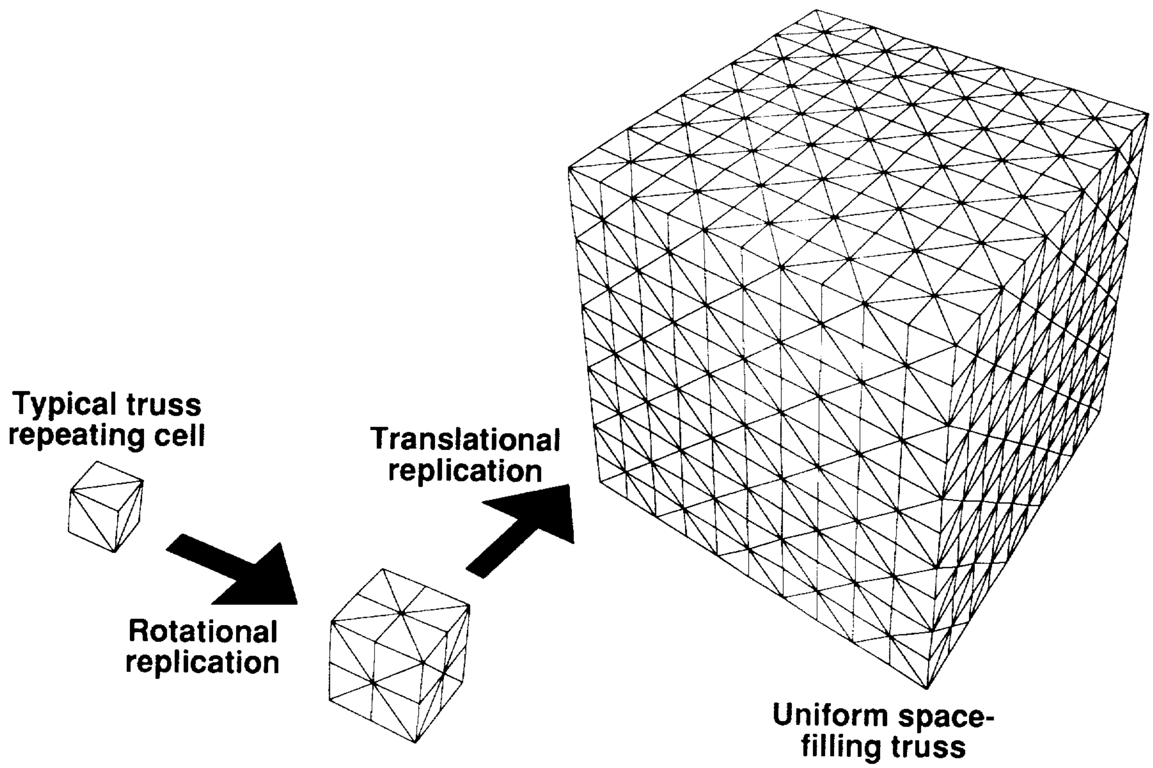



Figure 1. Large uniform trusses are generated from a repeating cell.

