

CSDL-T-1253

PERFORMANCE ANALYSIS OF A  
GPS INTERFEROMETRIC ATTITUDE  
DETERMINATION SYSTEM FOR A GRAVITY  
GRADIENT STABILIZED SPACECRAFT

by

John C. Stoll

May 1995

Master of Science Thesis  
Massachusetts Institute of Technology

(NASA-CR-1888372) PERFORMANCE  
ANALYSIS OF A GPS INTERFEROMETRIC  
ATTITUDE DETERMINATION SYSTEM FOR A  
GRAVITY GRADIENT STABILIZED  
SPACECRAFT M.S. Thesis (Draper  
(Charles Stark) Lab.) 128 p

N95-30036

Unclass

G3/18 0055360



The Charles Stark Draper Laboratory, Inc.  
555 Technology Square, Cambridge, Massachusetts 02139-3563

**Performance Analysis of a GPS Interferometric  
Attitude Determination System for a Gravity  
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by

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B.S., Pennsylvania State University  
(1993)

Submitted to the Department of Aeronautics and Astronautics in partial  
fulfillment of the requirements for the Degree of

**Master of Science in Aeronautics and Astronautics**

at the

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May 1995

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# **Performance Analysis of a GPS Interferometric Attitude Determination System for a Gravity Gradient Stabilized Spacecraft**

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Submitted to the Department of Aeronautics and Astronautics on  
May 12, 1995, in partial fulfillment of the requirements for the  
Degree of Master of Science in Aeronautics and Astronautics.

## **ABSTRACT**

The performance of an unaided attitude determination system based on GPS interferometry is examined using linear covariance analysis. The modelled system includes four GPS antennae onboard a gravity gradient stabilized spacecraft, specifically the Air Force's RADCAL satellite. The principal error sources are identified and modelled. The optimal system's sensitivities to these error sources are examined through an error budget and by varying system parameters. The effects of two satellite selection algorithms, Geometric and Attitude Dilution Of Precision (GDOP and ADOP, respectively) are examined. The attitude performance of two optimal-suboptimal filters is also presented. Based on this analysis, the limiting factors in attitude accuracy are the knowledge of the relative antenna locations, the electrical path lengths from the antennae to the receiver, and the multipath environment. The performance of the system is found to be fairly insensitive to torque errors, orbital inclination, and the two satellite geometry figures-of-merit tested.

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## Acknowledgments

I would like to express my appreciation to all those who have helped me in this effort. First, I owe great thanks to my advisors, Peter Kachmar, Stan Shepperd, and Dr. Tom Herring. Peter's guidance was instrumental in the development of this thesis. I would like to extend my deepest gratitude to Stan for his technical expertise, patience, and tireless support. I am grateful to Dr. Herring for his critical eye and for sharing his vast experience in the GPS realm.

I would also like to thank those at the Charles Stark Draper Laboratory who helped along the way, particularly Ken Spratlin and Tom DeSwarte for providing me with this wonderful opportunity. I thank Tony Bogner for his technical assistance and support, Don Pasik and Lee Norris for their valuable insight and suggestions, and Peter Neirinckx for helping me to thrash out points of confusion which inevitably arose in the course of my work.

I thank Steve Karner and Scott Wallace for their friendship, empathy, and encouragement, and Percell and Steve on the late shift for breaking up the nightly seriousness of these last few weeks. I also thank my ex-officemates, Matt Feulner and Rick Needham, for helping me in my first year here at MIT, even though they deserted me.

I would like to thank my parents for their unconditional love and support, and for the care packages. Thanks also go to my soon-to-be parents-in-law, Marsha and John Kramer, for bolstering my confidence and desire to achieve.

And a final special thanks to my fiancée, Kim. I thank her for her seemingly endless patience, thoughtfulness, and understanding, and for providing motivation for me to continue and succeed.

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This thesis was prepared at the Charles Stark Draper Laboratory, Inc., under NASA Contract #NAS9-18426.

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# Chapter 1

## Introduction

This introductory chapter first provides the motivation for and a brief background on the use of the Global Positioning System (GPS) in attitude determination. The objectives of this investigation are then outlined, followed by an introduction to the basics of GPS interferometry and the types of measurements which can be made using the interferometer to solve for three-axis attitude. Then follows a description of the interferometer considered in this investigation: four patch antennae onboard a gravity gradient stabilized spacecraft modelled after the RADCAL satellite.

### 1.1 Background

GPS technology provides a solid-state continual solution to the problem of solving for attitude, previously restricted to gyro-based systems. This technology offers high reliability, very little maintenance, and lower costs while potentially achieving milliradian-level accuracy [21]. Attitude determination using GPS is based on a simple geometric principle: two distinct points uniquely define a line and three noncolinear points define a plane. Treating GPS antennae, with known relative positions, as the noncolinear points, the orientation of the plane in which they reside can be determined, as well as the orientation of the antennae within the plane. This is achieved by using the differences in phase of the incoming GPS carrier signal between the antennae, hence the term "GPS interferometry."

The use of GPS carrier phase difference measurements for attitude determination of a stationary platform was first reported by Texas Instruments in 1981 [21]. Other manufacturers including Trimble Navigation, Ashtech, Adroit Systems, and Loral



have also developed GPS receivers for attitude determination. The capability of GPS-based attitude determination has been demonstrated in land-, air-, and marine-based applications [6, 19, 21, 27]. The first flight tests of a real-time GPS attitude determination system for an aircraft were conducted at Ohio University in 1991 [27]. Currently, there is no known use of GPS interferometry in a closed loop system.

The application of this method of attitude determination has recently been extended to space-based platforms. The GPS Attitude and Navigation Experiment (GANE), which will test a GPS interferometer intended for International Space Station Alpha (ISSA), is currently scheduled to fly onboard the Shuttle in April 1996 [9]. Mounted on a 1.5 by 3 m platform in the cargo bay will be a four-antenna interferometer, along with an Inertial Reference Unit (IRU) for attitude verification. The requirements for this stand-alone GPS interferometer are to estimate the station's attitude to within 0.3 deg ( $3\sigma$ ) per axis and attitude rates to within 0.01 deg/s ( $3\sigma$ ) per axis at 0.5 Hz.

An ongoing spaceborne experiment in GPS interferometric attitude determination is flying on the Air Force Space Test Program's RADar CALibration (RADCAL) satellite. RADCAL is the specific application of interest in this investigation. The RADCAL satellite was launched from Vandenberg Air Force Base on June 25, 1993. Its principal purpose is for calibration of civilian and military C-band radar stations. The spacecraft payload includes a radar transponder, a TRANET beacon for navigation, a magnetometer, and a Trimble Advanced Navigation Sensor (TANS) Quadrex receiver, specially adapted for attitude determination by Stanford University [8]. The spacecraft is gravity gradient stabilized, but no other source of attitude information exists on RADCAL, save the magnetometer.

Several research groups are involved in the RADCAL experiment in attitude determination, including Cohen, Lightsey, and Parkinson at Stanford University, Axelrad and Ward at the University of Colorado, and others at JSC. Cohen, et al, report a post-processed attitude solution accuracy ( $1\sigma$ ) on the order of one degree, though they ultimately expect to achieve an accuracy of 0.3 deg in the post-flight solution [8]. Axelrad and Ward are working to implement their "bootstrapping" algorithms, intended for onboard navigation, to provide high accuracy ( $\sim 0.2$  deg/axis) attitude solutions for near Earth satellites [2, 28] and are testing these algorithms on RADCAL flight data. The bootstrap process starts with an initial attitude estimator which resolves the integer ambiguities and approximates line biases, followed by a baseline estimator, and finally a Kalman filter which provides the best attitude estimate in real-time.

## **1.2 Objectives of This Investigation**

Developed in the following pages is a linear covariance analysis of a GPS interferometric attitude determination system for a gravity gradient stabilized spacecraft, using the RADCAL satellite configuration as an example. The goals of this study are to determine how well attitude can be maintained by processing GPS measurements with a Kalman filter and to understand the nature of the error sources which compromise the attitude solution. System parameters including noise levels, satellite selection criterion, and orbital inclination are varied to gain insight into the optimal filter's performance. An error budget is tabulated and used in determining the major contributors to the attitude rms errors. In anticipation of designing an onboard filter, the performance of a reduced-order or suboptimal filter is analyzed using consider states. Unfortunately, it was not possible to process actual RADCAL flight data in the form of raw

single phase differences, obtained from Johnson Space Center, in the time available. Contained in an appendix is a discussion of this data and some of the problems associated with it.

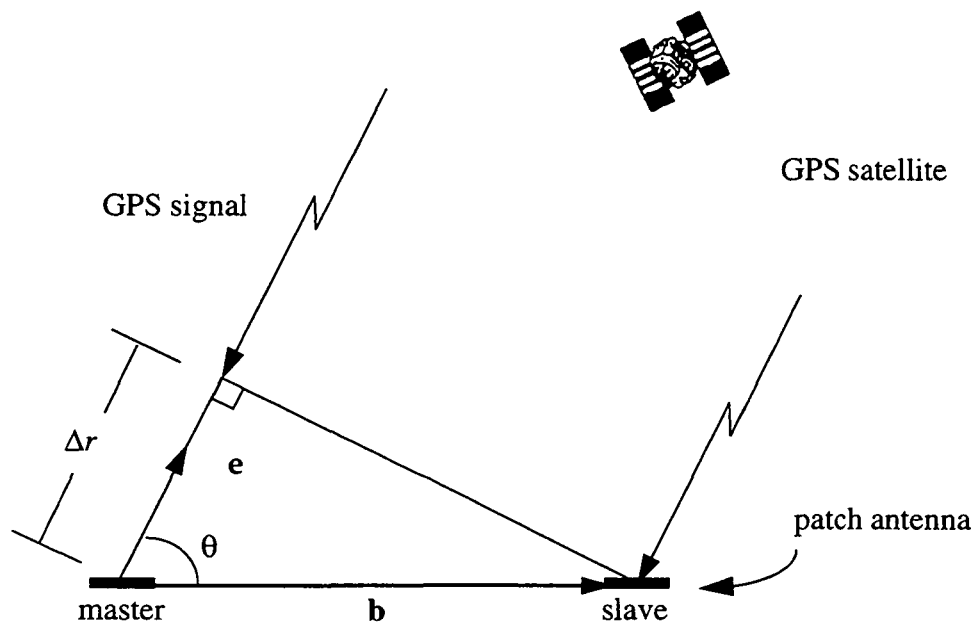
### 1.3 GPS Interferometry and Observables

The primary observable in attitude determination using GPS is the fractional difference in carrier phase,  $\Delta\phi$ , between two antennae. The carrier signal is normally at the L1 frequency for maximum accuracy, centered on 1575.42 MHz, corresponding to a wavelength of roughly 0.19 m. For multiple antennae, the phase differences are typically referred to a common antenna, identified as the master, and the others are designated as slaves. Figure 1.1 depicts the differential phase geometry of a GPS interferometer consisting of a single master-slave antenna pair. The phase difference is related to the difference in range,  $\Delta r$  (in cycles), from the satellite to each antenna via:

$$\Delta\phi = \Delta r - n + \beta + \xi + \nu \quad (1.1)$$

where  $n$  is the integer number of cycles in the differential range,  $\beta$  is the line bias,  $\xi$  the multipath, and  $\nu$  the receiver noise. Several techniques with varying degrees of complexity exist for determining the integer ambiguity, but it need only be resolved once, barring future cycle slips. The line bias given here is actually the difference in path delays from each antenna to the receiver. As can be seen from Figure 1.1, the range difference is merely the component of the baseline,  $\mathbf{b}$ , projected onto the unit line-of-sight (LOS) vector,  $\mathbf{e}$ , and is related to the attitude as follows:

$$\Delta r = \mathbf{b}^T \mathbf{e} = b \cos \theta \quad (1.2)$$



**Figure 1.1 GPS Interferometer Geometry**

Resolving the baseline and LOS vectors into a common frame is equivalent to determining the attitude of the baseline:

$$\Delta r = \mathbf{b}_B^T \mathbf{M}_{BE} \mathbf{e}_E \quad (1.3)$$

where  $\mathbf{M}_{BE}$  is the transformation from the frame in which the line-of-sight vector is expressed, usually an Earth-Centered-Earth-Fixed (ECEF) frame, to that of the baseline vector, usually a body-fixed frame.

Note that any rotation about the baseline axis is not observable, thus requiring a second baseline to determine the host body's three-axis attitude. Note further that a rotation about the LOS vector is also unobservable. Therefore, any number of baselines with just one satellite is insufficient to determine  $\mathbf{M}_{BE}$ . Expressing position only requires three pieces of information. For full attitude specification, however, this is not quite adequate. In general, attitude specification requires three Euler angles *and* the

sequence. This fact holds true even if the attitude is expressed as a quaternion or transformation matrix. However, only small corrections about a nominal attitude are dealt with here, and these corrections can be applied in any order. Therefore, only three pieces of information, i.e., measurements, are necessary to determine attitude errors (though two baselines are still required).

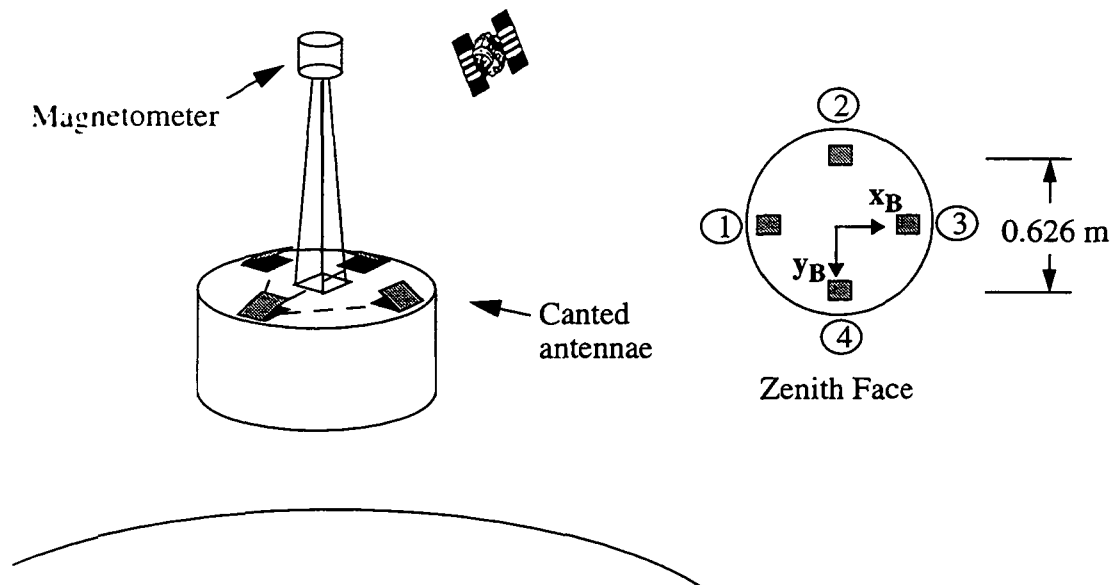
An attitude correction requires one less measurement than the four required for translational error state (three position, one clock) determination. This is a consequence of the differential nature of interferometry. The differencing operation is done with respect to a common oscillator in the receiver hardware, thus effectively eliminating user clock error. The single difference is free of the offset and the linear drift of the satellite frequency [17]. Small errors due to the effect of the changing satellite frequency between signal transmission times for the two antennae are present, but due to the proximity of the antennae and the high quality of the satellite oscillator, these effects are negligible. The ionospheric delay experienced by each signal is also essentially the same and cancels out in the single difference. These and other errors which corrupt the attitude solution will be discussed later in more detail.

Though only the single phase difference measurement is used in this investigation, two other combinations of the carrier phase measurement can be used in interferometric attitude determination: the double difference and the triple difference [17, 29]. The double difference is typically formed by differencing phase difference measurements from two satellites, though the difference can be a temporal one. In the between-satellite double difference, the error resulting from the difference in electrical path length to each antenna is removed. The double difference has its disadvantages, though, since the receiver noise is magnified in combining the phase difference measurements, and another satellite is required to obtain the same number of measurements. This might

pose a problem if the antennae do not view the same region of sky. The triple difference is formed by differencing over antennae, satellites, and time. The advantage of this observable is that the initial integer ambiguity, which is constant in time, is eliminated (assuming no cycle slips between epochs).

#### 1.4 Interferometer Configuration

The interferometer in this investigation resides on a vehicle modelled after the RADCAL satellite, which is in a polar, near-circular, 815 km altitude orbit. It is gravity gradient stabilized, requiring the incorporation of a dynamic model in the attitude determination process. Figure 1.2 depicts the RADCAL satellite and the GPS antenna array. Four patch antennae are mounted on the zenith face of the axially symmetric



**Figure 1.2 RADCAL Configuration**

craft and are equally spaced about the perimeter of the 30-inch diameter cylindrical bus. Each antenna is canted away from the zenith direction by 17.5 deg in order to maximize GPS satellite visibility while reducing multipath from the magnetometer

boom. The receiver time-multiplexes through the four antennae. It simultaneously closes 24 separate tracking loops and can therefore provide phase measurements to each of the four antennae for up to six satellites per antenna, or up to 18 phase difference measurements per epoch for the three baselines [8].

The Quadrex receiver onboard RADCAL is overly restrictive in that it records phase difference measurements only to satellites in view of all four antennae, even though a measurement could be processed if only two antennae view the satellite. The system is nevertheless still overdetermined given the 18 measurements to the three baselines. The simulation developed here employs the phase differences over three baselines with antenna 1 as the master. The biases from the oscillator and receiver electronics drop out in forming the phase differences, resulting in measurement noise errors of less than a millimeter [8]. A greater contributor to the error is multipath, the effect of which is speculated to be about 5 mm for the RADCAL configuration [8, 18]. These and other error sources which corrupt the attitude solution are described further in the following chapter.

## **Chapter 2**

### **Error Modelling**

A survey of the literature reveals that there are many error sources which can limit the attitude solution accuracy using GPS interferometry. This chapter outlines a few of the major contributors and discusses the modeling of those used in this analysis.

#### **2.1 Receiver Noise**

The calculation of the incoming signal's phase angle is subject to error induced by the hardware itself. The phase error depends on the receiver's quality, as well as the vehicle dynamics. The dynamics of the gravity gradient stabilized satellite are slow enough such that their risk of not being within the tracking loop bandwidth is negligible. The Quadrex receiver has 24 tracking loops and is capable of an output rate of 10 Hz, for a total of 240 phase measurements per second. According to Cohen, Lightsey, and Parkinson, the Quadrex's architecture provides very clean phase measurements [8]. As previously mentioned, the biases from the frequency standard and receiver hardware cancel in differencing these measurements. Thus, phase difference measurements should be corrupted by less than a millimeter after passing through the RF switch at the receiver's front-end.

#### **2.2 Vehicle Flexibility**

Knowledge of the spacecraft attitude can only be as good as that of the baseline vectors, for it is these vectors which define the orientation. A trade-off exists in choosing the baseline length. One might expect that greater angular resolution would be obtained with longer baselines, but more integer ambiguity and unaccounted vehicle



flexing between the antennae might negate that benefit [15]. An acknowledgment that the baselines are subject to flexures is therefore required for realistic attitude determination. One source of flexure would be thermal stresses experienced by the craft when passing in and out of eclipse. The effect of the flexure can be to change the baseline length, as well as its direction. The latter result may not be easily differentiated from a change in vehicle attitude. Flexures of the RADCAL baselines are kept reasonably small since the antennae are placed on the fairly rigid platform of the spacecraft bus and are less than a meter apart.

### **2.3 Line Bias**

The line bias over a baseline results from the difference in electrical path lengths from each antenna to the receiver. This "bias" may not be constant, however, since the effective path lengths may change with thermal variations, similar to the baseline variations previously mentioned. Minimizing the error can be accomplished by keeping the path lengths as short and as symmetric as possible, and by configuring the vehicle such that the cabling is subject to small temperature gradients [3, 6]. Fortunately, the phenomenon affecting the path delay to one antenna should be similar to that affecting the path delay to another; the changes in path delay are therefore mitigated somewhat. The line bias can be eliminated by utilizing the between-satellite double difference observable. In the sequel, the terms "path delay" and "line bias" will be used interchangeably to denote the path delay to one antenna, unless specified otherwise.

### **2.4 Multipath**

Multipath is the undesired reflection of the incoming GPS signal from the antenna's surroundings. The antenna may receive both the direct and reflected signal,

or could conceivably receive only the reflected one. Since the reflected signal travels a different path, its phase is shifted from the direct signal. The reflected signal then appears as an additive bias to the primary transmission, as indicated in Equation (1.1). For small baselines on a spacecraft, the multipath error experienced by each antenna could have a common component. In the RADCAL scenario, however, the antennae are canted outward and surround the base of a 6-meter boom; the majority of the multipath is then likely differential mode and does not drop out in the phase difference.

Multipath error can be diminished by various techniques. A low-multipath environment is most desirable; antennae, therefore, should obviously not be placed near multipath sources, though this is unavoidable in the RADCAL configuration. A suitable choice of coating on the mounting surface can also reduce reflectivity [21]. The antenna gain pattern can be appropriately shaped to mask out signals entering from directions of suspected multipath sources. Another way to mitigate multipath, described in great detail by Cohen and Parkinson, is calibrating out the repeatable part of the effect [6, 7]. The only source of multipath in a spacecraft application is the spacecraft itself. A wave front from a given direction will reflect off vehicle surfaces in a repeatable way, provided that the reflectivity does not change and there are no moving parts in the viewing environment. The multipath can then, in theory, be calibrated out as function of incidence direction, and whatever error remains is receiver noise. Cohen goes on to develop a spherical harmonic model of the repeatable multipath characteristic for an experimental configuration. He suggests that the residual carrier phase error can be reduced from 5 mm for the RADCAL configuration to less than 1 mm with multipath calibration [8].

## 2.5 Antenna Phase Center Variations

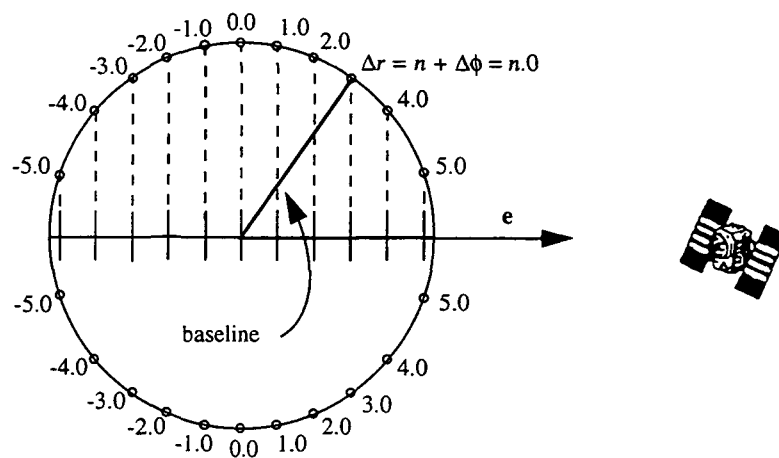
Another error source is the asymmetry of the antenna gain pattern [1, 18]. This asymmetry can cause significant differences in phase based on the signal's incidence direction, resulting in an apparent migration of the phase center. Two antennae of the same make can have similar gain patterns, and the error can be small if the two are similarly oriented with respect to the incoming signal. But if the signal arrives from two different antenna-relative directions caused by, say, the antennae being canted away from each other, the error in differencing the two phases may not be so small. Cohen asserts that this effect can be combined with multipath if one considers the entire vehicle to be the antenna, and due to their dependence on the direction of signal incidence only, the two error sources can be calibrated out [6, 7].

## 2.6 Dilution of Precision

The attitude solution is also compromised by the GPS satellite geometry, itself [18]. For translational space solutions, the familiar Geometric Dilution Of Precision (GDOP) figure-of-merit is used to assess the impact of satellite geometry on position and time determination. A smaller GDOP corresponds to a greater volume of the polyhedron formed by connecting the vertices of the unit vectors from the user to each GPS satellite. But for attitude determination, the best resolution occurs when the line of sight is perpendicular to the baseline vector; an alternate figure of merit, Attitude Dilution Of Precision (ADOP) [26], is therefore required to rate the effect of satellite geometry on the attitude solution. (The derivations of these criteria can be found in Section 4.2.) Note that with only one satellite visible, rotations about the LOS are unobservable. In RADCAL's viewing environment, this situation is not much of a problem, though satellite geometry will still affect the attitude solution.

## 2.7 Integer Ambiguity and Cycle Slip

Obviously, the GPS receiver cannot distinguish one cycle of the carrier signal from any other. Thus, for baselines longer than half of a wavelength (L1 carrier wavelength = 19 cm), an ambiguity in the integer number of cycles between the two antennae exists [15]. Translating the phase difference,  $\Delta\phi$ , into the range difference,  $\Delta r$ , and ultimately solving for the relative positions of the antennae requires resolution of this integer ambiguity. The concept of the ambiguity is illustrated in the two-dimensional example in Figure 2.1, for one satellite and a 1 m baseline [15]. The master antenna is at the center, and the slave resides somewhere on the 1 m radius circle. The vertical lines are the lines of constant phase. The range difference contains the unknown integer,  $n$ , and the measured (noiseless) phase difference for this example is  $\Delta\phi = 0.0$ . As one can see, there are 11 possible range differences (and  $n$ 's) to give 22 possible baseline orientations, i.e., attitudes.



**Figure 2.1 Integer Ambiguity for 1m Baseline & One Satellite**

In the three-dimensional problem, still with one baseline and one satellite, the set of possible attitudes defines a sphere with a 1 m radius, and the ambiguities define a

set of planes which are perpendicular to the LOS and intersect the sphere. The loci of intersections, in other words, are a family of parallel circles whose center is on the LOS. A second satellite creates another family of circles. The number of possible solutions is now reduced to the points of intersection between the two families of circles. In reality, however, the measurements are noisy - the points of intersection then become regions of uncertainty. The addition of more satellites will reduce the number of possible solutions. The added constraint introduced with another antenna can also be used to limit the search, by utilizing the baseline relative geometry.

For the three baselines which view up to six satellites at a time (typical of RAD-CAL), there will be 18 integer ambiguities. Once the ambiguities are resolved, they can be maintained until a cycle slip occurs, resulting from a loss of lock by the tracking loop. If the attitude prior to the slip is known well enough, the slip can be detected and resolved. For the longest RADCAL baseline, 62.6 cm, attitude accuracy better than 8 degrees is, in theory, sufficient to uniquely determine the integer ambiguity.

Many algorithms to resolve the integer ambiguity have been suggested with varying degrees of efficacy. One technique is the brute force integer search, in which different combinations of integers are checked. Methodical searches which utilize geometric constraints to limit the search have been proposed [4, 11, 14]. Knight describes additional time saving techniques in "pruning the decision tree" in the search and claims to resolve the integers instantaneously (using measurements at one epoch) for a four antenna array on a two-meter square with four satellites in track with 99.3% success rate in an average time of 0.24 seconds on a 486 PC [14]. The success rate may not be so great in a high-multipath environment. The problem with the integer search is that it may conclude with an incorrect result, since there may not be a unique solution [6], depending on the baseline geometry and the noise level. An alternative to

the integer search suggested by Cohen is a motion-based approach, which makes use of the additional information provided by the baseline motion [6]. The process involves mapping the phase differences into a set of baseline displacement vectors. Assuming the baseline length to be constant, and no cycle slips from one measurement to the next, the baseline solution may be obtained through a linear least-squares fit, for large angle motion. This solution is used as an initial guess and is refined using a non-linear least squares fit to best match the raw phase measurements. With an initial attitude, the integer ambiguities fall out. Cohen also outlines a scheme for small angle motion. Whatever the method, the ambiguity must be reconciled to determine attitude accurately.

## **2.8 GPS SV and User Clocks**

The clocks of both the GPS satellite and the user receiver are always in error to some extent. In forming the phase difference observable, however, the majority of that error is removed. The large terms resulting from the offset and linear drift of the GPS satellite frequency cancel [17]. The smaller terms from the errors in satellite frequency between the signal transmission times are negligible due to the proximity of the antennae and the stability of the GPS clocks. If the receiver position is determined from the GPS Coarse/Acquisition (C/A) code, the solution will be subject to the satellite clock dithering effect known as Selective Availability (SA). Virtually no accuracy is lost in the attitude solution due to SA, however, since the lines-of-sight are insensitive to the position errors induced by the intentional degradation. The user clock errors cancel altogether, since the signals from all antennae are referred to a common oscillator. Thus, clock errors tend not to affect attitude determination.

## 2.9 Propagation Media

The propagation media affect electromagnetic waves at all frequencies, resulting in refraction of the rays which manifests itself as a time delay of arriving signals. The troposphere, up to 40 km altitude, is a nondispersive medium, in which the refraction is independent of the GPS signal frequency, and it imposes an apparent increase in path length [17]. The ionosphere, from 100 to 1000 km, in which RADCAL resides, has an effect which depends on the total electron current (TEC) or the total number of electrons in the path between GPS satellite and receiver. This effect, arising from the dispersive nature of the medium, is to retard the modulation on the carrier wave while advancing the carrier phase [16]. Fortunately, the signals arriving at the interferometer travel nearly identical paths, and any delay or advance is effectively eliminated in differencing phase measurements. (The cabling from each antennae to the receiver is also dispersive, indicating that the path delays mentioned earlier could actually advance the carrier phase [12]. This dispersion, unlike that of the ionosphere, is likely to be nearly constant and is included in the line bias.)

## 2.10 Modelling of Error Sources

Several contributors from among the previous list of error sources are characterized in this investigation using statistical models. These are: receiver noise, baseline motion (flexibility), line bias, and multipath. The receiver noise, or total phase measurement error for each antenna is modelled as zero-mean, Gaussian white noise. Phase errors between antennae, and between satellites are assumed to be initially uncorrelated. Correlations do arise, however, when the phase difference measurements are formed. This topic will be discussed in more detail in Chapter 4.

The effects of baseline flexures and line biases are included in this analysis even though they are not explicitly modelled. Rather, each antenna's lever arm (from the center of mass) flexure and electronic path delay error state are modelled as first-order Markov processes. The flexure or delay becomes exponentially decorrelated over time as described by the autocorrelation function,  $R_x$ , of the Markov process [5, 10]:

$$R_x(\Delta t) = \sigma^2 e^{\frac{-|\Delta t|}{\tau}} \quad (2.1)$$

where  $\sigma^2$  is the mean-square value of the process, and  $\tau$  is its time constant, or the time separation required for the correlation to be reduced by 63%. In choosing an appropriate time constant, we must consider the physical phenomenon at work. In this case, we assume the driving force of the errors is thermal variation caused by solar heating on the rotating spacecraft. It therefore seems reasonable to choose a time constant of half an orbital period. A more sophisticated and accurate model of the flexure dynamics could be used, but the Markov description should adequately characterize the process.

Multipath is also modelled. Without a spherical harmonic model of the multipath environment for RADCAL, however, the simpler Markov model is assumed. Cohen [8] suggests that a mean-square value of 5 mm is appropriate. The time constant is the time required for the LOS to the satellite in view to 'significantly' change direction, defined as a rotation of 30 degrees [19, 26] in the body frame. The multipath is assumed to be independent from one satellite to another, though from any one satellite, the multipath may be correlated among the antennae. The degree of correlation depends on the environment's (i.e., vehicle's) local reflectivity and is quantified by a variable correlation coefficient,  $\rho_g$ . See Chapter 4 for more detail. The actual multipath behavior is more complicated, but this model can be used as a simple means to observe its effects.



As indicated earlier, clock and propagation media errors are considered negligible because of the differential nature of the problem. The antenna phase center variations can be lumped in with the multipath, since the two effects are practically indistinguishable. The error arising from poor satellite geometry is tempered by employing a satellite selection algorithm. The two algorithms considered are based on GDOP and ADOP. GDOP is used since RADCAL selects satellites based on this criterion for its translational state estimation. ADOP is used for comparison purposes because it more correctly assesses the impact of satellite geometry on the attitude solution. Note that the two algorithms may cause different groups of satellites to be selected.

The integer ambiguity is assumed known in the linear covariance analysis. Of course, in filtering real flight data, the ambiguity would have to be resolved at least in determining the initial attitude. Measurements were assumed to be cycle-slip free for the covariance analysis, because, although real data will contain cycle slips, they are assumed detectable.

## Chapter 3

### Attitude Error Dynamics

The reference frames used in this analysis are described herein, followed by an analytic development of the attitude error dynamics for gravity gradient stabilized spacecraft. Also presented is the analytical solution for the specific case in which the nominal attitude is coincident with the local-vertical-local-horizontal (LVLH) frame.

#### 3.1 Reference Frames

Four reference frames are employed in this investigation. The first is the Earth-Centered-Earth-Fixed (E) frame in which the GPS SV positions found in the International GPS Service (IGS) precise ephemeris files are expressed. (These precise ephemerides are expressed in the International Terrestrial Reference Frame (ITRF-93).) The  $z$  axis is aligned approximately with the spin axis of the Earth, the  $x$  axis lies in the equatorial plane and points toward the Greenwich Meridian, and the  $y$  axis completes the right-handed orthogonal set. An Earth-Centered-Inertial (I) frame is defined as the E frame at time zero - the transformation from the I to E frame is merely a rotation about the inertial  $z$  axis (see Figure 3.1). This approximation is valid only for short periods of time (on the order of days) due to the drift of the Earth's spin axis.

The third reference frame utilized is the LVLH (L) frame, centered on the vehicle. The  $z_L$  vector points in the negative radial direction, the  $y_L$  vector points in the negative orbital angular velocity direction, and the  $x_L$  vector is horizontal forward (see Figure 3.2).

Finally, the Body-Fixed (B) frame is aligned with the spacecraft's principal axes and nominally coincides with the L frame due to the gravity gradient stabilization and

RADCAL's axial symmetry (see Figure 3.2). The orientation of the B frame, and thus the body attitude, is defined by the successive Euler angle rotations: first yaw ( $\theta_3$ ) about the  $z_L$  axis, then pitch ( $\theta_2$ ) about the new  $y$  axis, and finally roll ( $\theta_1$ ) about the resulting  $x$  axis, now  $x_B$ . The transformation from the L frame to the B frame can be written:

$$\mathbf{M}_{BL} = \mathbf{T}_x(\theta_1) \mathbf{T}_y(\theta_2) \mathbf{T}_z(\theta_3)$$

where

$$\mathbf{T}_x(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad \mathbf{T}_y(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$\mathbf{T}_z(\theta_3) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The above attitude matrix can be linearized about an estimated (or nominal) attitude in the following manner:

$$\mathbf{M}_{BL} = \mathbf{M}_{B\hat{B}} \mathbf{M}_{\hat{B}L} \approx (\mathbf{I} - [\delta\theta \times]) \mathbf{M}_{\hat{B}L} \quad (3.1)$$

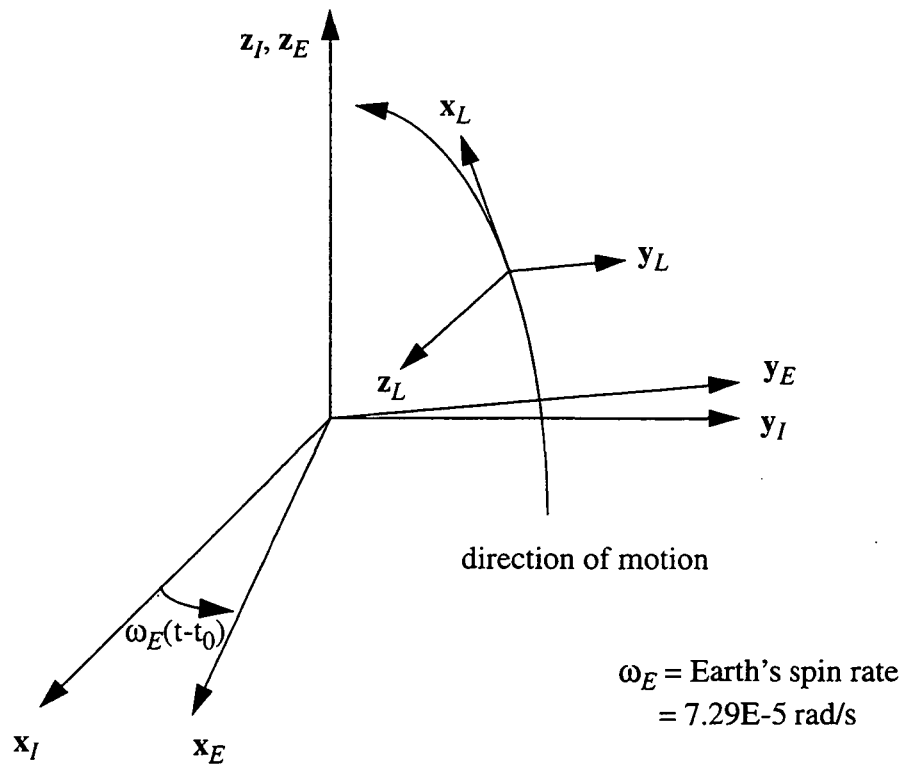
where  $\mathbf{M}_{B\hat{B}}$  is a transformation from the estimated body frame,  $\hat{B}$ , to the true body frame,  $B$ , through the small angles in the vector,  $\delta\theta$ :

$$\delta\theta = \theta - \hat{\theta} \text{ where } \theta = \begin{bmatrix} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{bmatrix} \quad (3.2)$$

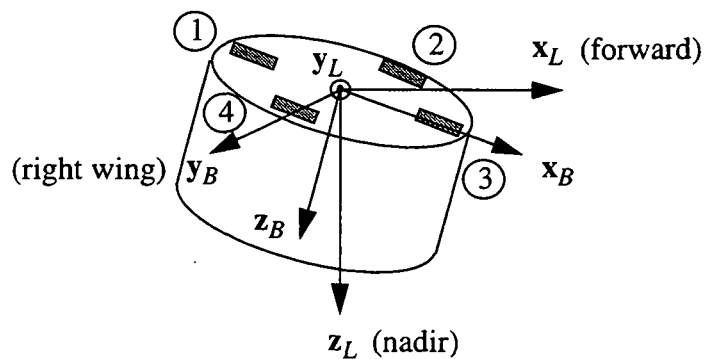
and the associated cross-product matrix is:

$$[\delta\theta \times] = \begin{bmatrix} 0 & -\delta\theta_3 & \delta\theta_2 \\ \delta\theta_3 & 0 & -\delta\theta_1 \\ -\delta\theta_2 & \delta\theta_1 & 0 \end{bmatrix}$$

Note that rotations through small angles are commutative.



**Figure 3.1 Inertial, Earth-Fixed, and Local Coordinate Frames**



**Figure 3.2 Local and Body Coordinate Frames**

### 3.2 Attitude Error Dynamics of a Gravity Gradient Stabilized Spacecraft

The focus of the linear covariance analysis is the error statistics' evolution, which depends on the error dynamics. The goal in this section is to derive the dynamics of

the errors given in Equation (3.2) and of their time derivatives. The starting point is Euler's vector equation of motion, expressed in a body-fixed frame:

$$\frac{d}{dt}(\omega) = \mathbf{I}_M^{-1} (\mathbf{T} - \omega \times (\mathbf{I}_M \omega)) \quad (3.3)$$

where

$\omega$  = angular velocity vector, expressed in body coordinates

$\mathbf{I}_M$  = moment of inertia matrix expressed in body coordinates

$\mathbf{T}$  = external torque on the rigid body expressed in body coordinates

### 3.2.1 The General Case

In rewriting Equation (3.3) in terms of the Euler angles and their errors from the nominal body frame, the following expressions will be required:

$$\frac{d}{dt}(\mathbf{M}_{IB}) = \mathbf{M}_{IB} [\omega \times] \quad (3.4)$$

$$\delta \omega = \omega - \hat{\omega} \quad (3.5)$$

The first is an expression for the time derivative of the attitude matrix,  $\mathbf{M}_{IB}$ , which is the transformation matrix from the body frame to the inertial frame. The second is the angular velocity vector error. Let us assume that there are no translational errors nor inertia matrix errors; that is, errors only arise from the uncertainty in the transformation to the true body frame. The derivation of the sensitivities proceeds [24].

Given the above definitions, we can express the transformation from the true body frame to the inertial frame in terms of the transformation from the nominal body frame and the attitude error, to first order:

$$\mathbf{M}_{IB} = \mathbf{M}_{I\hat{B}} \mathbf{M}_{\hat{B}B} = \mathbf{M}_{I\hat{B}} (\mathbf{I} + [\delta \theta \times]) \quad (3.6)$$

Substitute this expression and Equation (3.5) into Equation (3.4):

$$\frac{d}{dt}(\mathbf{M}_{I\hat{B}}(\mathbf{I} + [\delta\theta\mathbf{x}])) = \mathbf{M}_{I\hat{B}}(\mathbf{I} + [\delta\theta\mathbf{x}]) [(\hat{\omega} + \delta\omega)\mathbf{x}] \quad (3.7)$$

Expanding both sides, ignoring second-order terms, and cancelling where possible gives the following perturbation equation:

$$\left[ \left( \frac{d}{dt}(\delta\theta) \right) \mathbf{x} \right] = [\delta\omega\mathbf{x}] + [\delta\theta\mathbf{x}] [\hat{\omega}\mathbf{x}] - [\hat{\omega}\mathbf{x}] [\delta\theta\mathbf{x}] \quad (3.8)$$

Noting that  $[\mathbf{a}\mathbf{x}] [\mathbf{b}\mathbf{x}] - [\mathbf{b}\mathbf{x}] [\mathbf{a}\mathbf{x}] = [(\mathbf{a}\mathbf{x}\mathbf{b})\mathbf{x}]$ :

$$\left[ \left( \frac{d}{dt}(\delta\theta) \right) \mathbf{x} \right] = [\delta\omega\mathbf{x}] + [(\delta\theta\mathbf{x}\hat{\omega})\mathbf{x}] \quad (3.9)$$

or

$$\frac{d}{dt}(\delta\theta) = \delta\dot{\theta} = \delta\omega - [\hat{\omega}\mathbf{x}] \delta\theta \quad (3.10)$$

Take the time derivative of both sides:

$$\frac{d}{dt}(\delta\dot{\theta}) = \frac{d}{dt}(\delta\omega) - \left[ \frac{d}{dt}(\hat{\omega})\mathbf{x} \right] \delta\theta - [\hat{\omega}\mathbf{x}] \frac{d}{dt}(\delta\theta) \quad (3.11)$$

The first term on the right can be rewritten by linearizing Equation (3.3):

$$\frac{d}{dt}(\omega) = \frac{d}{dt}(\hat{\omega} + \delta\omega) = \mathbf{I}_M^{-1} [\mathbf{T} + \delta\mathbf{T} - (\hat{\omega} + \delta\omega) \times (\mathbf{I}_M(\hat{\omega} + \delta\omega))] \quad (3.12)$$

yielding the first-order perturbation equation:

$$\frac{d}{dt}(\delta\omega) = \mathbf{I}_M^{-1} [\delta\mathbf{T} + [ [(\mathbf{I}_M\hat{\omega})\mathbf{x}] - [\hat{\omega}\mathbf{x}] \mathbf{I}_M] (\delta\omega)] \quad (3.13)$$

The torque error can be decomposed into the contributions from the gravity gradient and other sources:

$$\delta\mathbf{T} = \frac{\partial \mathbf{T}_{GG}}{\partial \theta} \delta\theta + \frac{\partial \mathbf{T}_{GG}}{\partial \dot{\theta}} \delta\dot{\theta} + \delta\mathbf{T}_{other} \quad (3.14)$$

The second term on the right is zero since the gravity gradient torque depends only on the attitude and not its rate, as will be seen. The last term in the above equation embod-

ies all other sources of torques: solar radiation pressure, atmospheric drag, cosmic dust, and other perturbations. The modelled spacecraft has no control torques acting on it. Now insert equations (3.13), (3.14), and (3.10) into Equation (3.11) and factor out  $\delta\theta$ ,  $\delta\omega$ , and  $\delta\mathbf{T}_{other}$  to get:

$$\begin{aligned} \frac{d}{dt}(\delta\theta) = & \left[ \mathbf{I}_M^{-1} \left( \frac{\partial \mathbf{T}_{GG}}{\partial \theta} \right) - [\dot{\hat{\omega}}\mathbf{x}] + [\hat{\omega}\mathbf{x}] [\hat{\omega}\mathbf{x}] \right] \delta\theta \\ & + \left[ \mathbf{I}_M^{-1} [ [ (\mathbf{I}_M \hat{\omega}) \mathbf{x}] - [\hat{\omega}\mathbf{x}] \mathbf{I}_M ] - [\hat{\omega}\mathbf{x}] \right] \delta\omega \\ & + \mathbf{I}_M^{-1} \delta\mathbf{T}_{other} \end{aligned} \quad (3.15)$$

Now, substitute  $\delta\omega$  from Equation (3.10) and let  $\delta\mathbf{T} = \delta\mathbf{T}_{other}$ :

$$\frac{d}{dt}(\delta\theta) = \ddot{\delta\theta} = \mathbf{A}\delta\theta + \mathbf{B}\dot{\delta\theta} + \mathbf{C}\delta\mathbf{T} \quad (3.16)$$

where

$$\mathbf{A} = \mathbf{I}_M^{-1} \left[ [ [ (\mathbf{I}_M \hat{\omega}) \mathbf{x}] - [\hat{\omega}\mathbf{x}] \mathbf{I}_M ] [\hat{\omega}\mathbf{x}] + \frac{\partial \mathbf{T}_{GG}}{\partial \theta} \right] - [\dot{\hat{\omega}}\mathbf{x}] \quad (3.17)$$

$$\mathbf{B} = \mathbf{I}_M^{-1} [ [ (\mathbf{I}_M \hat{\omega}) \mathbf{x}] - [\hat{\omega}\mathbf{x}] \mathbf{I}_M ] - [\hat{\omega}\mathbf{x}] \quad (3.18)$$

$$\mathbf{C} = \mathbf{I}_M^{-1} \quad (3.19)$$

We still must get an expression for the sensitivity of the gravity gradient torque to the attitude in Equation (3.17). The gravity gradient torque is, in body coordinates [13, 30]:

$$\mathbf{T}_{GG} = \frac{3\mu}{|\mathbf{r}|^5} [\mathbf{r}\mathbf{x} (\mathbf{I}_M \mathbf{r})] \quad (3.20)$$

Since our knowledge of the body frame is in error, the position vector in (nominal) body coordinates is in error. Using the correction transformation from the nominal body frame to the true body frame given in Equation (3.6), express the true position

vector in terms of the attitude error,  $\delta\theta$ , and the position vector in nominal body frame coordinates,  $\hat{\mathbf{r}}$ :

$$\mathbf{r} = \mathbf{M}_{B\hat{B}} \hat{\mathbf{r}} = (\mathbf{I} + [\delta\theta \times])^T \hat{\mathbf{r}} = (\mathbf{I} - [\delta\theta \times]) \hat{\mathbf{r}} \quad (3.21)$$

Likewise, express the torque as a nominal term plus a correction and substitute the above into Equation (3.20) to get:

$$\hat{\mathbf{T}}_{GG} + \delta\mathbf{T}_{GG} = \frac{3\mu}{|\hat{\mathbf{r}}|^5} [\hat{\mathbf{r}} \times (\mathbf{I}_M \hat{\mathbf{r}}) - \hat{\mathbf{r}} \times (\mathbf{I}_M (\delta\theta \times \hat{\mathbf{r}})) - (\delta\theta \times \hat{\mathbf{r}}) \times (\mathbf{I}_M \hat{\mathbf{r}})] \quad (3.22)$$

again, ignoring terms of second-order and higher. The perturbation in the gravity gradient torque is therefore:

$$\delta\mathbf{T}_{GG} = \frac{\partial \mathbf{T}_{GG}}{\partial \theta} \delta\theta \quad (3.23)$$

where

$$\frac{\partial \mathbf{T}_{GG}}{\partial \theta} = \frac{3\mu}{|\hat{\mathbf{r}}|^5} [ [\hat{\mathbf{r}} \times] \mathbf{I}_M - [ (\mathbf{I}_M \hat{\mathbf{r}}) \times ] ] [\hat{\mathbf{r}} \times] \quad (3.24)$$

Substitute this result into Equation (3.17), and we now have an expression for the time rate of change of the attitude error rate in terms of  $\delta\theta$ ,  $\dot{\delta\theta}$  and  $\delta\mathbf{T}$ . The attitude error dynamics equation in matrix form is given as:

$$\begin{bmatrix} \dot{\delta\theta} \\ \ddot{\delta\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{A}_{3 \times 3} & \mathbf{B}_{3 \times 3} & \mathbf{C}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \delta\theta \\ \dot{\delta\theta} \\ \delta\mathbf{T} \end{bmatrix} \quad (3.25)$$

where  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are given in expressions (3.17), (3.18), and (3.19), respectively.