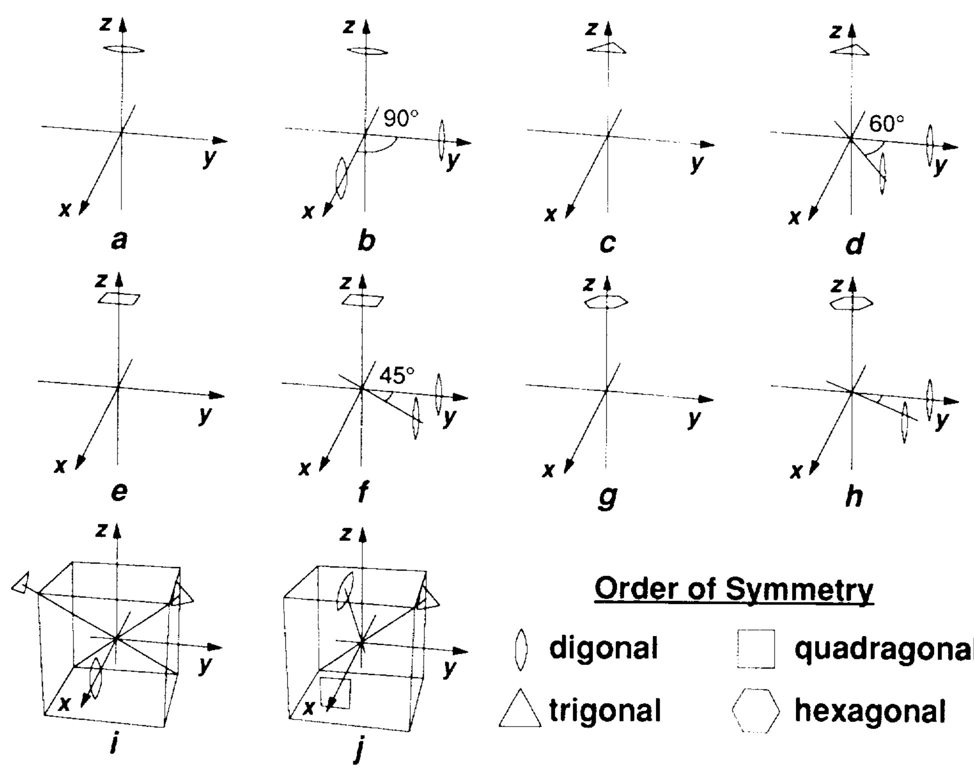
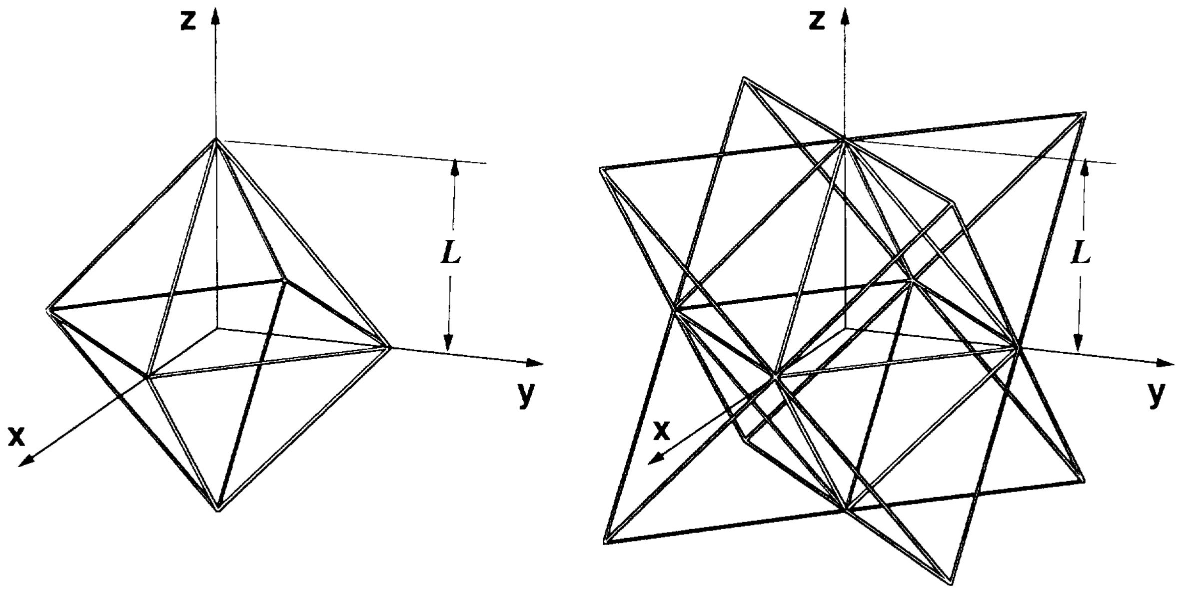



Figure 2. Possible rotational symmetry groups.




(a) regular octahedron.    (a) complete repeating cell with regular tetrahedrons.

Figure 3. Repeating cell for regular Octahedral truss.