### 3.2.2 Special Case

For the nominal case, in which the small rotations are from the LVLH frame, Euler's equations describing the motion of the gravity gradient stabilized spacecraft can be simplified. Let us assume the body frame coincides with the principal axes, the



orbit is circular, and only the gravity gradient torque is acting ( $\delta \mathbf{T} = \mathbf{0}$ ). The inertia matrix, nominal angular velocity and position vectors reduce to:

$$\mathbf{I}_M = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad \hat{\omega} = \begin{bmatrix} 0 \\ -\omega_{orb} \\ 0 \end{bmatrix} = \mathbf{const} \quad \hat{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}$$

where  $\omega_{orb}^2 = \mu/r^3$  is the orbital rate for a spacecraft in a circular orbit of radius,  $r$ , about a point mass with a gravitational constant,  $\mu$ . Inserting the above quantities into equations (3.17), (3.18), and (3.24), respectively, the linearized equations of motion in terms of the differential roll, pitch, and yaw angles are:

$$\delta \ddot{\theta}_1 + 4\omega_{orb}^2 \sigma_x \delta \theta_1 - \omega_{orb} k_x \delta \dot{\theta}_3 = 0 \quad (3.26)$$

$$\delta \ddot{\theta}_2 + 3\omega_{orb}^2 \sigma_y \delta \theta_2 = 0 \quad (3.27)$$

$$\delta \ddot{\theta}_3 + \omega_{orb}^2 \sigma_z \delta \theta_3 + \omega_{orb} k_z \delta \dot{\theta}_1 = 0 \quad (3.28)$$

where

$$\sigma_x = \frac{I_y - I_z}{I_x} \quad \sigma_y = \frac{I_x - I_z}{I_y} \quad \sigma_z = \frac{I_y - I_x}{I_z}$$

$$k_x = \frac{I_x - I_y + I_z}{I_x} \quad k_z = \frac{I_x - I_y + I_z}{I_z}$$

The modelled craft is axially symmetric with  $I_x = I_y = 26.4 \text{ kg m}^2$ , and  $I_z = 5.813 \text{ kg m}^2$  and resides in an 815 km altitude orbit (the principal moments of inertia and approximate altitude of RADCAL [18]). From Equation (3.27), we readily see that the pitch is decoupled and is a pure sinusoid of the form:

$$\delta \theta_2(t) = a_1 \sin((\omega_{orb} \sqrt{3\sigma_y})t + a_2) \quad (3.29)$$

where  $a_i$  is a constant of integration. The pitch is seen to have a period of oscillation of  $2\pi/(\omega_{orb} \sqrt{3\sigma_y}) \approx 3960 \text{ s}$ . The differential roll and yaw solutions can be found by first

rewriting equations (3.26) and (3.28) in matrix form [18]:

$$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\ddot{\theta}_1 \\ \delta\dot{\theta}_3 \\ \delta\ddot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4\omega_{orb}^2\sigma_x & 0 & 0 & \omega_{orb}k_x \\ 0 & 0 & 0 & 1 \\ 0 & -\omega_{orb}k_z - \omega_{orb}^2\sigma_z & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\dot{\theta}_1 \\ \delta\theta_3 \\ \delta\dot{\theta}_3 \end{bmatrix}$$

Noting that  $\sigma_z = 0$  and  $k_z = 1$  for this particular case gives the following eigenvalues for the roll and yaw modes:

$$\text{roll: } \lambda_{1,2} = \pm \omega_{orb} \sqrt{(3\sigma_x + 1)} i$$

$$\text{yaw: } \lambda_{3,4} = 0, 0$$

Taking these eigenvalues and their corresponding eigenvectors gives the roll and yaw solutions:

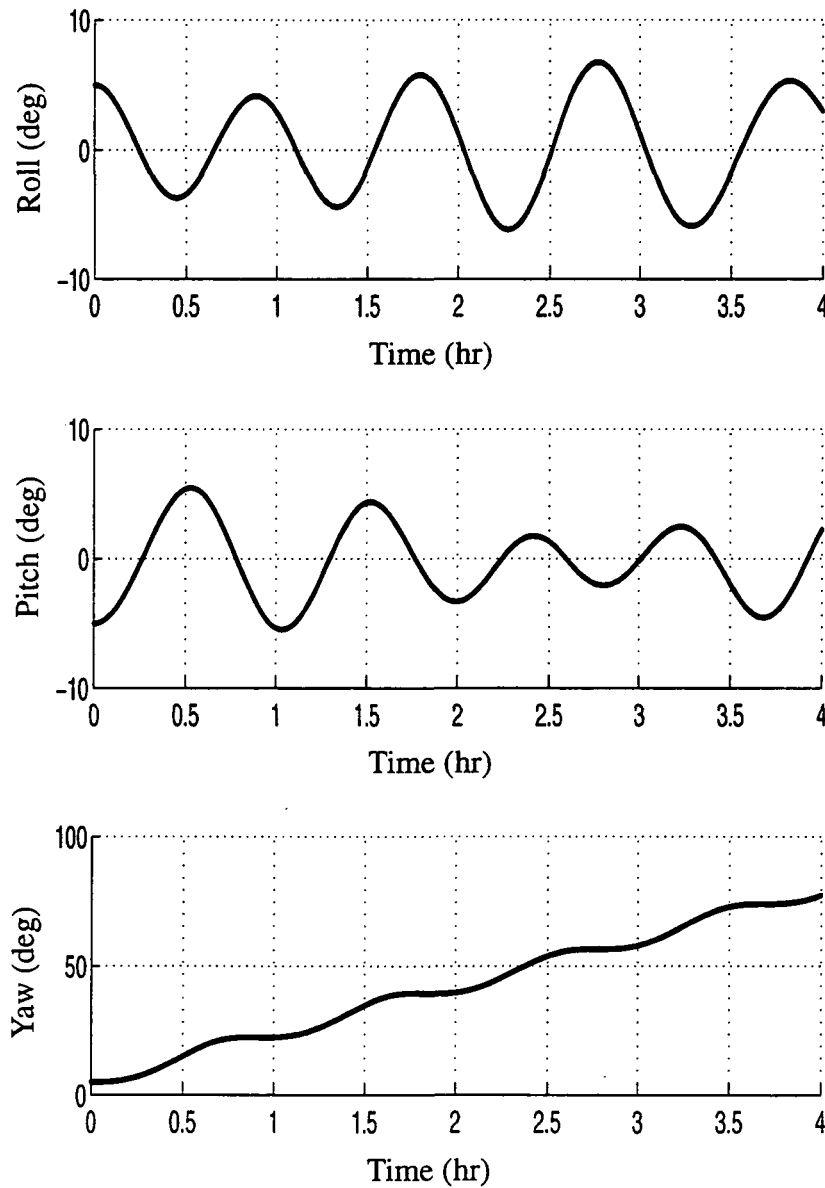
$$\delta\theta_1(t) = a_3 \sin((\omega_{orb} \sqrt{3\sigma_x + 1})t + a_4) \quad (3.30)$$

$$\delta\theta_3(t) = \frac{a_3}{\sqrt{3\sigma_x + 1}} \cos((\omega_{orb} \sqrt{3\sigma_x + 1})t + a_4) + a_5 t + a_6 \quad (3.31)$$

The roll and yaw are seen to have periods of oscillation of  $2\pi/(\omega_{orb} \sqrt{3\sigma_x + 1}) \approx 3320$  s, though the yaw has a constant offset and a linear term. Equations (3.29) through (3.31) may also be found in Reference [18]. This behavior is verified by directly integrating equations (3.3) and (3.4). A fourth-order Runge-Kutta scheme was used to integrate these two equations for a circular 815 km altitude orbit using a two-body gravity model, with the following initial conditions:

$$\delta\theta(t=0) = \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix}_0 = \begin{bmatrix} 5.0 \\ -5.0 \\ 5.0 \end{bmatrix} \text{ deg, } \omega_L(t=0) = \begin{bmatrix} 0 \\ -\omega_{orb} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0593 \\ 0 \end{bmatrix} \text{ deg/s}$$

where  $\omega_L$  is the orbital angular velocity expressed in the local frame. Figure 3.3 shows the resulting roll, pitch, and yaw, which are extracted from  $\mathbf{M}_{IB}$ , for a four-hour time span. Note that for the small angles in roll and pitch, the periods of oscillation approximate those predicted by the linearized equations of motion. The linear component of the yaw, dependent on the initial angular velocity, is also evident.



**Figure 3.3 Simulated Roll, Pitch, and Yaw**

## Chapter 4

### Linear Covariance Analysis

This chapter describes the analytic development of and some of the implementation issues involved in the linear covariance analysis of the RADCAL interferometer. An overview of the generic dynamics, measurement, and update equations is first provided. Also outlined are the analytic descriptions of the error budget and suboptimal filtering. The derivations of the two satellite selection criteria used in this investigation, GDOP and ADOP, are included. The error state and its dynamics as well as the measurement equation, specific to the gravity gradient stabilized spacecraft considered here, are described along with a brief discussion of their implementation.

#### 4.1 Overview of Filter Equations

An underlying tenet of the linear covariance analysis is that the system dynamics are linear or can be linearized, and can therefore be described by the following dynamics equation:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{w} \quad (4.1)$$

where  $\mathbf{x}$  is the state vector,  $\mathbf{F}$  is the dynamics matrix, and  $\mathbf{w}$  is the process noise. In the analysis, we are concerned not with the full state estimate itself, but rather with the evolution of the second-order statistics of the errors in that estimate (i.e., its error covariance). These statistics can be propagated in time with the knowledge of the dynamics given by Equation (4.1) and updated with measurements given the linear, or linearized, relationship of the state to the observation,  $\mathbf{z}_k$ , at time,  $t_k$ :

$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k \quad (4.2)$$

where  $\mathbf{H}_k$  is the measurement sensitivity matrix, and  $\mathbf{v}_k$  is the measurement noise. Equations (4.1) and (4.2) provide the bases from which the subsequent forms of the filter equations are derived.

#### 4.1.1 Standard Filter Equations

The true state,  $\mathbf{x}$ , is not known, however, so the best that can be done is to maintain an estimate of it,  $\hat{\mathbf{x}}$ . Define the error as the difference between the true and estimated states:

$$\Delta \mathbf{x} = \mathbf{x} - \hat{\mathbf{x}} \quad (4.3)$$

The error is assumed to be unbiased, i.e., the expected value is zero. The error covariance matrix, describing the second-order statistics of the error components and the correlations between them, is then:

$$\mathbf{P} = E [ (\Delta \mathbf{x}) (\Delta \mathbf{x})^T ] \quad (4.4)$$

where  $E[ \ ]$  is the expected value function. Given an initial covariance,  $\mathbf{P}_0$ , equations (4.1) and (4.2) can be used to compute future values of  $\mathbf{P}$  - this is the covariance analysis. The analysis is divided into two parts: 1) the propagation step and 2) the measurement update step.

Due to the presence of process noise which is used to allow for any inadequacies in the model of the true process, the covariance will grow over time if no measurements are taken. The propagation of the covariance matrix  $\mathbf{P}$  between measurements in continuous time is described via the matrix Ricatti equation:

$$\dot{\mathbf{P}} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{Q} \quad (4.5)$$

where  $\mathbf{Q}$  is the power spectral density matrix characterizing the process noise. This quantity is related to the noise covariance as follows:

$$E [ \mathbf{w} (t) \mathbf{w} (\tau)^T ] = \mathbf{Q} (t) \delta (t - \tau) \quad (4.6)$$

where  $\delta$  is the Dirac delta function with units of 1/time. The noise covariance represents the state uncertainties which accumulate over one time step due to errors in modelling the system.

To reduce the state uncertainties, we can incorporate a measurement and update the state estimate as well as the error covariance (although covariance analysis is only concerned with the latter). We seek to use the measurement,  $z_k$ , to improve the state estimate by a linear combination of the noisy measurement residuals applied to the prior estimate:

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k + \hat{\mathbf{K}}_k (z_k - \mathbf{H}_k \hat{\mathbf{x}}_k) \quad (4.7)$$

where the "+" indicates immediately after the update and  $\hat{\mathbf{K}}_k$  is (for the moment) an arbitrary gain or weighting factor. The covariance of the *a posteriori* error,  $\Delta \mathbf{x}_k^+ = \mathbf{x}_k - \hat{\mathbf{x}}_k^+$ , is:

$$\mathbf{P}_k^+ = E [ (\Delta \mathbf{x}_k^+) (\Delta \mathbf{x}_k^+)^T ] \quad (4.8)$$

Expanding this out using Equation (4.7) gives, after some manipulation, the Joseph form of the error covariance update:

$$\mathbf{P}_k^+ = (\mathbf{I} - \hat{\mathbf{K}}_k \mathbf{H}_k) \mathbf{P}_k (\mathbf{I} - \hat{\mathbf{K}}_k \mathbf{H}_k)^T + \hat{\mathbf{K}}_k \mathbf{R}_k \hat{\mathbf{K}}_k^T \quad (4.9)$$

where  $\mathbf{R}_k$  is the measurement noise covariance. The measurement noise is assumed to be a white sequence with a known covariance structure. It is also assumed that there is no cross correlation between the measurement noise and the process noise [5, 10]:

$$E[\mathbf{v}_k \mathbf{v}_i^T] = \begin{cases} \mathbf{R}_k & i = k \\ \mathbf{0} & i \neq k \end{cases} \quad (4.10)$$

$$E[\mathbf{w}(t) \mathbf{v}_k^T] = \mathbf{0} \text{ for all } t \text{ and } k \quad (4.11)$$

We wish to find the gain factor,  $\mathbf{K}_k$ , which will give the optimal update. Defining this optimality as the minimum mean-square error, or the minimum trace of  $\mathbf{P}$ , gives the Kalman gain,  $\mathbf{K}_k$ :

$$\mathbf{K}_k = \mathbf{P}_k \mathbf{H}_k^T \left( \mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \quad (4.12)$$

If the elements of the measurement error vector,  $\mathbf{v}_k$ , are not correlated with each other, the measurement noise covariance matrix then assumes a diagonal form, and each scalar measurement can be incorporated sequentially. In this case, the measurement sensitivity and weighting matrices reduce to vectors, the measurement noise covariance matrix reduces to a scalar,  $\alpha^2$ , and the Joseph form of the measurement update becomes:

$$\mathbf{P}_k^+ = \left( \mathbf{I} - \mathbf{k}_k \mathbf{h}_k^T \right) \mathbf{P}_k \left( \mathbf{I} - \mathbf{k}_k \mathbf{h}_k^T \right)^T + \alpha^2 \mathbf{k}_k \mathbf{k}_k^T \quad (4.13)$$

and the matrix inverse operation becomes a simple division:

$$\mathbf{k}_k = \frac{\mathbf{P}_k \mathbf{h}_k}{\mathbf{h}_k^T \mathbf{P}_k \mathbf{h}_k + \alpha^2} \quad (4.14)$$

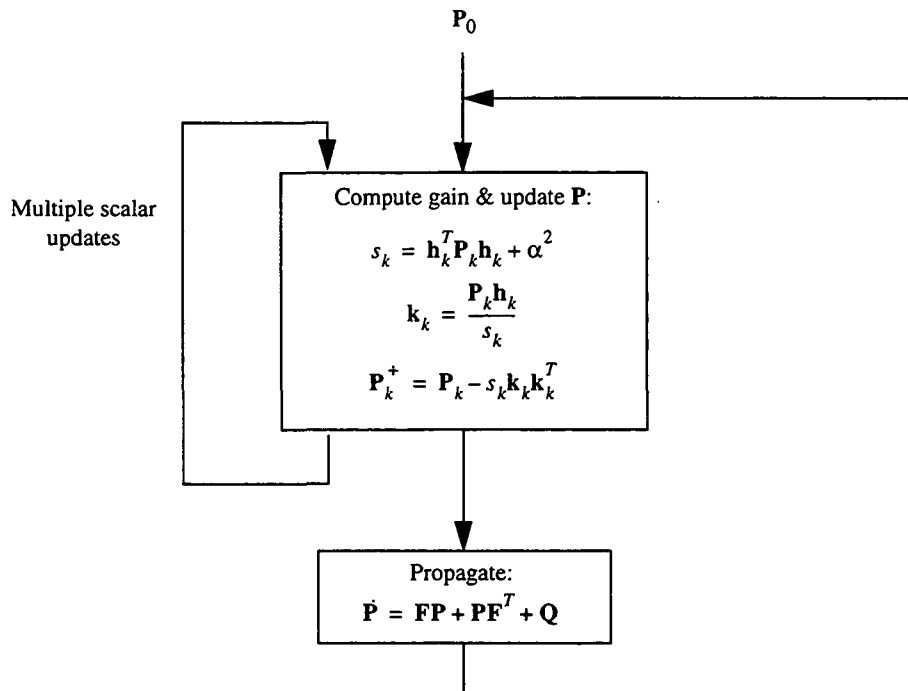
Substituting Equation (4.14) into Equation (4.13) gives the standard form of the optimal update:

$$\mathbf{P}_k^+ = \mathbf{P}_k - s_k \mathbf{k}_k \mathbf{k}_k^T \quad (4.15)$$

where

$$s_k = \mathbf{h}_k^T \mathbf{P}_k \mathbf{h}_k + \alpha^2$$

The recursive loop for the optimal error covariance is summarized in the figure below. Note that the  $\mathbf{P}$  matrix does not depend on the actual measurements themselves, just their statistics and the measurement geometry. Multiple measurements are processed sequentially using the scalar update form in Equation (4.15).



**Figure 4.1 Recursive Loop for Optimal Error Covariance**

The optimal Kalman filter gain minimizes the error covariance at the present measurement update as well as during the following extrapolation [22]. In other words, the gain which yields optimum performance at some later point in time is the same gain which yields optimum performance at the present measurement update.

#### 4.1.2 Error Budget

Using consider state analysis, all of the error sources in the “truth” model, whether modelled or not in the onboard filter, affect the errors in the filter estimate. The error budget is a tabulation of the relative contributions of individual error sources, or groups of error sources, to system accuracy. In one form of the error budget calculations, the time history of the filter gain is recorded and then processed repeatedly in determining the individual effects of each error source; multiple runs are therefore



required. The ensuing equations provide a more elegant method in tabulating an error budget with a single run [24].

We wish to decompose the total error covariance,  $\mathbf{P}_{Total}$ , into its constituents:

$$\mathbf{P}_{Total} = \Phi \mathbf{P}_0 \Phi^T + \mathbf{P}_Q + \mathbf{P}_M \quad (4.16)$$

The first term on the right is the contribution from the initial conditions, the middle term is the contribution from the error states' process noise, and the final term is the contribution from the measurement uncertainty.  $\Phi$  is the state transition matrix from the initial time,  $t_0$ , to the final time,  $t_f$ . Note that there can be as many  $\mathbf{P}_Q$  and  $\mathbf{P}_M$  terms as desired. The quantities of interest are the effect of the above contributions on the final roll, pitch, and yaw mean-square errors - the first three diagonal terms in each component will therefore be examined. Note that in the form given in Equation (4.16), the initial condition contribution to the final attitude error variances is the contribution from the *entire initial covariance*, not just the initial attitude error variances. Equation (4.16) will require a slight modification to extract this information subset (see Appendix A). Continuing with the general equations of the error budget, each component of the covariance matrix is propagated and updated, though in a slightly different form from the total covariance matrix, as shown in the following sections.

#### 4.1.2.1 Propagation

The usual discrete form in extrapolating the error covariance between measurements is, dropping the subscript,  $k$ , for convenience:

$$\mathbf{P}^+ = \phi \mathbf{P} \phi^T + \mathbf{N} \quad (4.17)$$

where  $\mathbf{P}$  and  $\mathbf{P}^+$  are the covariances just after the last measurement update at  $t_{k-1}$  and after extrapolation to  $t_k$ , respectively, and  $\phi = \phi(t_k, t_{k-1})$  is the state transition

matrix between the two times (not to be confused with the total phase measurement,  $\phi$ ) given by the relationship:

$$\mathbf{x}(t_k) = \Phi(t_k, t_{k-1}) \mathbf{x}(t_{k-1})$$

where for a stationary process in which  $\mathbf{F}$  is constant [5, 10]:

$$\frac{d}{dt}\Phi(t_k, t_{k-1}) = \mathbf{F}\Phi(t_k, t_{k-1}) \quad (4.18)$$

$$\Phi(t_k, t_{k-1}) = e^{\mathbf{F}(t_k - t_{k-1})}$$

The last term in Equation (4.17) is the process noise covariance over the time interval and is related to the spectral density,  $\mathbf{Q}$ , by [5, 10]:

$$\mathbf{N} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) \mathbf{Q} \Phi^T(t_k, \tau) d\tau \quad (4.19)$$

(See Appendix A for the details on calculating  $\mathbf{N}$ .) Substituting Equation (4.17) into Equation (4.16) gives the extrapolation of each component of the covariance:

$$\Phi^+ \mathbf{P}_0 \Phi^{+T} + \mathbf{P}_Q^+ + \mathbf{P}_M^+ = \Phi [\Phi \mathbf{P}_0 \Phi^T + \mathbf{P}_Q + \mathbf{P}_M] \Phi^T + \mathbf{N}$$

The extrapolation of each component is seen from the above equation to be:

$$\Phi^+ = \Phi \Phi$$

$$\mathbf{P}_Q^+ = \Phi \mathbf{P}_Q \Phi^T + \mathbf{N} \quad (4.20)$$

$$\mathbf{P}_M^+ = \Phi \mathbf{P}_M \Phi^T$$

#### 4.1.2.2 Measurement Update

The Joseph form of the measurement update resembles the extrapolation step shown in Equation (4.17):

$$\mathbf{P}^+ = \mathbf{S} \mathbf{P} \mathbf{S}^T + \mathbf{M} \quad (4.21)$$

where, for the scalar update with an arbitrary weighting,  $\hat{\mathbf{k}}$  :

$$\mathbf{S} = \mathbf{I} - \hat{\mathbf{k}}\mathbf{h}^T \quad \text{and} \quad \mathbf{M} = \alpha^2 \hat{\mathbf{k}}\hat{\mathbf{k}}^T$$

Thus, similar to propagation, the update of each component of the decomposed covariance matrix takes the following form:

$$\begin{aligned} \Phi^+ &= \mathbf{S}\Phi \\ \mathbf{P}_Q^+ &= \mathbf{S}\mathbf{P}_Q\mathbf{S}^T \\ \mathbf{P}_M^+ &= \mathbf{S}\mathbf{P}_M\mathbf{S}^T + \mathbf{M} \end{aligned} \tag{4.22}$$

### 4.1.3 Suboptimal Filter Equations

The ultimate goal of covariance analysis is to design a filter for an onboard navigator. The previously described error budget provides a breakdown of the relative contribution of each error source to the final state uncertainties and allows the states of lesser consequence to be identified. Due to hardware constraints, i.e., limited storage capacity and processing power, it is often necessary to reduce the order of the filter used in an onboard system. We therefore seek the suboptimal filter giving the best performance, or the minimum-variance reduced-order (MVRO) filter. To assess the performance of this suboptimal filter, we can cycle the suboptimal gains through the truth-model update equation, in which the states to be omitted are not estimated. Their effects, however, are felt in the states that remain; these omitted states are thus referred to as “consider states.”

Equation (4.15), since it represents an optimal update, is inappropriate for rating the suboptimal filter's performance. The Joseph form in Equation (4.13) is used in processing a suboptimal weighting vector,  $\hat{\mathbf{k}}$ , and can be reworked into a form which requires fewer operations and is intuitively clear [23]:

$$\mathbf{P}^+ = \mathbf{P} - s\mathbf{k}\mathbf{k}^T + s(\hat{\mathbf{k}} - \mathbf{k})(\hat{\mathbf{k}} - \mathbf{k})^T \tag{4.23}$$

where the subscripts have been omitted for convenience. From Equation (4.23) we readily see that any gain which is not optimal results in worse performance.

In consider state analysis, the  $nx1$  optimal weighting vector,  $\mathbf{k}$ , is computed, and then zeros are inserted into the components of the  $n-m$  states that will not go onboard:

$$\begin{aligned} \hat{\mathbf{k}} &= \overbrace{\mathbf{W}\mathbf{W}^T\mathbf{P}\mathbf{h}(\mathbf{h}^T\mathbf{P}\mathbf{h} + \alpha^2)^{-1}}^{nx1 \text{ optimal gain, } \mathbf{k}} \\ &= \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_m \\ 0 \\ \dots \\ 0 \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} k_1 \\ k_2 \\ \dots \\ k_m \end{array} \right\} m \text{ elements of } \mathbf{k} \\ \left. \begin{array}{c} 0 \\ \dots \\ 0 \end{array} \right\} n-m \text{ zeros for} \\ \text{consider states} \end{array} \end{aligned}$$

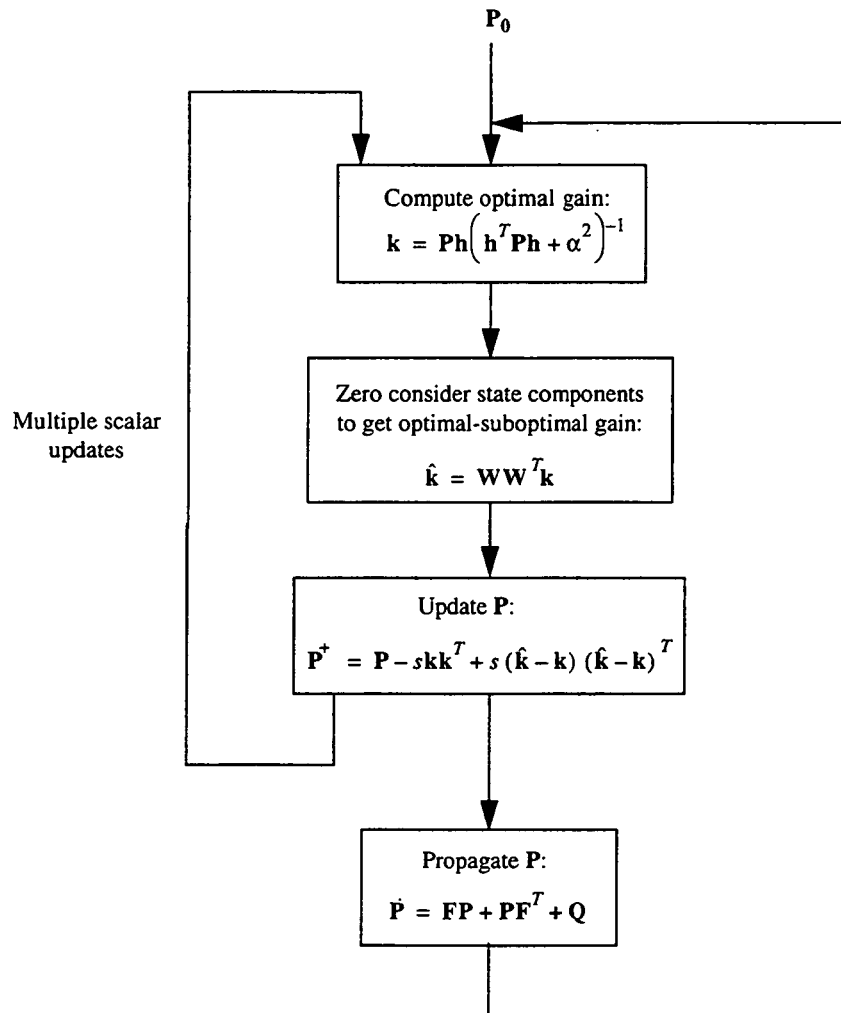
where

$$\mathbf{W} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}_{n \times m}$$

This weighting vector is an “optimal-suboptimal” gain to be used in Equation (4.23) in computing the actual error statistics of the “best” reduced-order, or optimal-suboptimal, filter. The term “optimal-suboptimal” is somewhat of a misnomer since this filter will give minimum variance in a global sense, not necessarily at every time. That is, another suboptimal gain can be computed to give a smaller variance at a particular time [22]. This other gain, however, will result in performance which is worse than that of the optimal-suboptimal overall. The optimal-suboptimal algorithm provides the best performance, in general, which can be achieved in reducing the order of the filter,

and is useful as a benchmark for “tuning” the actual suboptimal filter to go onboard.

The process of consider state analysis is summarized in Figure 4.2.



**Figure 4.2 Consider State Analysis Block Diagram**

The actual onboard filter will not give the optimal-suboptimal performance because it does not consider the omitted states. To analyze the performance of this reduced-order filter, its “optimal” weighting vector,  $\mathbf{k}_{sub}$ , is computed and padded with zeros for use in the “truth” filter:

$$\hat{\mathbf{k}} = \mathbf{W}\mathbf{k}_{sub}$$

Figure 4.3 shows the major steps in the suboptimal filter analysis. This time, however, two covariance matrices must be computed:

- $P_{sub}$  : mxm covariance matrix of the suboptimal filter. Its only relevance is in generating the suboptimal gain sequence.
- $P$  : nxn covariance matrix of the truth model based on suboptimal filter gains. It represents the statistical performance of the suboptimal filter.

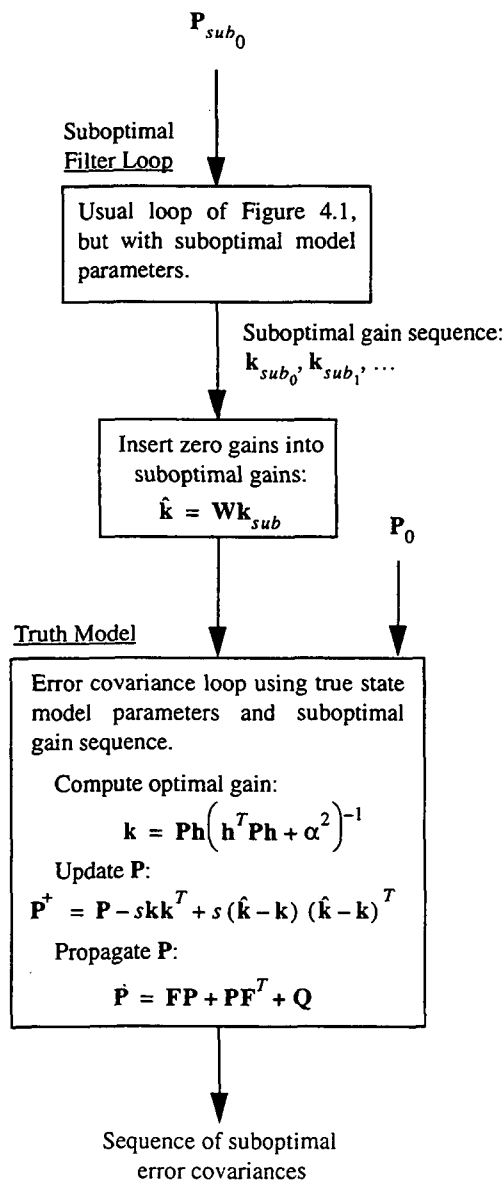


Figure 4.3 Steps in Suboptimal Filter Performance [5]

The suboptimal filter can then be “tuned” to approach the best performance given by the optimal-suboptimal filter. This tuning, which usually involves measurement underweighting and adjusting the process noise, will most likely require multiple runs through the process shown in Figure 4.3.

Performance results of optimal-suboptimal filtering using consider states are discussed in Section 5.4.

## 4.2 Satellite Selection Algorithms

Two criteria were used to rate the effect of GPS satellite geometry on the attitude solution: Geometric Dilution Of Precision (GDOP) and Attitude Dilution Of Precision (ADOP). GDOP is primarily used for translational state estimation and is considered here only because the satellites that RADCAL selected were chosen based on this figure-of-merit. Consequently, the satellites in RADCAL’s “best view” may not necessarily be best in an attitudinal sense. For this reason, the ADOP figure-of-merit was also employed in the linear covariance analysis for purposes of comparison. A brief explanation of the two criteria is provided herein.

The DOP factor is a measure of how satellite geometry degrades accuracy - of user position and time estimates when using GDOP, or of user attitude when using ADOP. It is, in effect, the mapping of measurement errors into state errors due to the effect of satellite geometry:

$$\sigma_x = DOP \cdot \sigma_v \quad (4.24)$$

where  $v$  is the measurement error and  $x$  is the RSS state error. The DOP scaled by the measurement noise is the instantaneous RSS state error using only the measurement and no *a priori* knowledge. Satellites are chosen so as to minimize the DOP.

The DOP is derived as follows. Let each measurement error be zero-mean, have a unity  $1\sigma$  error, and be uncorrelated with errors in measurements to other satellites. Assuming the system is not underdetermined, Equation (4.2) can be solved for the error state vector,  $\mathbf{x}$ , using the method of least squares. The error state covariance is then given by:

$$Cov(\mathbf{x}) = E[\mathbf{x}\mathbf{x}^T] = (\mathbf{H}^T Cov(\mathbf{v})^{-1} \mathbf{H})^{-1} = (\mathbf{H}^T \mathbf{H})^{-1} \quad (4.25)$$

since  $Cov(\mathbf{v})$  is an  $m \times m$  identity matrix, where  $m$  is the number of satellites in the best set. In the above equation,  $\mathbf{H}$  is the  $m \times n$  measurement sensitivity matrix:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial}{\partial x_1} obs_1 & \frac{\partial}{\partial x_2} obs_1 & \dots & \frac{\partial}{\partial x_n} obs_1 \\ \frac{\partial}{\partial x_1} obs_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial}{\partial x_1} obs_m & \dots & \dots & \frac{\partial}{\partial x_n} obs_m \end{bmatrix} \quad (4.26)$$

where  $obs_i$  is the  $i^{\text{th}}$  observation or measurement and  $x_j$  is the  $j^{\text{th}}$  component of the state -  $\mathbf{H}$  will differ depending on the DOP type. For GDOP, the measurement is the pseudo-range,  $\rho$ , which can be expressed as a function of user position and clock error:

$$\rho_{obs} = |\mathbf{r}^{GPS} - \mathbf{r}_{user}| + c\Delta t_{user} \quad (4.27)$$

Row  $i$  of  $\mathbf{H}$  in this case is merely the unit line-of-sight vector to satellite  $i$ , the sensitivity of the pseudo-range measurement to user position, and a 1 for the clock bias term:

$$\mathbf{H}_i = \begin{bmatrix} e_1^i & e_2^i & e_3^i & 1 \end{bmatrix} \quad (4.28)$$

RADCAL's best selection consists of a set of up to six satellites, thus making  $\mathbf{H}$  a  $6 \times 4$  matrix in computing GDOP.



For ADOP, on the other hand, the observation becomes the single, double, or triple phase difference. Ignoring integer ambiguity, line bias and multipath for the moment, the single phase difference to satellite  $j$  is:

$$\Delta\phi^j = \Delta r = \mathbf{b}^T \mathbf{e}^j \quad (4.29)$$

Recalling Equation (1.3), the sensitivity of the phase difference to the  $i^{\text{th}}$  component of the attitude is:

$$\frac{\partial}{\partial \theta_i} \Delta\phi^j = \mathbf{b}_B^T \frac{\partial \mathbf{M}_{BE}}{\partial \theta_i} \mathbf{e}_E^j \quad (4.30)$$

and the  $j^{\text{th}}$  row of  $\mathbf{H}$  is:

$$\mathbf{H}_j = \left[ \frac{\partial}{\partial \theta_1} \Delta\phi^j \quad \frac{\partial}{\partial \theta_2} \Delta\phi^j \quad \frac{\partial}{\partial \theta_3} \Delta\phi^j \right] \quad (4.31)$$

With six satellites in the best view and three baselines, as in the RADCAL scenario,  $\mathbf{H}$  then becomes an  $18 \times 3$  matrix. Section 4.4.1 addresses the above measurement sensitivity at greater length.

Having computed the measurement sensitivity matrix, we can calculate the DOP factor, which is defined as the square root of the trace of  $\text{Cov}(\mathbf{x})$  when  $\text{Cov}(\mathbf{v})$  is identity:

$$DOP = \sqrt{\text{trace}[(\mathbf{H}^T \mathbf{H})^{-1}]} \quad (4.32)$$

that is, for the positioning problem:

$$GDOP = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + \sigma_{c\Delta t}^2} \quad (4.33)$$

and for the attitude problem:

$$ADOP = \sqrt{\sigma_{\theta_1}^2 + \sigma_{\theta_2}^2 + \sigma_{\theta_3}^2} \quad (4.34)$$

GDOP is unitless, or [length/length]; so a pseudo-range measurement uncertainty in meters maps directly into a position-time uncertainty in meters. ADOP, however, maps the phase uncertainty into attitude uncertainty - its units are therefore slightly different. The phase uncertainty is given as a fraction of a cycle, or length in meters. The units of ADOP are then rad/m.

If the measurements are correlated,  $Cov(\mathbf{v})$  is not identity, and the DOP is weighted accordingly. Section 4.4.2 discusses this measurement correlation. As we shall see later, choice of the DOP type will affect the satellites chosen and, consequently, the evolution of the attitude errors.

### 4.3 The Error State and Its Dynamics

The state of primary interest is, of course, the attitude error - the error in roll, pitch, and yaw angles from the LVLH frame. We must include, however, as many other states as we can think of and reasonably model which affect that attitude error. For this investigation, the entire error state vector is chosen to be:

$$\mathbf{x} = \left[ \delta\theta^T \quad \dot{\delta\theta}^T \quad \delta\mathbf{T}^T \quad \delta\mathbf{l}_1^T \quad \dots \quad \delta\mathbf{l}_{N_{ant}}^T \quad \delta\beta_1 \quad \dots \quad \delta\beta_{N_{ant}} \quad \xi^T \right]^T \quad (4.35)$$

where

$\delta\theta$  = attitude error vector (3x1)

$\dot{\delta\theta}$  = attitude error rate vector (3x1)

$\delta\mathbf{T}$  = torque error vector (3x1)

$\delta\mathbf{l}_i$  = lever arm vector, i.e., vector from origin to  $i^{\text{th}}$  antenna in body frame (3x1)

$\delta\beta_i$  = path delay error from  $i^{\text{th}}$  antenna to receiver (1x1)

$\xi$  = multipath vector (24x1)

where  $N_{ant}$  = number of antennae = 4. The individual lever arm vectors to each antenna are chosen as states to avoid modelling the spatial correlations that exist between baseline vectors due to the common master antenna. The same reasoning applies to the path delays. Note that the full  $i^{\text{th}}$  path delay may not be zero, but its error is assumed to be zero mean. Let there be one multipath state per antenna per channel. The Quadrex receiver on-board RADCAL has  $N_{chan} = 6$  channels, giving rise to 24 multipath states:

$$\xi^T = \left[ \xi_1^1 \ \xi_2^1 \ \xi_3^1 \ \xi_4^1 \ \xi_1^2 \ \dots \ \xi_i^j \ \dots \ \xi_4^{N_{chan}} \right] \quad (4.36)$$

The subscript and the superscript in the above definition refer to the antenna and satellite (channel), respectively. The error state therefore consists of 49 elements total, whose time evolutions are described by the system dynamic equation, given in Equation (4.1).

The attitude dynamics result from the linearization of Euler's moment equations including the gravity gradient torque. The error torque is then due to any residual torques not modelled including higher-order gravity gradient, solar radiation, atmospheric drag, and other perturbative affects, characterized as a Markov process with a half orbital period time constant, or approximately 3000 s for RADCAL's 815 km altitude orbit. The lever arm and path delay errors are also characterized as Markov processes with time constants equal to that of the error torque. Each multipath state is also modelled as a time-correlated Markov process, with the time constant defined as the time required for the LOS to the satellite to rotate through 30 degrees with respect to the vehicle. There are then six multipath time constants, corresponding to each of the six satellites being tracked. For any one satellite, however, the multipath to the four

antennae are spatially correlated, the degree of correlation quantified by the coefficient,  $\rho_\xi$ .

It is convenient to partition the dynamics matrix as such:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F1}_{9 \times 9} & \mathbf{0}_{9 \times 40} \\ \mathbf{0}_{40 \times 9} & \mathbf{F2}_{40 \times 40} \end{bmatrix}_{49 \times 49} \quad (4.37)$$

where  $\mathbf{F1}$  depends on the attitude error dynamics, and  $\mathbf{F2}$  defines the dynamics of the lever arm, path delay, and multipath errors. These will be discussed separately in the next two sections.

### 4.3.1 Attitude Error State Dynamics

We wish to derive the form of  $\mathbf{F1}$ , which defines the dynamics of the state sub-vector:

$$\mathbf{x1} = \begin{bmatrix} \delta\theta \\ \delta\dot{\theta} \\ \delta\mathbf{T} \end{bmatrix} \quad (4.38)$$

where  $\delta\theta$  is the vector of error angular displacements about the nominal body frame axes, and  $\delta\mathbf{T}$  is the vector of error torques about those same axes and causes attitude error accelerations. Then  $\mathbf{F1}$  is:

$$\mathbf{F1} = \begin{bmatrix} \frac{\partial(\delta\ddot{\theta})}{\partial(\delta\theta)} & \frac{\partial(\delta\ddot{\theta})}{\partial(\delta\dot{\theta})} & \frac{\partial(\delta\ddot{\theta})}{\partial(\delta\mathbf{T})} \\ \frac{\partial(\delta\dot{\theta})}{\partial(\delta\theta)} & \frac{\partial(\delta\dot{\theta})}{\partial(\delta\dot{\theta})} & \frac{\partial(\delta\dot{\theta})}{\partial(\delta\mathbf{T})} \\ \frac{\partial(\delta\ddot{\mathbf{T}})}{\partial(\delta\theta)} & \frac{\partial(\delta\ddot{\mathbf{T}})}{\partial(\delta\dot{\theta})} & \frac{\partial(\delta\ddot{\mathbf{T}})}{\partial(\delta\mathbf{T})} \end{bmatrix} \quad (4.39)$$

which results from linearizing the dynamic equations given in equations (3.3) and (3.4) and repeated here:

$$\frac{d}{dt}(\omega) = \mathbf{I}_M^{-1}(\mathbf{T} - \omega \times (\mathbf{I}_M \omega))$$

$$\frac{d}{dt}(\mathbf{M}_{IB}) = \mathbf{M}_{IB}[\omega \times]$$

The derivation of the 6x9 upper block of **F1** was given in Section 3.2.1. Now only the sensitivities of the error torque need be obtained. This will be easy since the error torque is modelled as a Markov process. The dynamic equation of the error torque then takes the form:

$$\delta \dot{\mathbf{T}} = \frac{-1}{\tau_T} \delta \mathbf{T} + \mathbf{w}_T \quad (4.40)$$

where  $\tau_T$  is the half-orbital-period time constant and  $\mathbf{w}_T$  is the noise on the error torque process. The sensitivity of the error torque rate to the error torque is then simply:

$$\frac{\partial(\delta \dot{\mathbf{T}})}{\partial(\delta \mathbf{T})} = \frac{-1}{\tau_T} \mathbf{I} \quad (4.41)$$

where **I** is the 3x3 identity matrix.

We now have all nine 3x3 block matrices which comprise the attitude error dynamics matrix, **F1**, shown in Equation (4.39). In summary, this matrix is shown:

$$\mathbf{F1} = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{A}_{3 \times 3} & \mathbf{B}_{3 \times 3} & \mathbf{C}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \frac{-1}{\tau_T} \mathbf{I}_{3 \times 3} \end{bmatrix} \quad (4.42)$$

where **A**, **B**, and **C** are given in expressions (3.17), (3.18), and (3.19), respectively.

### 4.3.2 Lever Arm, Line Bias, and Multipath State Dynamics

We now focus our attention on the dynamics of the second state sub-vector, which consists of the measurement-type states: lever arm, line bias, and the multipath errors. Thus,

$$\mathbf{x}_2^T = [\delta \mathbf{l}_1^T \dots \delta \mathbf{l}_4^T \delta \beta_1 \dots \delta \beta_4 \xi^T] \quad (4.43)$$

Each of these elements are modelled as a Markov process. The time rates of change of the lever arm and line bias errors can then be expressed as:

$$\frac{\partial (\delta \mathbf{l}_i)}{\partial (\delta \mathbf{l}_j)} = -\frac{1}{\tau_l} \mathbf{I}_{3 \times 3} \delta_{ij} \quad (4.44)$$

$$\frac{\partial (\delta \beta_i)}{\partial (\delta \beta_j)} = -\frac{1}{\tau_\beta} \delta_{ij} \quad (4.45)$$

where  $\tau_l = \tau_\beta = \text{half-orbital-period}$  and  $\delta_{ij}$  is the Kronecker delta function which is 1 for  $i = j$  and 0 for  $i \neq j$  for  $i, j = 1, \dots, 4$ . Modelling the lever arm and line bias states in this way does not take into account the correlations resulting from the physical phenomenon assumed to drive these errors. The expansion caused by heating, for instance, will cause all four lever arms to expand outward in concert. So the model used here assumes a pessimistic performance in neglecting these correlations.

The final set of states, multipath, are also modelled as exponentially correlated (in time) random variables. Since we have assigned one multipath time constant for the antenna array for each satellite, the time rates of change of the multipath states for the four antennae to satellite  $i$  are given by:

$$\frac{\partial (\xi^i)}{\partial (\xi^i)} = -\frac{1}{\tau_{\xi^i}} \mathbf{I}_{4 \times 4} \text{ for } i = 1, \dots, 4 \quad (4.46)$$

The multipath time constant,  $\tau_\xi$ , is computed at each measurement time by calculating the angular velocity of the selected satellite relative to the body frame.

The dynamics matrix for these states is:

$$\mathbf{F2} = \begin{bmatrix} -\frac{1}{\tau_l} \mathbf{I}_{12 \times 12} & & & \mathbf{0} \\ & -\frac{1}{\tau_\beta} \mathbf{I}_{4 \times 4} & & \\ & & \begin{bmatrix} -\frac{1}{\tau_{\xi^1}} \mathbf{I}_{4 \times 4} & \mathbf{0} \\ & \dots \\ \mathbf{0} & -\frac{1}{\tau_{\xi^6}} \mathbf{I}_{4 \times 4} \end{bmatrix} & \\ \mathbf{0} & & & \end{bmatrix}_{40 \times 40} \quad (4.47)$$

We now have the constituents of the entire (49x49) dynamics matrix,  $\mathbf{F}$ .

### 4.3.3 Process Noise

The process noise vector,  $\mathbf{w}$ , accounts for inadequate modelling of the system dynamics. To be on the conservative side, a zero-mean Gaussian white noise process is incorporated into the attitude acceleration process, resulting in a random walk component in the attitude rate. The nominal value of the noise power spectral density for the three components of the acceleration is  $10^{-7} \text{ (deg/s}^2\text{)}^2/\text{Hz}$ , resulting in a random walk in attitude rate with a standard deviation growth of  $3.2 \times 10^{-4} \text{ (deg/s)/sqrt(s)}$ . This level of noise is chosen to give steady-state rms attitude error rates in the nominal run that are commensurate with the results given in Reference [2].