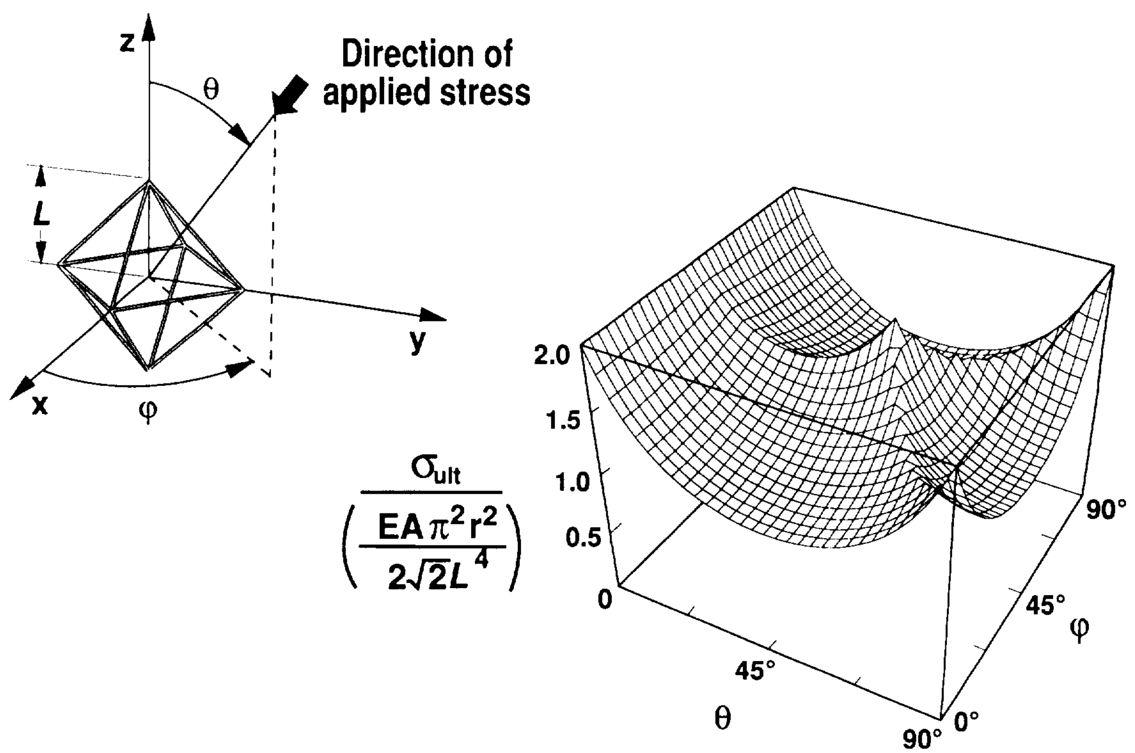
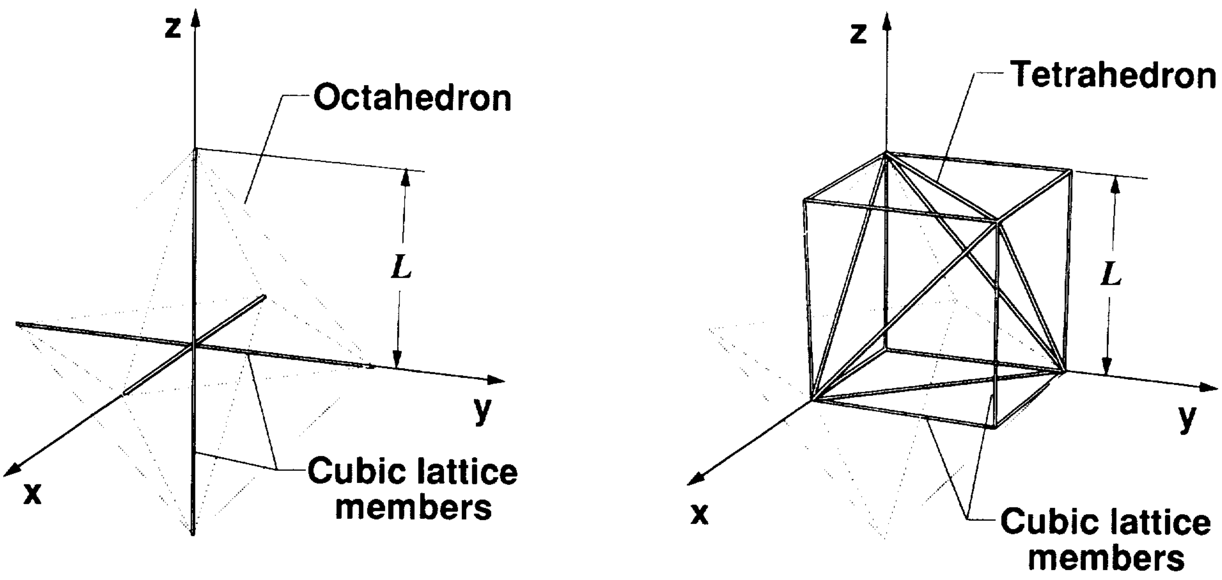



Figure 4. Strength of Octahedral truss under uniaxial compression.