The torque, lever arm, and line bias error states are modelled as Markov processes, or exponentially correlated random variables (ECRV's). The white noise spectral density, which generates the process described by the autocorrelation function of the

ECRV in Equation (2.1), is given in terms in terms of the variance of the state,  $x$ , as [10]:

$$Q = \frac{2\sigma_x^2}{\tau}$$

The multipath error states are also assumed to behave as Markov processes, though their noise terms differ from those of the torque, lever arm, and line bias error states in that spatial correlations between the antennae are assumed to exist. To determine the form that the multipath white noise PSD will take, let us consider the multipath to two antennae from one satellite and prescribe that the steady-state covariance of these two states is of the form:

$$P_\infty = \begin{bmatrix} \sigma_\xi^2 & \rho_\xi \sigma_\xi^2 \\ \rho_\xi \sigma_\xi^2 & \sigma_\xi^2 \end{bmatrix}$$

where

$\rho_\xi$  = multipath correlation coefficient

Insert this into the matrix Riccati equation with unknown dynamics and process noise matrices:

$$\dot{P} = FP + PF^T + Q$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 2 \begin{bmatrix} f_a & f_b \\ f_b & f_a \end{bmatrix} \begin{bmatrix} \sigma_\xi^2 & \rho_\xi \sigma_\xi^2 \\ \rho_\xi \sigma_\xi^2 & \sigma_\xi^2 \end{bmatrix} + \begin{bmatrix} q_a & q_b \\ q_b & q_a \end{bmatrix}$$

Now if it assumed that each multipath state has the dynamics of a typical ECRV, both having the same time constant, then  $f_a = -1/\tau_\xi$  and  $f_b = 0$ , and the above equation can be used as a constraint to solve for  $q_a$  and  $q_b$ :



$$q_a = \frac{2\sigma_\xi^2}{\tau_\xi}, \quad q_b = \frac{2\rho_\xi\sigma_\xi^2}{\tau_\xi}$$

In our case, the process noise PSD associated with one satellite and the four antennae is a 4x4 matrix,  $\mathbf{Q}_{\xi^i}$ , with  $q_a$  for each of the diagonal elements and  $q_b$  for each of the off-diagonal elements.

The noise spectral density in full is:

$$\mathbf{Q} = \begin{bmatrix} 0 & & & & 0 \\ & \mathbf{Q}_\theta & & & \\ & & \mathbf{Q}_T & & \\ & & & \mathbf{Q}_l & \\ & & & & \mathbf{Q}_\beta \\ 0 & & & & & \begin{bmatrix} \mathbf{Q}_{\xi^1} & 0 \\ 0 & \dots & \mathbf{Q}_{\xi^6} \end{bmatrix} \end{bmatrix} \quad (4.48)$$

#### 4.4 Measurement Equation

The observation or measurement of the process occurs at discrete points in time as dictated by the linear relationship:

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$

where, for the case considered here,

$\mathbf{z}_k$  = measurement vector (3x1)

$\mathbf{H}_k$  = measurement sensitivity matrix (3x49)

$\mathbf{v}_k$  = measurement noise vector (3x1)

The measurement vector used is the difference between the observed and predicted phase differences:

$$\mathbf{z}_k = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{bmatrix} - \begin{bmatrix} \hat{\Delta\phi}_1 \\ \hat{\Delta\phi}_2 \\ \hat{\Delta\phi}_3 \end{bmatrix} \quad (4.49)$$

The predicted values are denoted with a “^”, and the subscript refers to the baseline. There will be up to six of these measurement vectors per epoch, one for each satellite in view. Note that it should be possible to take a measurement if any two antennae view a particular satellite, but for some reason RADCAL only processes measurements to a satellite if it is in view of all four antennae at once. Nonetheless, the system is overdetermined with 18 phase difference measurements per epoch.

The actual phase difference between antennae 1 (master) and i (slave) to satellite j can be expressed as:

$$\Delta\phi_{1 \rightarrow i}^j = \Delta r_{1 \rightarrow i}^j - n_{1 \rightarrow i}^j + \beta_1 - \beta_i + \xi_1^j - \xi_i^j + \nu_1^j - \nu_i^j \quad (4.50)$$

Equation (4.50) will be used in the following section to derive the sensitivity of the phase difference to each of the error states, with the integer ambiguity,  $n$ , assumed to be known. The final two terms in the expression, which comprise the measurement noise, will be addressed as well.

#### 4.4.1 Measurement Sensitivity

Let us consider just one phase difference so we need only deal with a measurement sensitivity vector:

$$\mathbf{h}^T = \frac{\partial(\Delta\phi)}{\partial \mathbf{x}} = \left[ \frac{\partial(\Delta\phi)}{\partial \theta} \quad \frac{\partial(\Delta\phi)}{\partial \dot{\theta}} \quad \frac{\partial(\Delta\phi)}{\partial \mathbf{T}} \quad \frac{\partial(\Delta\phi)}{\partial \mathbf{l}} \quad \frac{\partial(\Delta\phi)}{\partial \beta} \quad \frac{\partial(\Delta\phi)}{\partial \xi} \right]$$

or

$$\mathbf{h}^T = \begin{bmatrix} \mathbf{h}_\theta^T & \mathbf{h}_{\dot{\theta}}^T & \mathbf{h}_T^T & \mathbf{h}_l^T & \mathbf{h}_\beta^T & \mathbf{h}_\xi^T \end{bmatrix}$$

where it is understood that there are four terms each for the lever arm and line bias corresponding to each of the four antennae. We proceed to derive each term in the sensitivity vector.

Recalling the range difference from Equation (1.3):

$$\Delta r = \mathbf{b}_B^T \mathbf{M}_{BE} \mathbf{e}_E \quad (4.51)$$

where  $\mathbf{b}$  is the baseline vector in body coordinates,  $\mathbf{e}$  is the LOS in ECEF coordinates, and  $\mathbf{M}_{BE}$  is the transformation between the two frames in which is embedded the roll, pitch, and yaw from the LVLH frame. Express this true transformation in terms of a transformation to the nominal frame followed by a correction transformation:

$$\mathbf{M}_{BE} = \mathbf{M}_{B\hat{B}} \mathbf{M}_{\hat{B}E} = (\mathbf{I} - [\delta\theta \times]) \mathbf{M}_{\hat{B}E}$$

and substitute it into Equation (4.51) to get:

$$\hat{\Delta r} + \delta(\Delta r) = (\hat{\mathbf{b}}_B + \delta\hat{\mathbf{b}}_B)^T \mathbf{M}_{\hat{B}E} \mathbf{e}_E - \hat{\mathbf{b}}_B^T [\delta\theta \times] \mathbf{M}_{\hat{B}E} \mathbf{e}_E$$

The perturbation in the range difference is therefore:

$$\delta(\Delta r) = \delta\hat{\mathbf{b}}_B^T \mathbf{M}_{\hat{B}E} \mathbf{e}_E + \hat{\mathbf{b}}_B^T [(\mathbf{M}_{\hat{B}E} \mathbf{e}_E) \times] \delta\theta$$

From this expression, we can get the sensitivity to both the attitude and the lever arms in use:

$$\mathbf{h}_\theta^T = \frac{\partial(\Delta\phi)}{\partial\theta} = \frac{\partial(\Delta r)}{\partial\theta} = -[(\mathbf{M}_{\hat{B}E} \mathbf{e}_E) \times] \hat{\mathbf{b}}_B = -[(\mathbf{M}_{\hat{B}E} \mathbf{e}_E) \times] (\hat{\mathbf{l}}_i - \hat{\mathbf{l}}_1)$$

$$\mathbf{h}_{l_i}^T = \frac{\partial(\Delta\phi)}{\partial l_i} = \frac{\partial(\Delta r)}{\partial l_i} = -\mathbf{e}_E^T \mathbf{M}_{E\hat{B}} \quad \text{and} \quad \mathbf{h}_{l_i}^T = \frac{\partial(\Delta\phi)}{\partial l_i} = \frac{\partial(\Delta r)}{\partial l_i} = \mathbf{e}_E^T \mathbf{M}_{E\hat{B}}$$

where we have used the fact that the baseline of interest can be expressed in terms of the nominal and error lever arms in body coordinates:

$$\mathbf{b}_B = (\hat{\mathbf{l}}_i + \delta\mathbf{l}_i) - (\hat{\mathbf{l}}_1 + \delta\mathbf{l}_1)$$

Referring once again to Equation (4.50), we see that for the line biases:

$$h_{\beta_1} = \frac{\partial(\Delta\phi)}{\partial\beta_1} = 1 \quad \text{and} \quad h_{\beta_i} = \frac{\partial(\Delta\phi)}{\partial\beta_i} = -1$$

and for the multipath:

$$h_{\xi_1} = \frac{\partial(\Delta\phi)}{\partial\xi_1} = 1 \quad \text{and} \quad h_{\xi_i} = \frac{\partial(\Delta\phi)}{\partial\xi_i} = -1$$

and the sensitivities associated with the other antennae and other satellites are zero.

The full 24 element multipath term for antennae 1 (master) and 2 (slave) to satellite 2, for example, is:

$$\mathbf{h}_{\xi}^T = \begin{bmatrix} \mathbf{0}_{1 \times 4} & 1 & -1 & 0 & 0 & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} \end{bmatrix}$$

Finally, we note that the phase difference is not sensitive to the attitude rate nor the torque:

$$\mathbf{h}_{\dot{\theta}}^T = \frac{\partial(\Delta\phi)}{\partial\dot{\theta}} = \mathbf{0}_{1 \times 3} \quad \text{and} \quad \mathbf{h}_{\mathbf{T}}^T = \frac{\partial(\Delta\phi)}{\partial\mathbf{T}} = \mathbf{0}_{1 \times 3}$$

We now have our fully populated 49x1 measurement sensitivity vector.

#### 4.4.2 Measurement Noise

The measurement noise is assumed to be a white sequence with a known covariance structure. It is also assumed that there is no cross correlation between the measurement noise and the process noise [5, 10]:

$$E[\mathbf{v}_k \mathbf{v}_i^T] = \begin{cases} \mathbf{R}_k & i = k \\ \mathbf{0} & i \neq k \end{cases} \quad (4.52)$$

$$E[\mathbf{w}(t) \mathbf{v}_k^T] = \mathbf{0} \quad \text{for all } t \text{ and } k \quad (4.53)$$

The noise term in Equation (4.50) for the phase difference is a combination of the total phase measurement noises from the master and slave antennae. The noises on the

three phase differences will be correlated due to the common master antenna. Let us first make some assumptions about the nature of the *total* phase measurement noise:

$$E[v_i^k v_j^k] = \sigma_{v_\phi}^2 \delta_{ij} \quad (= 0 \text{ for } i \neq j)$$

$$E[v_i^j v_i^k] = \sigma_{v_\phi}^2 \delta_{jk} \quad (= 0 \text{ for } j \neq k)$$

The first indicates that the measurement noises of two antennae to the same satellite are independent, and the second states that the noises between channels are independent. Based on the first assumption, we can say that the 4x4 total phase measurement noise covariance associated with each satellite,  $\mathbf{R}_\phi$ , is diagonal:

$$\mathbf{R}_\phi = E[\mathbf{v}_\phi \mathbf{v}_\phi^T] = \sigma_{v_\phi}^2 \mathbf{I}_{4 \times 4}$$

To characterize the correlated measurement noises of the phase differences, we can use the single difference operator,  $\mathbf{SD}$ , defined as [12]:

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \mathbf{SD}\boldsymbol{\phi}$$

Then the phase difference measurement noise vector becomes:

$$\mathbf{v}_{\Delta\phi} = \Delta\mathbf{v}_\phi = \mathbf{SD}\mathbf{v}_\phi$$

and the 3x3 measurement noise covariance of the differences is:

$$\mathbf{R}_{\Delta\phi} = E[\mathbf{v}_{\Delta\phi} \mathbf{v}_{\Delta\phi}^T] = E[\mathbf{SD}\mathbf{v}_\phi \mathbf{v}_\phi^T \mathbf{SD}^T] = \mathbf{SD} \mathbf{R}_\phi \mathbf{SD}^T$$

This operation results in a covariance matrix with off-diagonal terms:

$$\mathbf{R}_{\Delta\phi} = \sigma_{v_\phi}^2 \mathbf{SD} \mathbf{SD}^T = \sigma_{v_\phi}^2 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(The matrix,  $\mathbf{SDSD}^T$ , is used in weighting the ADOP, as discussed in Section 4.2.) The sequential filter that we wish to use to process one phase difference at a time, however, requires uncorrelated measurements. We therefore need to decorrelate the measurements by diagonalizing  $\mathbf{R}_{\Delta\phi}$ , as suggested by Sullivan [26].

Any  $n \times n$  symmetric matrix,  $\mathbf{A}$ , can be diagonalized into a matrix,  $\mathbf{D}$ , by constructing a matrix,  $\mathbf{U}$ , the columns of which are the eigenvectors of  $\mathbf{A}$  and are linearly independent. The non-zero elements of  $\mathbf{D}$  are the eigenvalues of  $\mathbf{A}$ . The matrix  $\mathbf{A}$  can then be diagonalized as follows [25]:

$$\mathbf{U}^{-1}\mathbf{AU} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \dots \\ & & & \lambda_n \end{bmatrix}$$

where the  $\lambda_i$ 's are the eigenvalues. Unit eigenvectors give an orthonormal matrix  $\mathbf{U}$ ; therefore,  $\mathbf{U}^{-1} = \mathbf{U}^T$ . Using this logic, we can diagonalize  $\mathbf{R}_{\Delta\phi}$ :

$$\mathbf{U}^T\mathbf{R}_{\Delta\phi}\mathbf{U} = \begin{bmatrix} \alpha_1^2 & 0 & 0 \\ 0 & \alpha_2^2 & 0 \\ 0 & 0 & \alpha_3^2 \end{bmatrix}$$

Decorrelating the phase difference measurement noises in this way has altered the measurement equation. Since

$$\mathbf{U}^T\mathbf{R}_{\Delta\phi}\mathbf{U} = E[\mathbf{U}^T\mathbf{v}_{\Delta\phi}\mathbf{v}_{\Delta\phi}^T\mathbf{U}],$$

the new vector of uncorrelated phase difference measurements is:

$$\mathbf{z}' = \mathbf{U}^T\mathbf{z} = \mathbf{U}^T\mathbf{H}\mathbf{x} + \mathbf{U}^T\mathbf{v}_{\Delta\phi}$$

and each scalar measurement can now be processed sequentially by realizing that the new measurement sensitivity vectors,  $\mathbf{h}'^T$ , are the rows of  $\mathbf{U}^T\mathbf{H}$ . The 3x3 modal

matrix,  $U$ , can be precomputed since the measurement statistics do not change. Processing scalar measurements, as opposed to incorporating the three correlated phase differences in batch, results in about a 50% reduction in computation time for six-second propagation and measurement intervals.

#### **4.5 Simulation Overview and Implementation Issues**

The flow diagram shown in Figure 4.4 illustrates the major steps in the covariance analysis. First, the initial conditions are set. These include definition of the nominal trajectory, which is a circular orbit and a local vertical attitude hold. The spacecraft is nominally in a polar orbit and starts at zero degrees latitude. Also initialized are the error covariance, process noise spectral density, and measurement noise covariance matrices.

The GPS constellation and the spacecraft states are then propagated to the current time. The GPS satellite positions are found by interpolating over the precise ephemerides made available by the International GPS Service, rather than from a simulated GPS constellation. The precise ephemerides files contain the ITRF-93 positions, accurate to within 10 cm, of all active satellites at regular time intervals. This high degree of position accuracy is not required for attitude determination (e.g., only a three degree error in phase difference, about 1.6 mm, over a one meter baseline results from a position error of 30 km). The precise ephemerides are used in anticipation of filtering the actual flight data and are easier to manipulate than the broadcast ephemerides in RINEX format. The file obtained for this investigation corresponds to GPS week 708, day 6, or August 7, 1993, that of the RADCAL flight data. (See Appendix B for discussion of RADCAL data.) Calculating the user satellite state is trivial for the circular orbit and LVLH attitude.

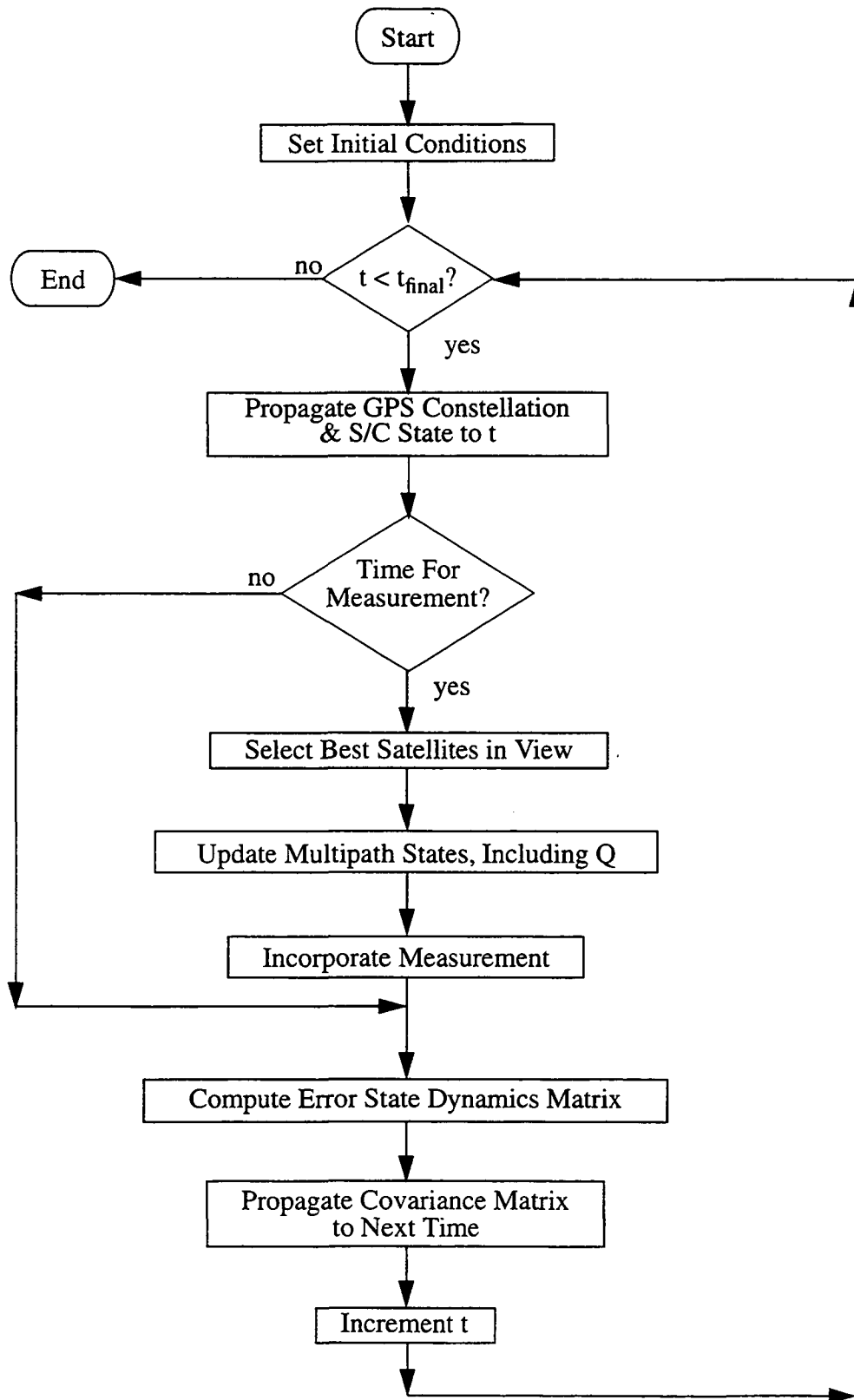


Figure 4.4 Simulation Flow Diagram



If it is time for a measurement, the best group of satellites are selected based on either GDOP or ADOP, described earlier. GDOP is the default criterion since RAD-CAL's best view is defined in this way. The multipath states in both the covariance and process noise spectral density matrices are then updated. If a satellite in view has switched or dropped out since the last measurement time, the multipath elements corresponding to that channel must be reinitialized. The 4x4 block diagonal matrix in the covariance matrix is set to its initial, steady-state value as given in Section 4.3.3, and the correlations with all other states are zeroed. If the satellite was replaced, the new multipath time constant is calculated from the relative angular velocity. If the satellite dropped out and was not replaced, the multipath variances for that channel will remain at their steady-state values, since no measurements are taken. The noise spectral density associated with that channel must also be updated, since it depends on the time constant.

With the satellites selected and the multipath states updated, the measurement sensitivity and thus the weighting, optimal or otherwise, can be calculated. The measurement can then be incorporated into the total covariance matrix via Equation (4.23).

The dynamics matrix given in Section 4.3 is computed, and the covariance is then propagated forward to the next time. The propagation of the total error covariance matrix as described in Section 4.1.1 is accomplished in this investigation via fourth-order Runge-Kutta integration of the Ricatti equation. The dynamics matrix is assumed to be constant over the integration time - a small time step compared to the system dynamics is thus required. Upon propagation, the process repeats until the designated final time.

Some of the implementation issues involved in performing the error budget are discussed in Appendix A.

## Chapter 5

### Simulation Results

The following sections present the results obtained from the simulation outlined in the previous chapter. A nominal run is defined and its results are discussed. The performance sensitivity to various parameters such as orbital inclination, satellite selection method, and noise levels are examined, as well. Finally, the performances of two optimal-suboptimal filters are discussed.

#### 5.1 Nominal Run Description

Table 5.1 shows the nominal run assumptions consistent with the RADCAL system. The error terms are, of course, only a best estimate and will be varied in a later section to determine performance sensitivity to these assumptions. States with associated time constants ( $\tau_x$ ) are modelled as Markov processes. The performance results presented in this section are fully optimal and will show what might be possible onboard given enough computation power. They will provide insight into system behavior.

**Table 5.1 Nominal Run Conditions**

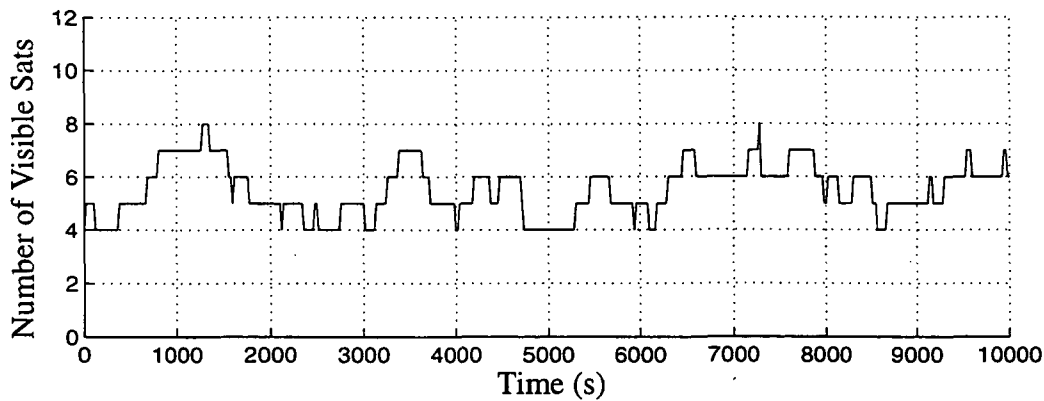
| Parameter                      | Condition                               |
|--------------------------------|-----------------------------------------|
| Orbital Altitude (circular)    | 815 km                                  |
| Orbital Inclination            | 90.0°                                   |
| Attitude Hold                  | LVLH                                    |
| Antenna Baseline Configuration | 3 Baselines as described in Section 1.4 |
| Antenna Half-Cone Angle        | 85.0°                                   |
| Antenna Boresite Cant Angle    | 17.5°                                   |
| Satellite Selection Criterion  | GDOP                                    |

**Table 5.1 Nominal Run Conditions**

| Parameter                                               | Condition                                                |
|---------------------------------------------------------|----------------------------------------------------------|
| Principal Moments of Inertia                            | 26.400, 26.400, 5.813 kg m <sup>2</sup>                  |
| Initial Attitude, Attitude Rate Uncertainty (each axis) | 5.0°, 0.1°/s                                             |
| Torque Error                                            | $\sigma_T = 5.0 \times 10^{-7}$ Nm,<br>$\tau_T = 3035$ s |
| Lever Arm Error                                         | $\sigma_l = 2$ mm, $\tau_l = 3035$ s                     |
| Line Bias Error                                         | $\sigma_\beta = 2$ mm, $\tau_\beta = 3035$ s             |
| Multipath                                               | $\sigma_\xi = 5$ mm, $\tau_\xi \approx 500$ s            |
| Multipath Correlation                                   | 50%                                                      |
| Attitude Error Rate Process Noise PSD                   | $10^{-7}$ (°/s) <sup>2</sup> /s                          |
| Total Phase Error                                       | 3.0°                                                     |
| Propagation Step Size & Time Between Measurements       | 6 s                                                      |

The nominal conditions in the orbital altitude and inclination, measurement frequency, antenna baseline configuration, antenna boresite cant angle, satellite selection criterion, and principal moments of inertia are all chosen to be consistent with those of RADCAL. The 85° antenna half-cone angle chosen for the baseline case is based on GPS satellite visibility. A satellite is considered in view if it is visible to all four antennae simultaneously. Figure 5.1 reveals the simulated visibility for the 85° half-cone angle to be comparable to its real counterpart. See Figure B.2 in appendix for the number of satellites in RADCAL's best view. At least four satellites will be in view more than 90% of the time in both cases. The simulated spacecraft is over the poles at  $t = 1500$  s and 4500 s, respectively.

The nominal attitude is chosen to be the ideal LVLH orientation of a gravity gradi-



**Figure 5.1 Simulated GPS Satellite Visibility: 85° Half-Cone Angle**

that the craft can be off by 20 degrees in roll and pitch, and that the yaw grows secularly in time [2, 8, 28]. The LVLH assumption will do for the purposes of linear covariance analysis. (Not to imply that the performance is insensitive to attitude.) The assumed nominal torque error corresponds to 5% of the modelled gravity gradient torque arising from 5° roll and pitch angles using the inertias shown.

The relative antenna movement and changes in electrical path length, which introduce errors in the attitude solution, are attributed to temperature variations. In assigning values for the lever arm and line bias error processes, the following simplistic argument is made. The change in length of the lever arm,  $\Delta l$ , is a function of the temperature change,  $\Delta T$ , and the material's thermal expansion coefficient,  $\alpha$ :

$$\Delta l = \hat{l} \alpha (\Delta T)$$

where  $\hat{l}$  is the nominal length. The temperature of a typical spacecraft ranges from -34°C to 71°C [31]. Assuming  $\alpha = 20\text{E-}6 \text{ m}/(\text{m } ^\circ\text{C})$ , representative of aluminum, this temperature difference results in a change in length of two parts per thousand, or 2 mm for a 1 m lever arm. This conservative number is used for the lever arms here, which are less than a meter. The steady-state line bias rms error is also assumed to be 2 mm in the nominal case.

The characteristic multipath level is assumed to be 5 mm, as suggested by Cohen, Lightsey, and Parkinson [8, 18], and its correlation between antennae is nominally assumed to be 50%.

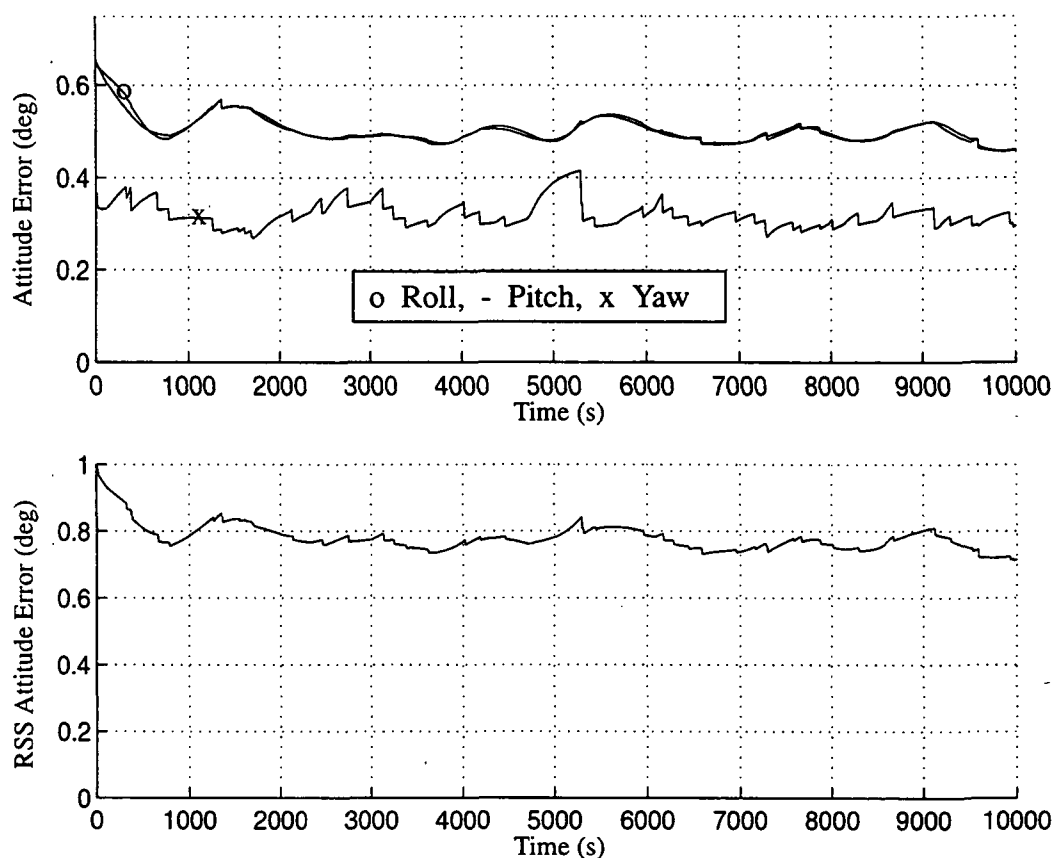
Though less than 1 mm of phase measurement error has been suggested for RADCAL's patch antennae [8], a slightly more conservative figure of  $3^\circ$  or about 1.6 mm is used, instead, in an attempt to absorb the error induced by the antenna's asymmetric reception pattern.

The step size shown is the average time between RADCAL measurements. In general, an appropriate integration step size is  $1/100^{\text{th}}$  of the minimum period of the system dynamics or smaller. Given the slow dynamics of the gravity gradient stabilized spacecraft as seen in Chapter 3, it would seem reasonable to assume that the attitude error dynamics can be considered constant over an integration time step up to 30 - 40 s. The torque, lever arm, and line bias error dynamics behave on a similar time scale. The multipath, however, as modelled here, operates on a smaller time scale (higher dynamics) with a time constant on the order of 500 s. Results from runs with various step sizes, not shown here, reveal no significant difference (0.01% maximum difference in variances over 10000 s) in performance between time steps from 0.1 to 6 s. The dynamics matrix is therefore safely assumed to be constant over the nominal 6 s time step. This is also the maximum time step possible because measurements normally occur at 6 s intervals.

## **5.2 Nominal Run Results**

### **5.2.1 Attitude Error and Error Rate**

Figure 5.2 shows the post-measurement rms errors in roll, pitch, and yaw and their root-sum-squares (RSS) for a 10000 s interval, more than an orbital period. The stan-



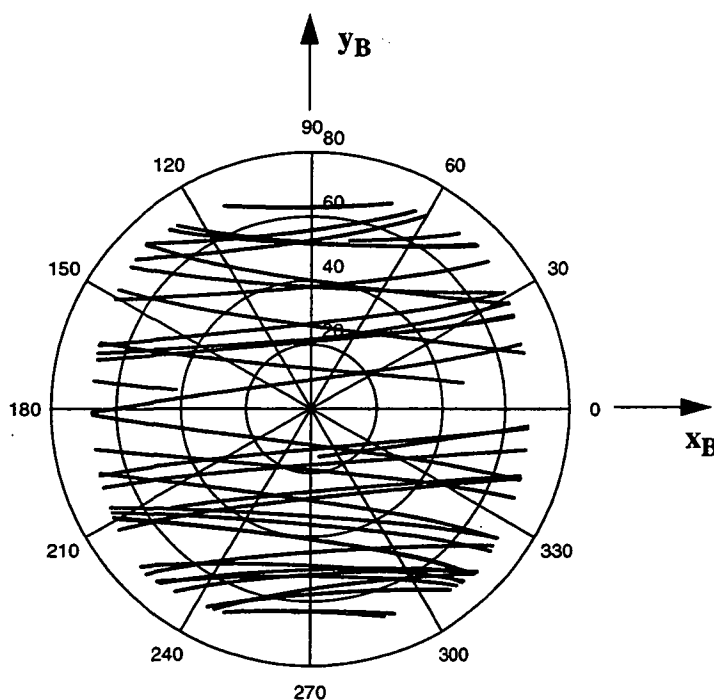
**Figure 5.2 Attitude Errors for the Nominal Run**

dard deviations in the roll, pitch, and yaw jump quickly from their initial 5 degrees to nearly their “steady-state” values, limited by process and measurement noises.

Notice from Figure 5.2 that the yaw angle is the most observable (i.e., its error is smallest), whereas the roll and pitch angles are less observable and have very similar error characteristics. This can be explained geometrically as a result of the fact that the three baselines lie in the horizontal plane. With this geometry, satellites directly overhead are best for determining roll and pitch, while satellites in the horizontal plane are best for determining yaw (disregarding antenna limitations and susceptibility to multipath, of course).

Clearly, a conflict exists in choosing the best satellites for yaw determination versus those for roll and pitch determination. By means of ADOP it is possible to resolve this conflict in an optimal way. Nevertheless, roll and pitch estimates tend to suffer relative to yaw, as illustrated in Figure 5.2, simply because there are less overhead satellite opportunities.

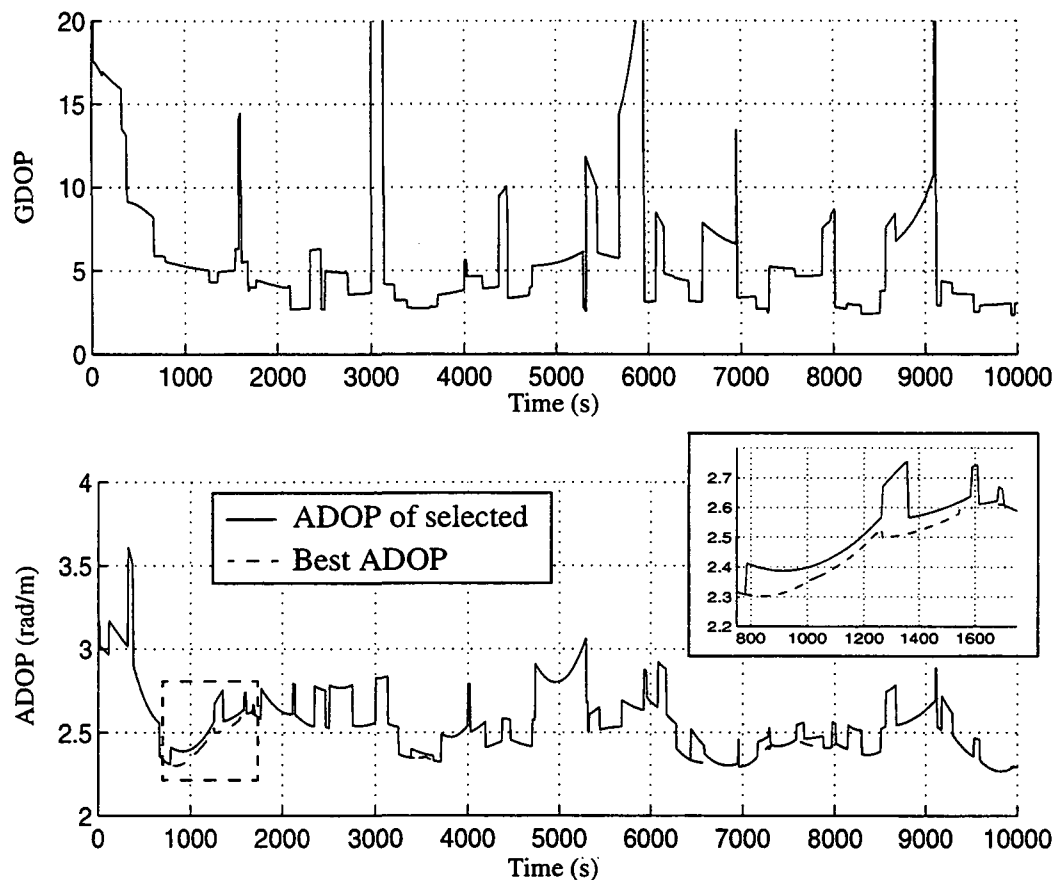
Figure 5.3 shows a sky map for the 10000 s of the nominal run. The arcs are the paths of the GPS satellites in view when looking to the zenith. Vehicle motion is in the +x direction, which is at 0 degrees azimuth, so the satellites move from right to left across the map. The +y direction is at 90 deg, and the +z direction (nadir) comes out of



**Figure 5.3 Satellite Motion (Azimuth & Angle from Zenith) Sky Map**

the page. The radial component on the plot is the angle from zenith. Note that for this time interval in a polar orbit, no satellites can be seen directly overhead. The visibility “cone” is seen from the plot as being somewhat squared-off as a result of the outwardly canted antennae.

The ragged appearance of the plots in Figure 5.2 indicate that the attitude errors may be at the mercy of the satellite geometry. To corroborate this possibility, two measures of satellite geometry, GDOP and ADOP, are shown in Figure 5.4. Two curves are



**Figure 5.4 GDOP and ADOP for Nominal Run**

shown in the ADOP chart - one is the ADOP calculated for the geometry of the satellites selected using the GDOP criterion, and the other is the ADOP using the ADOP criterion to select the best satellites. Note the units of ADOP: to get the instantaneous attitude uncertainty, the phase uncertainty is first converted to meters then scaled by the ADOP to give radians, then converted to degrees.

A glance at Figure 5.1 reveals that between 800 and 1500 s, more than six satellites are in view. Looking to the insert in the above figure at the same time interval, we see



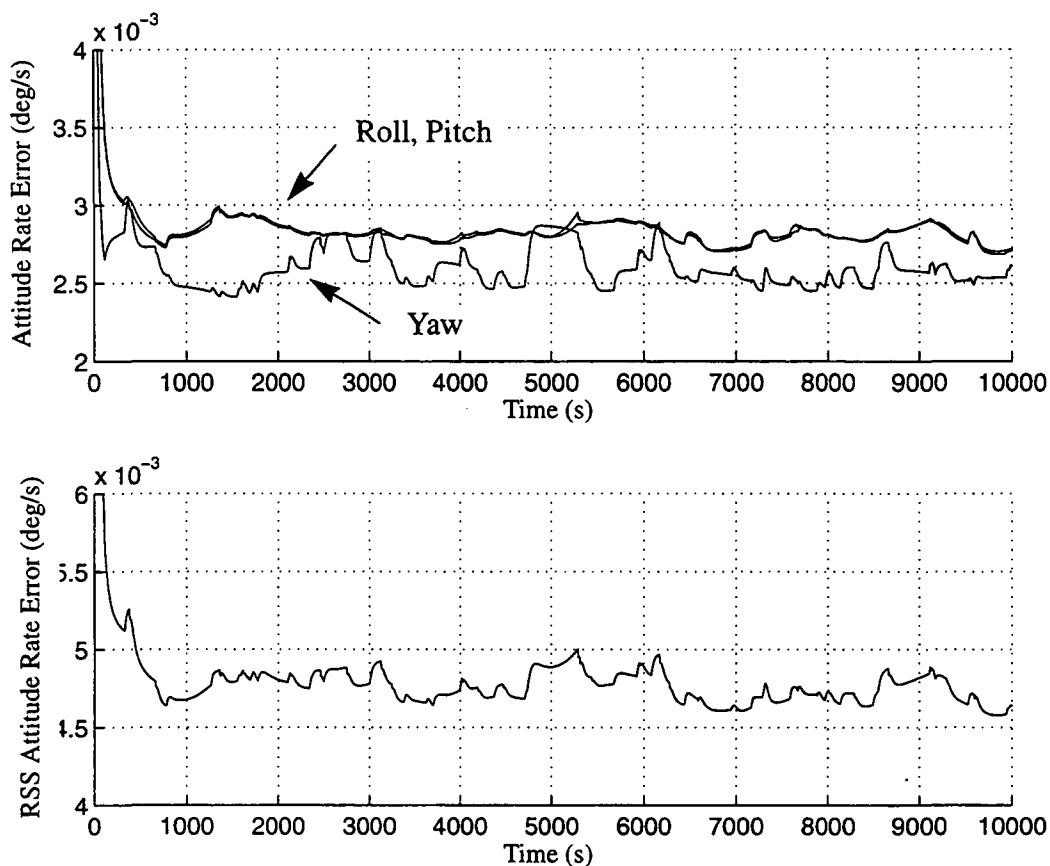
that use of the two criteria result, not surprisingly, in different combinations of the “best” six satellites, as evidenced by the discrepancy in the ADOP’s. The implications of the two figures-of-merit will be discussed further in Section 5.3.2. A comparison of Figure 5.2 and Figure 5.4 shows that the attitude errors tend toward the ADOP, which represents the “steady-state” condition at any given time, dependent upon instantaneous geometry.

Since the evolution of the attitude errors depend largely on that of the satellite geometry, perhaps it is more appropriate to cite the averages, minima, and maxima of the “steady-state” rms errors over, say, an orbital period, rather than the final uncertainties. Table 5.2 shows these values over the final 6000 s of the nominal run. The maximum values in each component do not occur at the same epoch. For each component, the average error as given here is actually the square root of the average of the post-measurement variances over the 6000 s.

**Table 5.2 Attitude Errors Over Final 6000 s**

| Component | Min RMS Error (deg) | Max RMS Error (deg) | Average RMS Error (deg) |
|-----------|---------------------|---------------------|-------------------------|
| Roll      | 0.457               | 0.536               | 0.496                   |
| Pitch     | 0.457               | 0.533               | 0.495                   |
| Yaw       | 0.271               | 0.415               | 0.317                   |

The attitude error rates and the magnitude, shown in Figure 5.5, exhibit similar behaviors to the attitude errors in that the yaw component is generally smaller than the roll and pitch components, which are very much alike. Note that the attitude rate is not directly observable from the phase differences, but rather is inferred from the attitude itself.



**Figure 5.5 Attitude Error Rates of the Nominal Run**

### Closer Inspection of Attitude RMS Errors

Let us examine the evolution of the errors in greater detail. In Figure 5.6 are close-ups of the attitude rms error, its magnitude, and the ADOP for the selected satellites for the first 1000s of the nominal case. The Roll, Pitch, and Yaw Dilutions Of Precision, are also shown in the figure, indicated as RDOP, PDOP (not to be confused with Position Dilution Of Precision), and YDOP, respectively. These component-wise DOP's are simply the square roots of the diagonal components of  $(\mathbf{H}^T \mathbf{H})^{-1}$ , i.e., their RSS is the ADOP. They represent the rms errors in roll, pitch, and yaw that would be obtained if only the measurements were used (no *a priori* knowledge) and the measurement uncertainty was unity. Inspection of the six events labeled on the plot will

provide more insight into the behavior of the attitude errors. Table 5.3 is a list of the descriptions of the satellite changes causing the events.

**Table 5.3 Description of Events Shown in Figure 5.6**

| Event Number | Event Description                                                        |
|--------------|--------------------------------------------------------------------------|
| 1            | Number of satellites in view increases from 4 to 5                       |
| 2            | Number of satellites in view decreases from 5 to 4                       |
| 3            | Satellite switches - 4 satellites still in view, better GDOP, worse ADOP |
| 4            | Number of satellites in view increases from 4 to 5                       |
| 5            | Number of satellites in view increases from 5 to 6                       |
| 6            | Satellite switches - 6 satellites still in view, better GDOP, worse ADOP |

Obviously, the addition of another usable satellite (usable in that we are not required to throw another satellite away) brings only a better knowledge of each state, as seen in events 1, 4, and 5. One would expect attitude accuracy to suffer when a satellite drops out of view (event 2), as indicated by the ADOP. The yaw error does in fact start to increase, albeit slowly because of filter memory, as it tries to track the YDOP, but the roll and pitch errors continue to decrease. The satellite which dropped out of view was on the horizon, and as previously mentioned, the phase differences to this lower-elevation satellite are fairly insensitive to roll and pitch, but very sensitive to yaw; the roll and pitch DOP's are not affected as strongly by the loss and continue to improve after it. The behavior of the roll and pitch rms errors reflects this insensitivity.

Some unexpected results arise from a satellite switch as in events 3 and 6. In each case, the GDOP of the new combination is smaller than that of the old combination, but the ADOP is larger. Each component of the ADOP is larger in event 3 as well, and yet the attitude uncertainties are seen to decrease sharply, more noticeably in yaw.

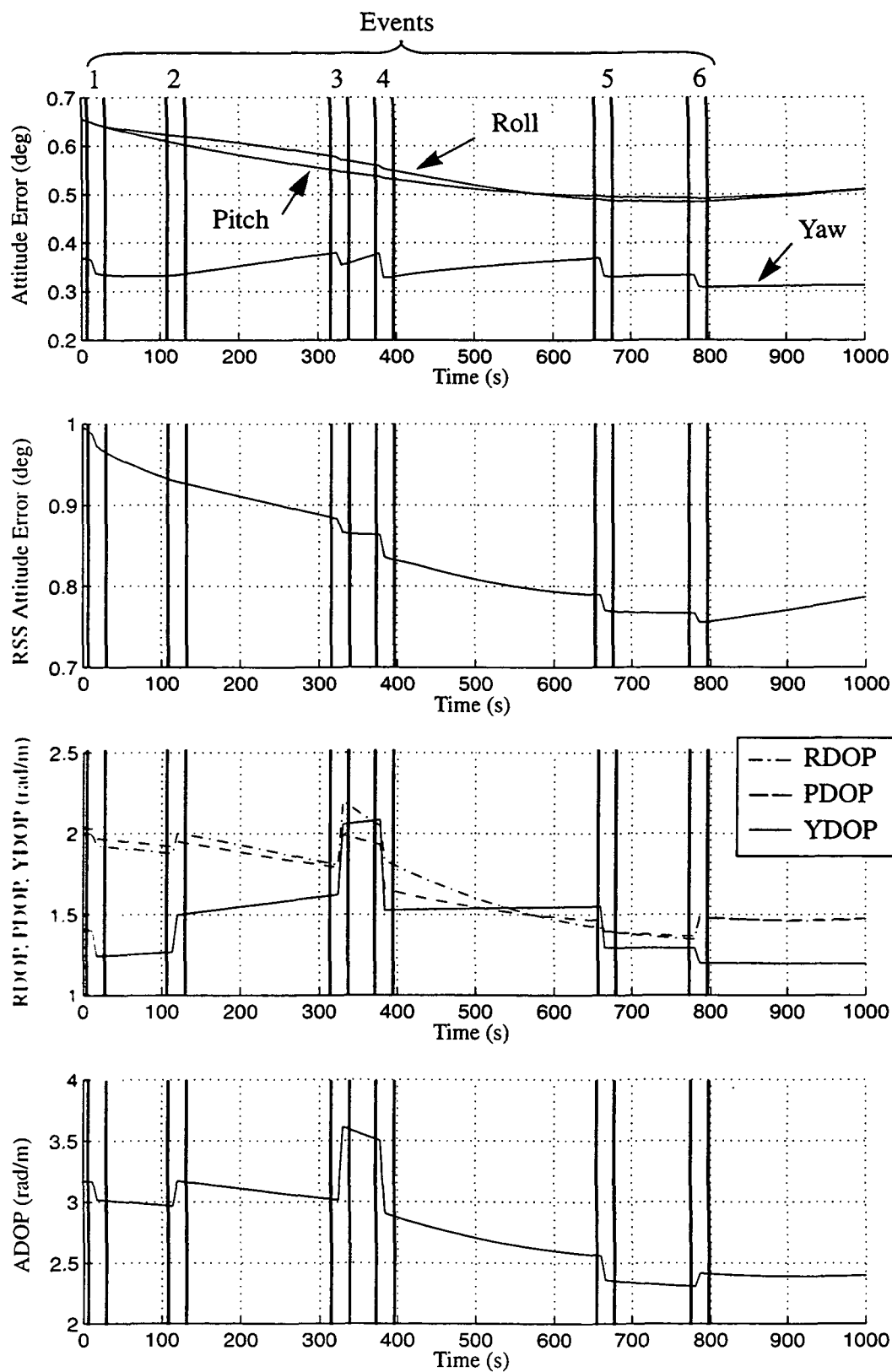
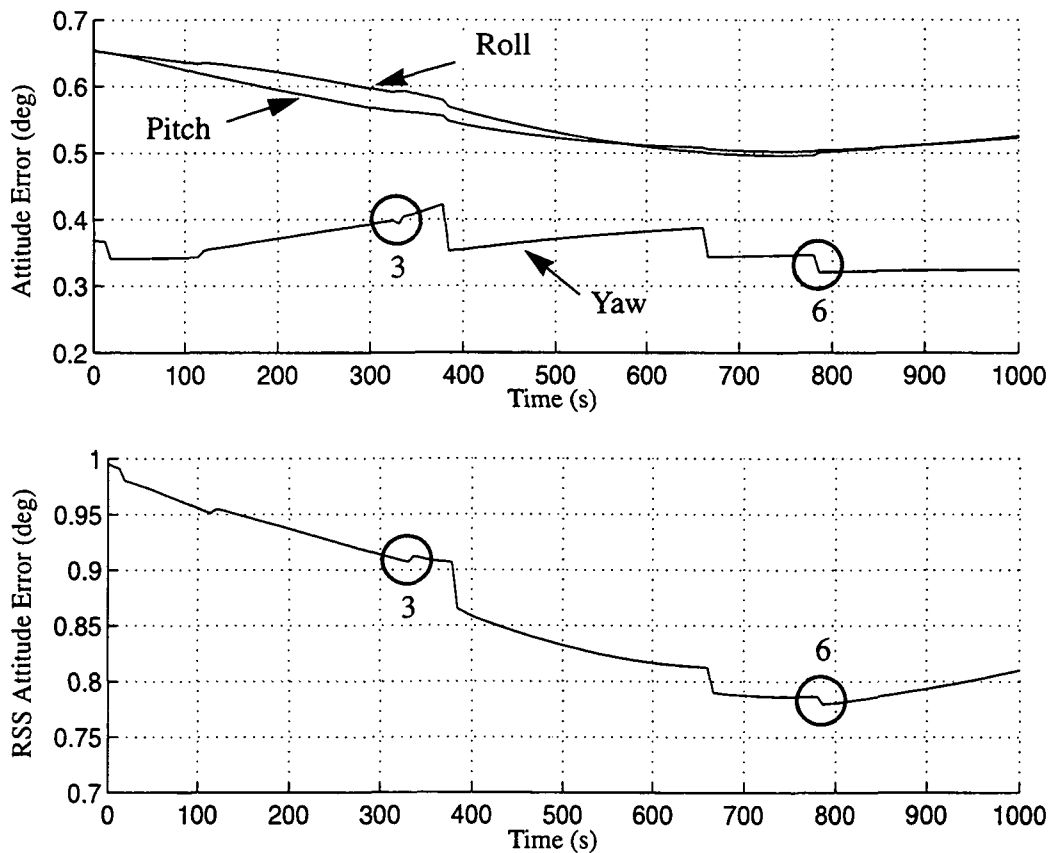