(a) members added to octahedral lattice.

(b) resulting cubic lattice.

Figure 5. Repeating cell for Warren truss.

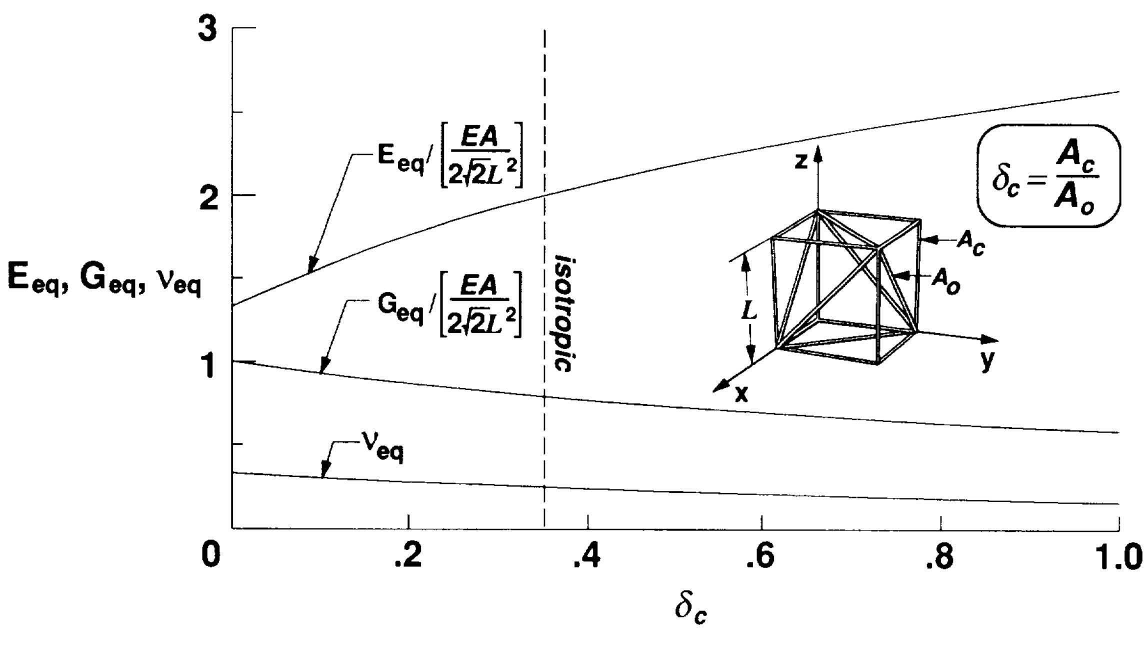



Figure 6. Stiffness tailoring of the Warren truss.