Figure 5.6 Attitude Errors and ADOP for 1000s of Nominal Run

These decreases at the switches are due not to the filter taking advantage of the new geometry with the memory of the previous geometry, but rather the correlation, or more appropriately, the decorrelation of the attitude with the multipath state at the switch. Up to the time of the switch, correlations build between the attitude error and the multipath - the uncertainties in each state will decrease to a point, but the two states can not be separated because of these correlations. When a satellite switch occurs, a reinitialization takes place - the 4x4 matrix in the covariance corresponding to the multipath from the new satellite to the four antennae is set to its initial, steady state, and the correlations between those multipath states and all other states are zeroed. The effect of this decorrelation is to temporarily add new information to the system, and the immediate result is a smaller uncertainty in both states. Correlations then build up again, and the uncertainties again tend toward a level consistent with the ADOP. The reason we do not see a subsequent increase is that the noise driving the attitude process is too small to be seen over short time intervals. One might contend with the realism of the model and could argue that a new satellite should not come in uncorrelated with the system, but this was the assumption made.

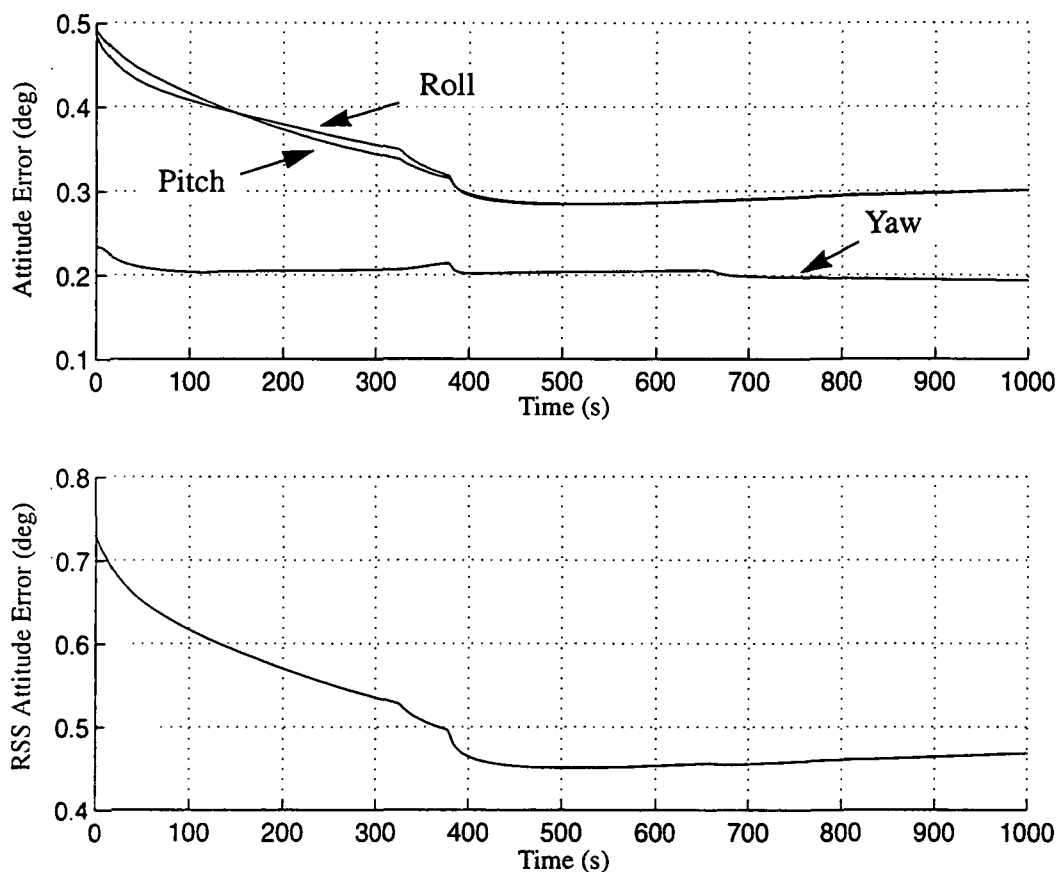
The nature of the interaction between the attitude and multipath states can be substantiated by examining some more cases. A larger amount of process noise feeding into the attitude causes the filter to weight the measurements more heavily, and as the noise level increases, the error magnitude approaches the ADOP scaled by the measurement uncertainty. Figure 5.7 shows the attitude errors resulting from a larger attitude rate process noise spectral density,  $10^{-3} \text{ (deg/s)}^2/\text{s}$ . The dip associated with the switch at event 3 is still visible in yaw, but the errors are tending more toward the ADOP than in the nominal run case with a smaller rate noise. A satellite switch also occurs at event 6, but note that the errors only slightly increase in roll and pitch. In



**Figure 5.7 Attitude Errors With Larger Attitude Rate Process Noise**

yaw, however, the uncertainty remains at the lower level, consistent with the lower YDOP after the switch. The reason that the decorrelation apparently has a greater impact on the yaw component has yet to be determined.

Figure 5.8 shows the effect of eliminating multipath from the state by zeroing its uncertainty. We no longer see the sudden decrease at the satellite switch. The yaw error is seen to increase with the switch at 3, while the roll and pitch errors decrease. Observing the trend in the DOP's in the previous discussion and since the multipath effect has been removed, these phenomena must be due to the lever arm and line bias states.



**Figure 5.8 Attitude Errors With Multipath Eliminated**

We conclude that the simulation results concerning the attitude-multipath correlation are consistent with the assumptions made in the model, and that the unrealistic phenomena shown in the plots are not due to a programming flaw. The “problem” lies with the model itself, which assumes that: 1) multipath is a zero-mean process, and 2) the multipath associated with a newly introduced satellite is not correlated with the attitude. Taken over an extended period of time, perhaps it is zero-mean in reality, but at any one point, multipath is a bias. This bias is entirely dependent on the local reflective environment and the signal’s direction of incidence with respect to that environment. In other words, real-world multipath associated with a newly introduced satellite is in fact correlated with the attitude, contrary to the model’s assumption.

### 5.2.2 Torque Errors

Recall from Section 4.4.1 that the phase difference measurement is insensitive to the torque, as well as to the attitude rate. Figure 5.9 below illustrates this insensitivity. The GPS measurements do not measure torque directly - it must be inferred from correlations that have built up. Since the torques are small, the correlations are small, and the observability is low. After 10000 s the torque rms error magnitude decreases by less than 5%. But again, like the attitude error and error rate, the yaw component is slightly more observable than the torques about the roll and pitch axes.

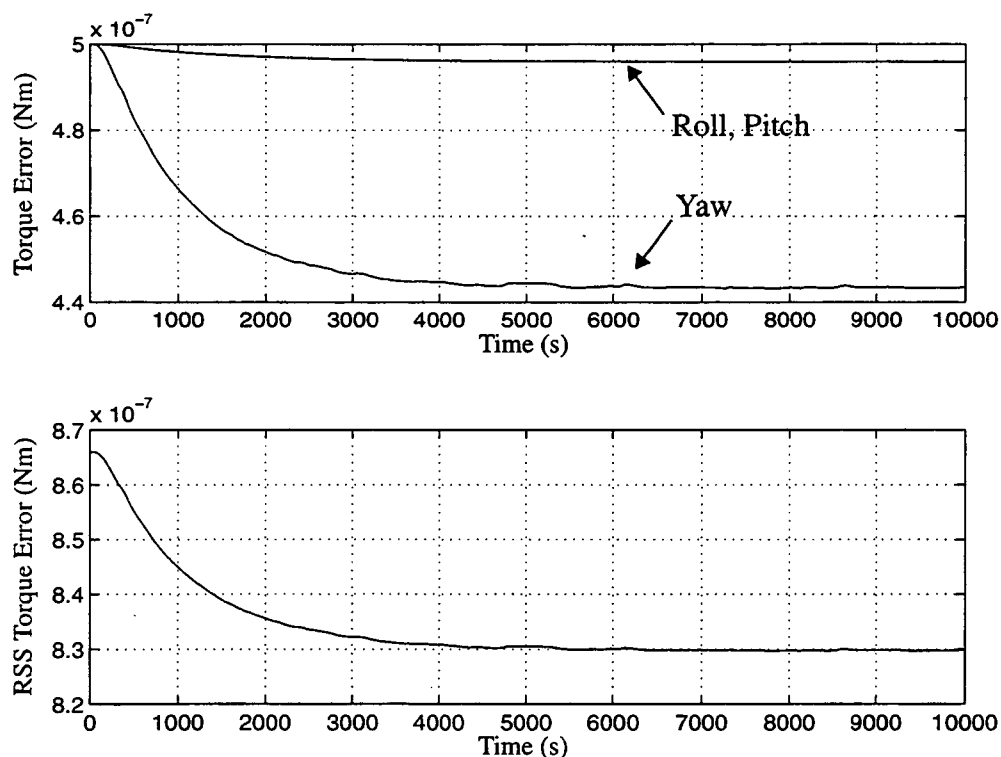


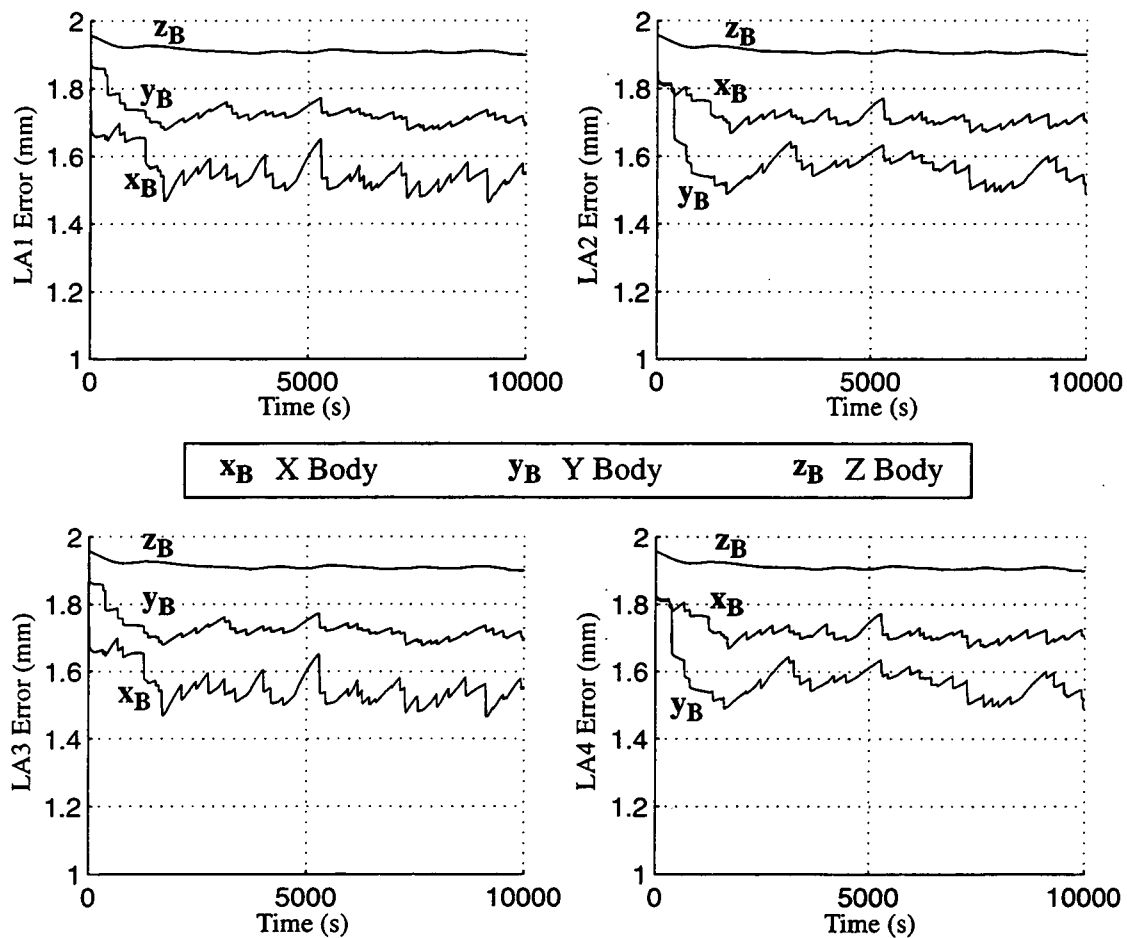
Figure 5.9 Torque Errors of the Nominal Run

### 5.2.3 Lever Arm and Baseline Errors

The filter has a difficult time distinguishing one lever arm from another given the phase difference measurement between the antennae. Figure 5.10 illustrates this diffi-



culty by showing the component errors for each of the four lever arms, denoted LA1, LA2, LA3, and LA4. The z components of all lever arm errors are equally difficult to observe - note that they are all the same and larger than the x and y components. This outcome is consistent with the results of the attitude, attitude rate, and torque error states. All four antennae nominally lie in the body x-y plane (shown in Figure 1.2), which nominally coincides with the local-horizontal plane. A flexure in the z direction for lever arms 1 and 3, which are aligned respectively with the -x and +x directions, is



**Figure 5.10 Nominal Run Errors of the Four Lever Arms**

equivalent to a rotation about the y axis, or a pitch. Similarly, a flexure in the z direction for lever arms 2 and 4, which are aligned with the -y and +y directions, is tanta-

mount to a rotation about the x axis, or a roll. These apparent rolls and pitches are not easily estimated for the reason given in Section 5.2.1 and are difficult to distinguish from actual vehicle roll and pitch.

Another item of note is that the component along the lever arm axis in each case (e.g., the x component of LA1) has a smaller rms error than the other two components. The phase difference is insensitive to changes in baseline length when the LOS is perpendicular to that baseline. There is a one-to-one correspondence between a change in length and a change in phase difference when the LOS is coincident with the baseline axis. The plots indicate that the lever arm length is more observable than its misalignment, which is reasonable since misalignment of the lever arm can be misinterpreted as vehicle rotation.

We are more interested, however, in the errors over the baselines, not the lever arms. Estimating the lever arm to each antenna was merely a convenience to avoid modelling the correlations between baseline flexures due to the common reference. The baseline 1 error is the error in lever arm 2 with respect to the error in lever arm 1:

$$\mathbf{b}_1 = \mathbf{l}_2 - \mathbf{l}_1 \Rightarrow \delta \mathbf{b}_1 = \delta \mathbf{l}_2 - \delta \mathbf{l}_1$$

The baseline error covariance is, by definition, the expected value of the outer product of the baseline errors:

$$\mathbf{P}_{\mathbf{b}_1} = E[\delta \mathbf{b}_1 \delta \mathbf{b}_1^T] = E[(\delta \mathbf{l}_2 - \delta \mathbf{l}_1)(\delta \mathbf{l}_2 - \delta \mathbf{l}_1)^T]$$

Expanding the right side in terms of the lever arm error covariances gives:

$$\mathbf{P}_{\mathbf{b}_1} = E[\delta \mathbf{l}_1 \delta \mathbf{l}_1^T] + E[\delta \mathbf{l}_2 \delta \mathbf{l}_2^T] - E[\delta \mathbf{l}_1 \delta \mathbf{l}_2^T] - E[\delta \mathbf{l}_2 \delta \mathbf{l}_1^T]$$

The above formulation gives the error covariance in the body frame, but it may be more illustrative to transform to a local baseline frame - one in which the coordinate axes are along and perpendicular to the baseline axis. Since the three baselines are in

the x-y plane, the transformation is simply about the z axis through the angle,  $\alpha$ , from the body frame (subscript "B") to the local baseline frame (subscript "A"):

$$\mathbf{b}_A = \mathbf{T}_{AB} \mathbf{b}_B$$

where

$$\mathbf{T}_{AB} = \mathbf{T}_z(\alpha)$$

The covariance in the A frame is then:

$$\mathbf{P}_{b_A} = E[\delta \mathbf{b}_A \delta \mathbf{b}_A^T] = E[\mathbf{T}_{AB} \delta \mathbf{b}_B \delta \mathbf{b}_B^T \mathbf{T}_{AB}^T] = \mathbf{T}_{AB} \mathbf{P}_{b_B} \mathbf{T}_{BA}$$

For baselines 1, 2, and 3,  $\alpha = -45, 0, +45$  degrees, respectively. The square roots of the diagonal elements of the resulting three baseline error covariance matrices are shown in Figure 5.11.

Note that for each of the three baselines, the uncertainty is smallest in the "along-track" component, that is, the component along the axis of the respective nominal baseline. The "cross-track," or transverse, component in each baseline, in the direction perpendicular to the baseline in the horizontal plane, is somewhat less observable. These  $y_A$  components, as shown on the plots, are difficult to distinguish from yaw motion. Notice the one-sided saw-tooth characters of the along- and cross-track components, similar to the profile of the yaw error itself - a reasonable result since these two components are in the plane defined by yaw rotation. The  $z_A$  component, which is in the nadir direction, is seen to be the least observable for all three baselines. This component in baselines 1 and 3 can be construed as a combination of roll and pitch motions, while for baseline 2, this vertical component appears indistinguishable from vehicle pitch.

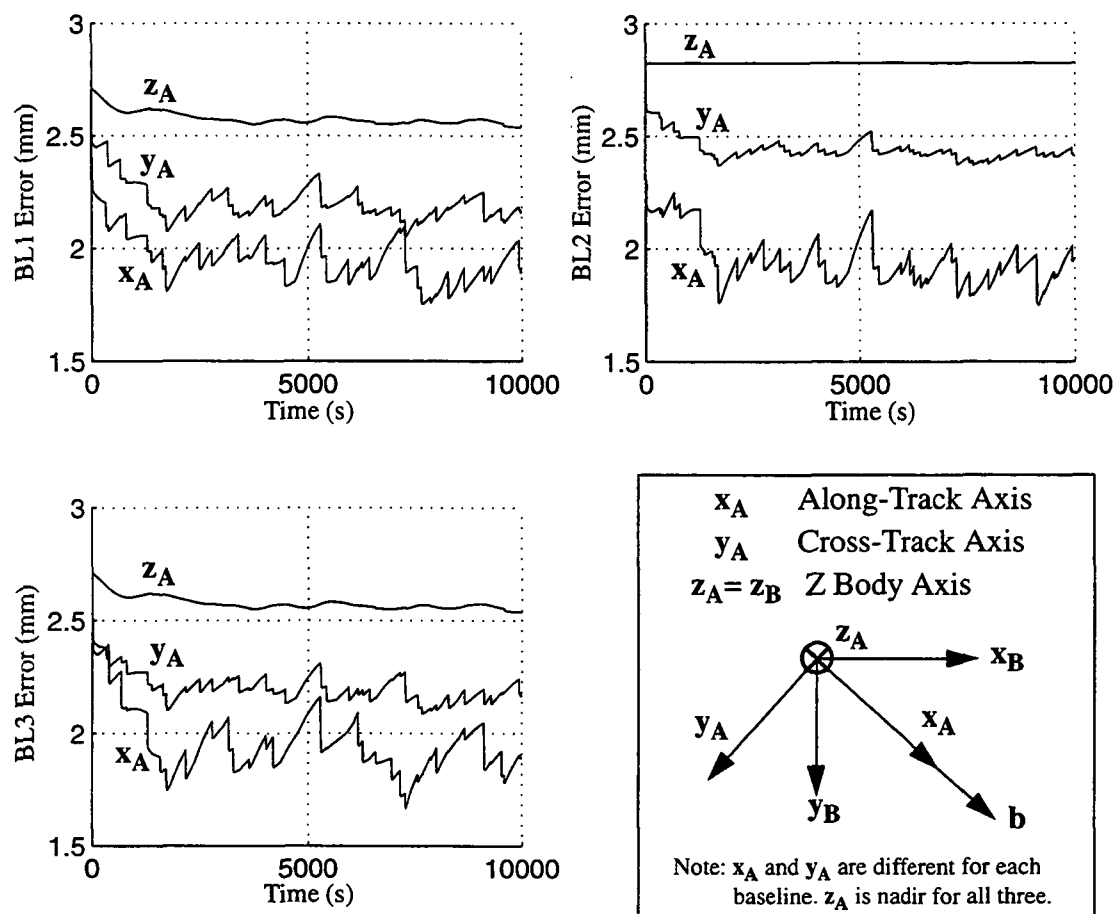
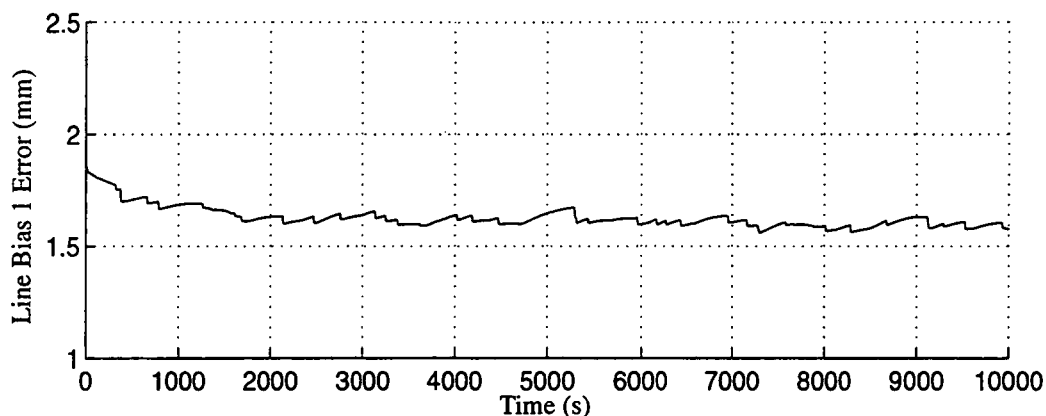


Figure 5.11 Nominal Run Baseline Errors

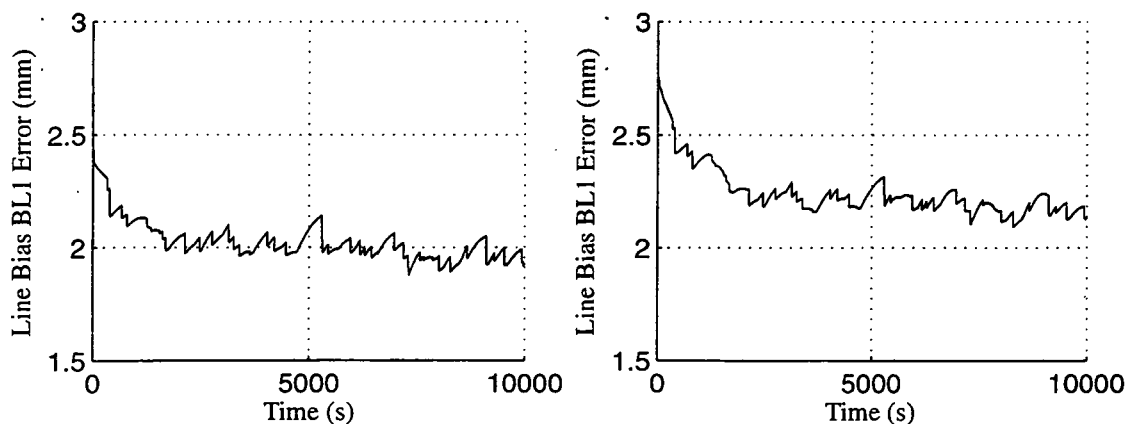
## 5.2.4 Line Bias Errors

The line bias errors associated with each antennae are indistinguishable, similar to the lever arm errors. Figure 5.12 shows that the line bias rms error for antenna 1 hangs up at a  $1\sigma$  value of approximately 1.6 mm. This profile is identical to that of antenna 3, since those two antennae lie along the same body axis. For the same reason, the errors for antennae 2 and 4 are identical and are both very close to those of antenna 1 shown in the figure.