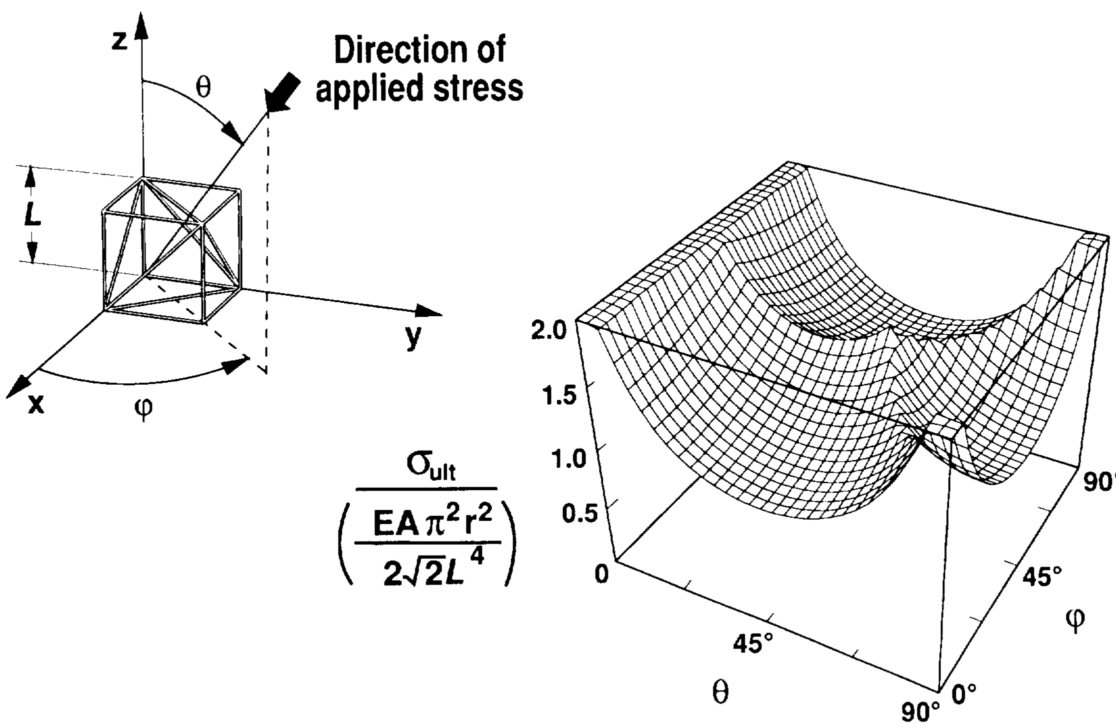



Figure 7. Uniaxial compression strength of isotropic Warren truss (all members have same radius of gyration).

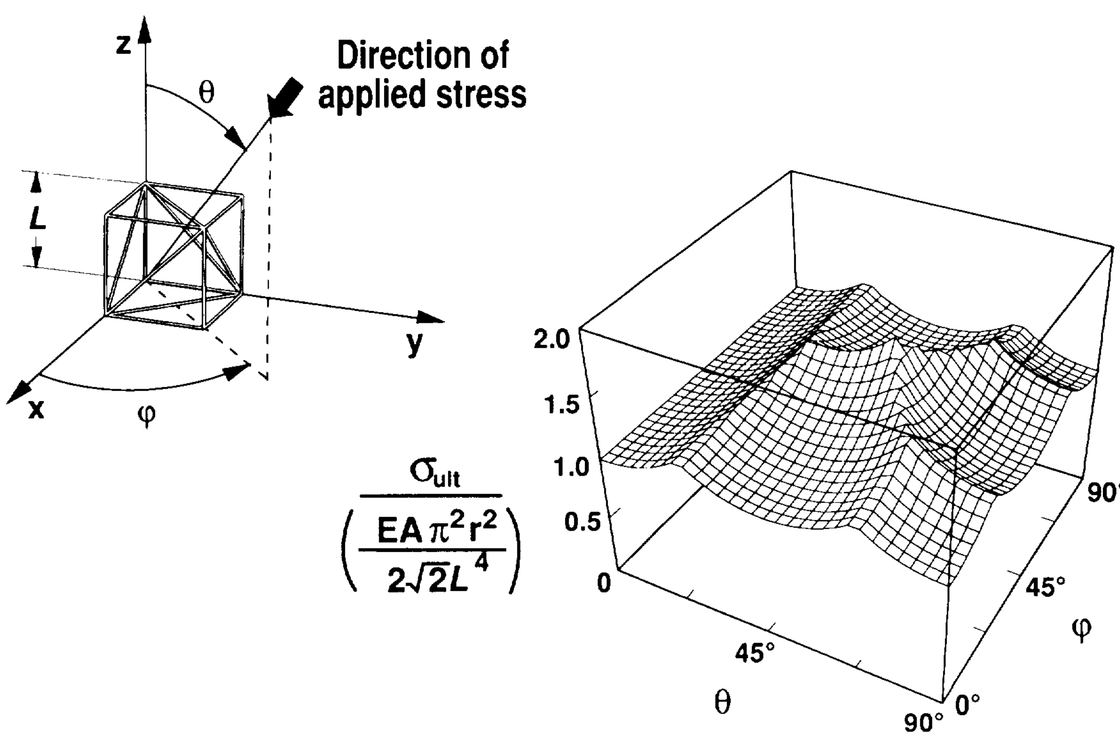



Figure 8. Variation in strength diminished by tailoring all members to have the same buckling load.

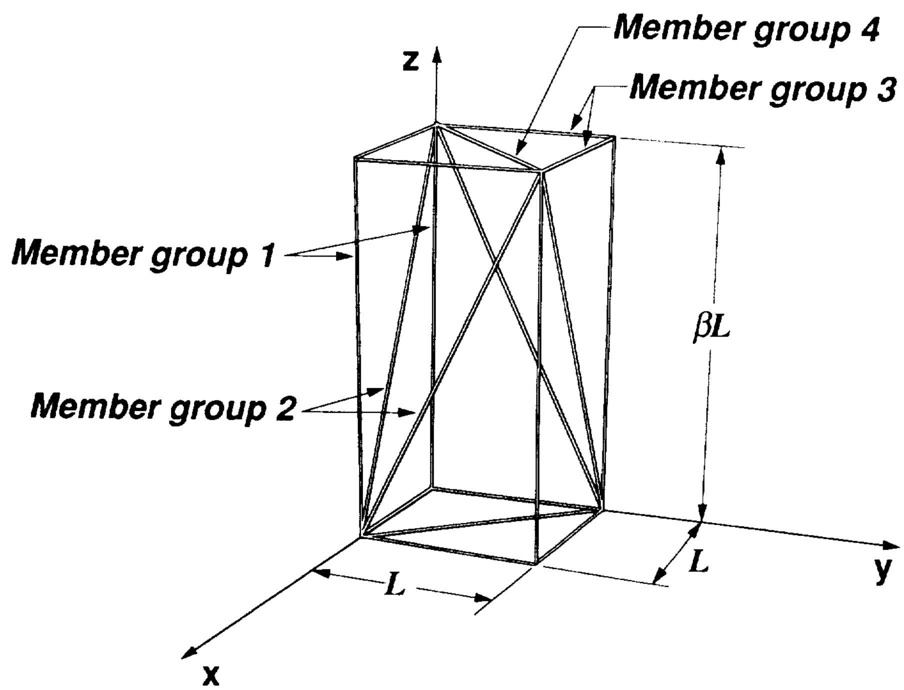



Figure 9. Repeating cell for orthotropic Warren truss.

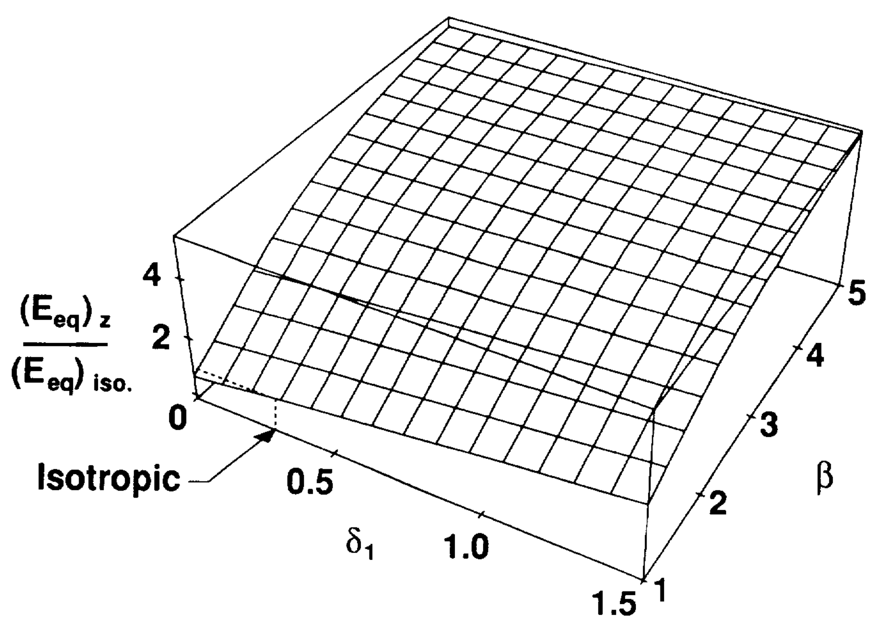



Figure 10. Variation in z modulus with cell length ratio, $\beta$ and member area ratio, $\delta_1$.

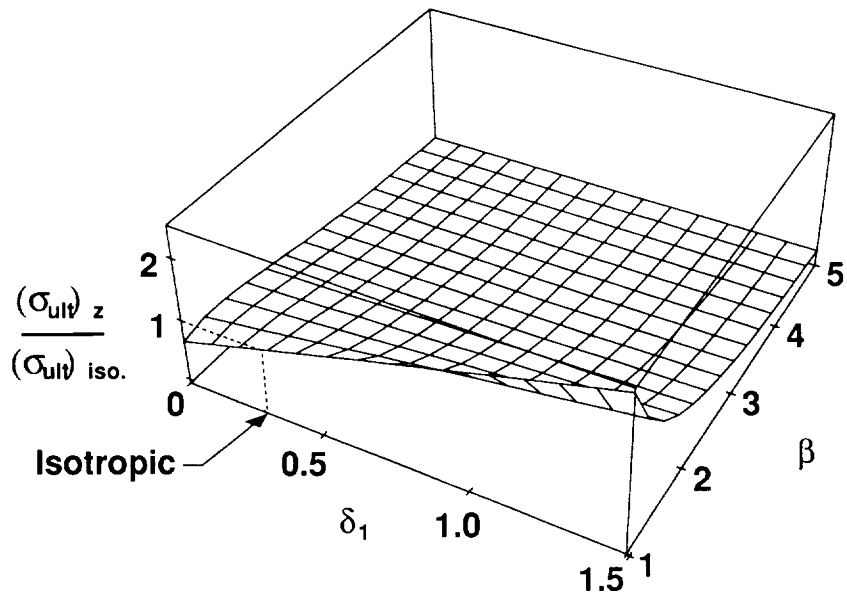



Figure 11. Variation in z strength with cell length ratio, β and member area ratio, $\delta_1$.

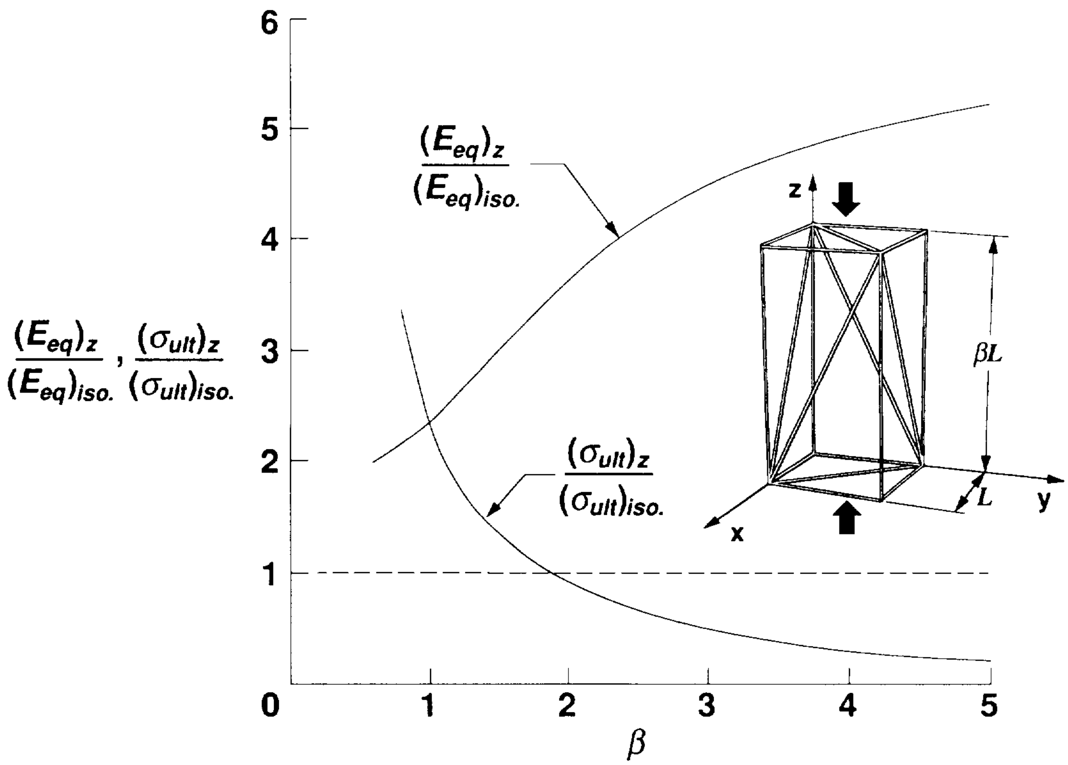



Figure 12. Variation in z direction stiffness and strength with cell length.

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