**Figure 5.12 Nominal Run Line Bias Error for Antenna 1**

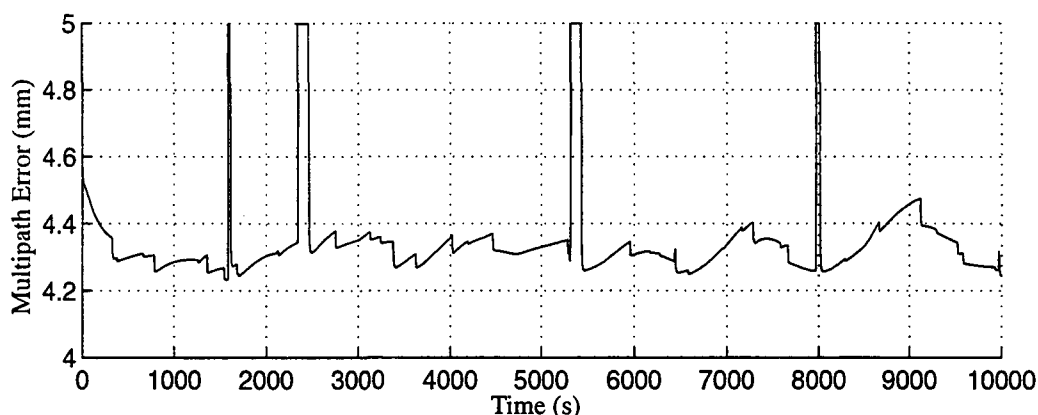
The line bias errors over the baselines can be obtained in the same manner as the baseline errors in the previous section - by differencing the appropriate antenna line biases and accounting for their correlations. These differential path delay errors are shown in Figure 5.13 for baselines 1 and 2 - the baseline 3 results are the same as those of baseline 1 due to the symmetry of the two baselines about the x axis. The line bias errors for the first baseline appear to reach a steady-state value of approximately 2 mm. The errors for the second baseline are seen to level off near 2.2 mm. The reason for this slightly higher value is not known.



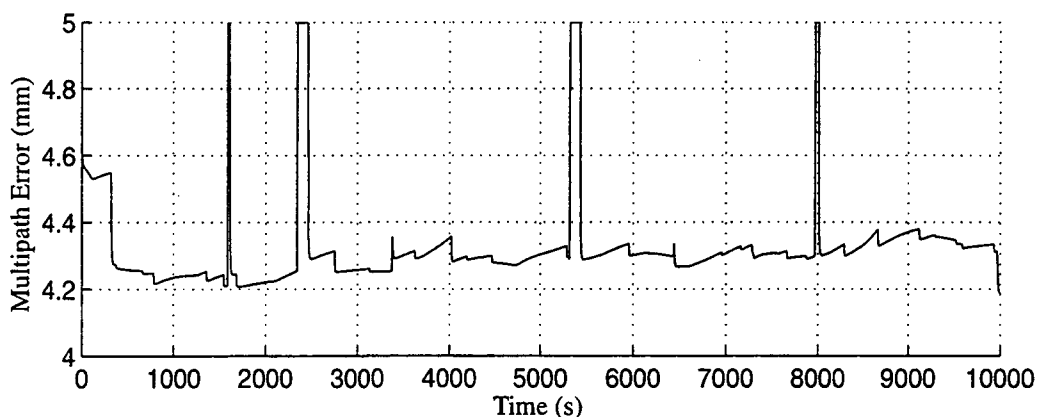
**Figure 5.13 Nominal Run Line Bias Errors for Baselines 1 & 2**

### 5.2.5 Multipath

The multipath errors from one satellite are modelled as correlated between antennae, but the errors between satellites is assumed independent. Figure 5.14 shows the multipath for antennae 1 and 3 on channel 2 to be estimable to no better than about 4.2 mm, even with the best geometry. The multipath errors on these two antennae, which



**Figure 5.14 Nominal Run Multipath Errors for Antennae 1 & 3 on Channel 2**



**Figure 5.15 Nominal Run Multipath Errors for Antennae 2 & 4 on Channel 2**

lie on the x axis, are indistinguishable, as are the errors on antennae 2 and 4 on the y axis. The multipath uncertainties associated with the latter antennae for the same channel are shown in Figure 5.15. The jumps to the 5 mm uncertainty level at 1600, 2300, 5300, and 8000 s occur when there are less than 6 satellites in view. At those times, no

satellite is visible on channel 2, and the  $1\sigma$  multipath error defaults to its initial value. The nature of the first-order Markov process keeps the uncertainty at this steady-state value, since no measurements are taken to update it. The sudden dip near 300 s corresponds to event 3 in Section 5.2.1, in which a satellite switch occurred. As previously described, the decrease in uncertainty is due to the fact that the new satellite multipath errors are initially uncorrelated with the rest of the system. Note that the error did not jump to 5 mm because the errors shown are after the measurement.

### 5.2.6 Error Budget

An error budget was tabulated for the nominal run. Table 5.4 provides the percentage breakdown of each component to the roll, pitch, and yaw error variances, as well as the absolute attitude errors in degrees after 10000 seconds. Note that the first contribution listed is the effect from the initial conditions (IC) of *all states*, not just from those of the attitude uncertainties. The fact that the percentages do not total to 100 is due partly to round-off error, and partly to the truncation of the Taylor series in the process noise covariance,  $\mathbf{N}$ . The values quoted are after the measurement since we are interested in the best filter performance. If we had merely propagated to the final time and not taken a final measurement, the process and measurement noise terms would have commanded a greater percentage of the attitude uncertainties.

Immediately apparent from the tabulation is that the contributions to the yaw are quite distinct from their counterparts in roll and pitch, consistent with previously presented results. It should not be surprising that the effects of initial conditions diminish significantly over time. From the table, we see that the impact of the initial conditions on roll and pitch uncertainties after 10000 s is a fairly insignificant 2%. The yaw component, on the other hand, seems to be considerably more sensitive to the IC's, which

**Table 5.4 Error Budget for Nominal Run**

| Contribution          | Per Cent of Total Variance |       |       | RMS Errors (deg) |       |       |
|-----------------------|----------------------------|-------|-------|------------------|-------|-------|
|                       | Roll                       | Pitch | Yaw   | Roll             | Pitch | Yaw   |
| 1) Initial Conditions | 1.97                       | 1.94  | 15.09 | 0.065            | 0.064 | 0.115 |
| 2) Process Noise:     |                            |       |       |                  |       |       |
| Attitude Rate         | 1.34                       | 0.59  | 1.55  | 0.053            | 0.036 | 0.037 |
| Torque                | 0.03                       | 0.00  | 0.08  | 0.008            | 0.001 | 0.008 |
| Lever Arm             | 31.22                      | 31.60 | 39.60 | 0.257            | 0.259 | 0.187 |
| Line Bias             | 23.16                      | 23.22 | 1.01  | 0.221            | 0.222 | 0.030 |
| Multipath             | 39.77                      | 39.99 | 38.49 | 0.290            | 0.292 | 0.184 |
| 3) Measurement Noise  | 2.11                       | 2.29  | 4.11  | 0.067            | 0.070 | 0.060 |
| Totals                | 99.60                      | 99.63 | 99.93 | 0.460            | 0.461 | 0.297 |

constitute 15% of the total mean square error. This is partly due to the fact that the overall yaw error is smaller; therefore, the percentage effect of the initial conditions is larger. But this is not the whole story since the absolute contribution of the IC's to yaw is larger (see last column, first row). This is a suspicious result since the yaw angle is more observable (as indicated by the smaller total uncertainty) than the roll and pitch angles and should therefore be less influenced by the initial conditions. To investigate this curious outcome, the IC contribution to the final roll, pitch, and yaw rms errors was broken down into the contributions from each of the six different groups of error states, as described in Appendix A. This breakdown, given in Table 5.5, shows that the final attitude uncertainties, after 10000 seconds, are insensitive to the initial attitude, attitude rate, torque, and line bias uncertainties and are only minutely sensitive to the initial lever arm conditions. The bulk of the IC contribution is due to multipath, as a result of reinitialization when a satellite dropout or switch occurs. Recall that in the



state transition matrix, the 4x4 block matrix, corresponding to the channel with the switch, is set to identity and all associated off-diagonal elements are zeroed. The last switch occurred only 30 seconds from the end of the run, and naturally, the effects of that reinitialization are still felt: the process noise (PN) relative contribution decreases, while the IC relative contribution increases. The effect on the yaw is larger than on the roll and pitch, consistent with the larger dips in yaw uncertainty of Figure 5.6. It is unclear why the yaw is apparently more correlated with the multipath than are roll and pitch.

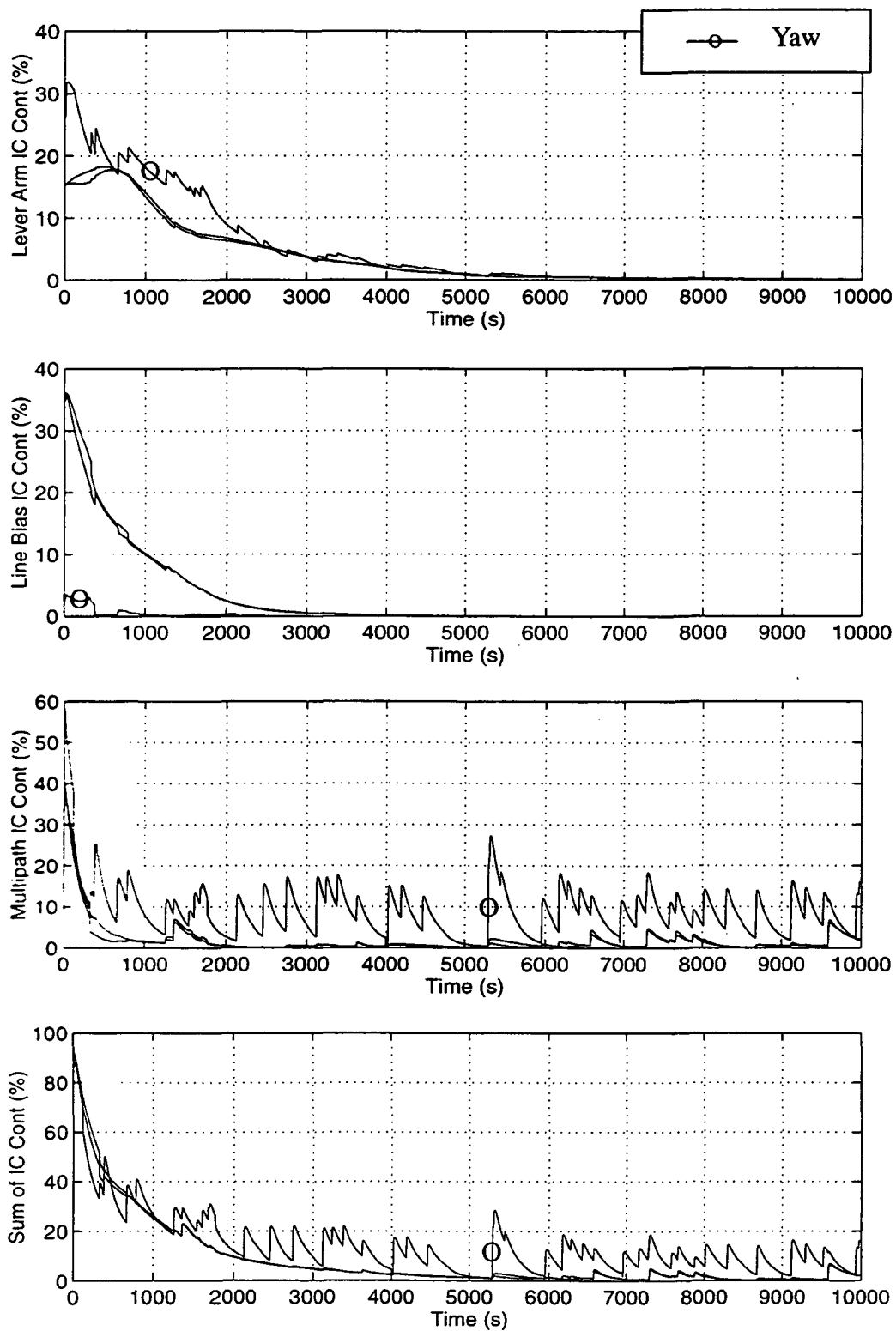
**Table 5.5 Initial Condition Contribution Breakdown: Percentages**

| Initial Condition Contribution                   | Per Cent of Total Variance |       |       |
|--------------------------------------------------|----------------------------|-------|-------|
|                                                  | Roll                       | Pitch | Yaw   |
| Attitude<br>Attitude Rate<br>Torque<br>Line Bias | 0.00                       | 0.00  | 0.00  |
| Lever Arm                                        | 0.04                       | 0.04  | 0.05  |
| Multipath                                        | 1.93                       | 1.90  | 15.04 |
| Sub-Totals                                       | 1.97                       | 1.94  | 15.09 |

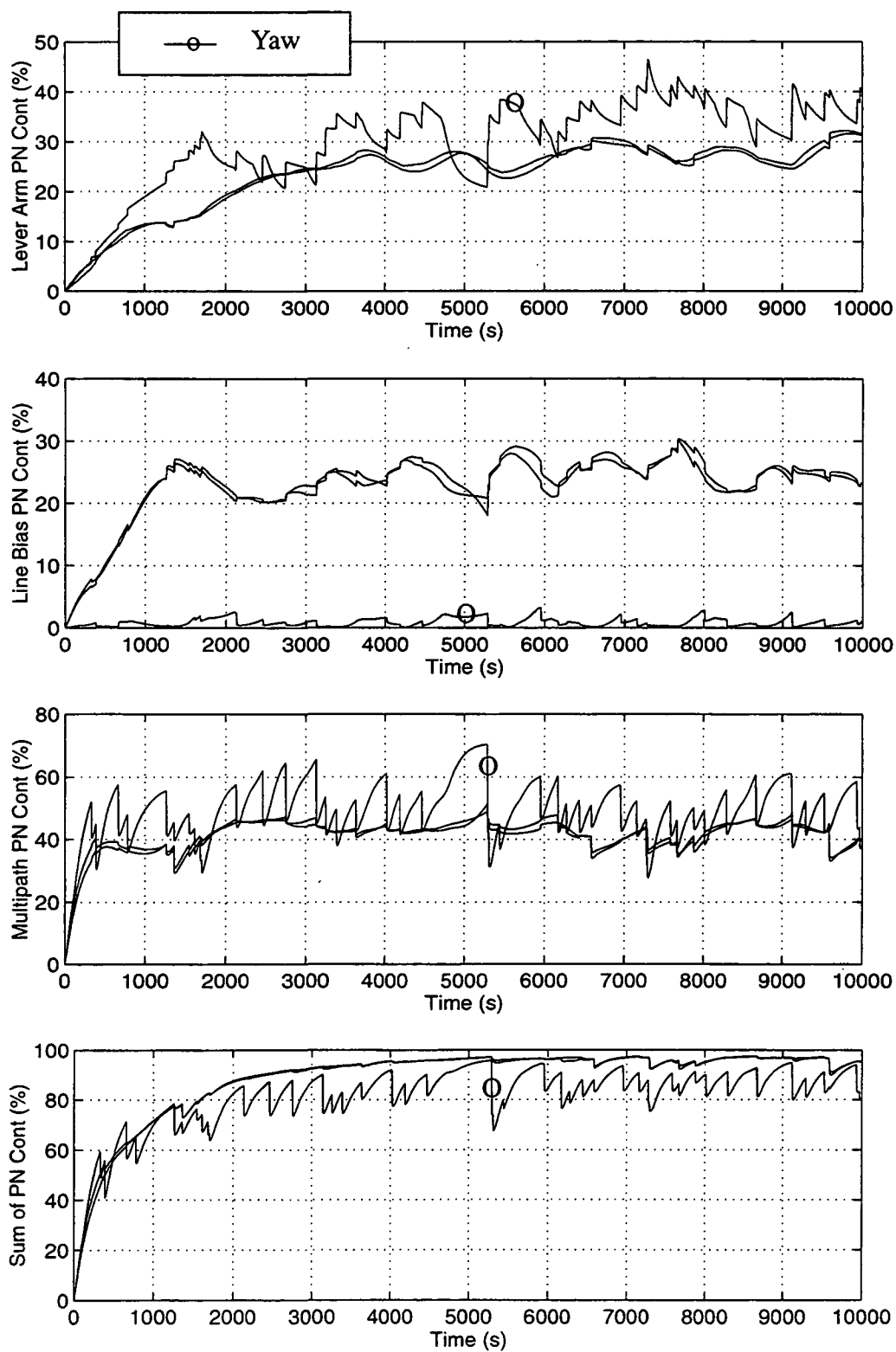
Despite the reinitialization, which occurs typically on one channel only, the final uncertainties are seen to be dominated by the process noises from the bias-type states - the lever arm, multipath, and line bias except for the yaw component. The yaw's insensitivity to line bias may result from the distribution of the satellites in azimuth. A rotation could be interpreted as a change in line bias, or vice versa. Concerning yaw, a positive change in line bias observed from one satellite would be counteracted by a negative change in line bias observed from a satellite 180 deg in azimuth from the

first. All observable satellites, however, are in one hemisphere, causing only one-signed changes in line bias for roll and pitch rotations.

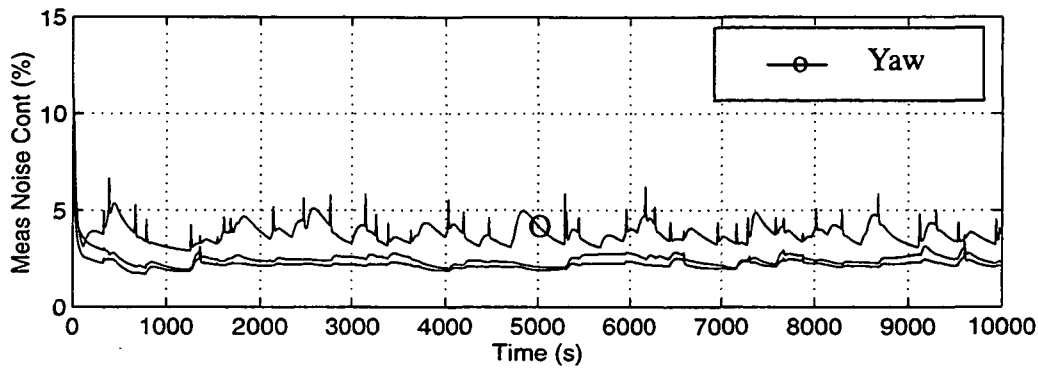
Figures 5.16, 5.17, and 5.18 show the IC, PN, and measurement noise contributions to the roll, pitch, and yaw variances for the nominal run. Each curve is the percentage of the total variance of that component. The yaw component is easily discerned from the roll and pitch, which have very similar profiles. Note the evidence of the reinitialization in the multipath contributions. Also note that the line bias IC and PN contributions to the yaw component are less than 5% for all time.



**Figure 5.16 Initial Condition Contributions to Attitude Variances**



**Figure 5.17 Process Noise Contributions to Attitude Variances**



**Figure 5.18 Measurement Noise Contribution to Attitude Variances**

### 5.3 Off-Nominal Run Results

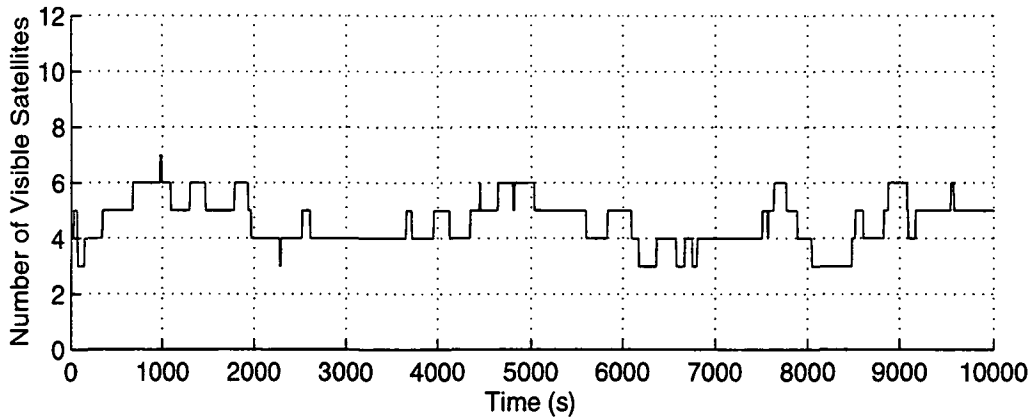
A better understanding of system performance requires examination of its sensitivity to various filter parameters. This section presents and discusses the results of a few of these off-nominal cases, summarized in Table 5.6, for the 49 state optimal filter. The “SS Error” for each Markov process is the steady-state, maximum uncertainty used to characterize it.

**Table 5.6 Off-Nominal Conditions**

| Parameter                             | Conditions Considered                                           |
|---------------------------------------|-----------------------------------------------------------------|
| Orbital Inclination                   | 0, 28.5, 51.6, 90 deg                                           |
| Satellite Selection Criterion         | GDOP, ADOP                                                      |
| Multipath SS Error                    | 0, 2, 5, 10 mm                                                  |
| Multipath Correlation                 | 0, 50, 100%                                                     |
| Lever Arm SS Error                    | 0, 2, 5, 10 mm                                                  |
| Line Bias SS Error                    | 0, 2, 5, 10 mm                                                  |
| Torque SS Error                       | 0, $5 \times 10^{-7}$ , $10^{-5}$ , $10^{-2}$ Nm                |
| Phase Measurement Error               | 1, 3, 10, 50 deg                                                |
| Attitude Error Rate Process Noise PSD | 0, $10^{-7}$ , $10^{-2}$ (deg/s <sup>2</sup> ) <sup>2</sup> /Hz |
| Antenna Half-Cone Angle               | 70, 85, 95 deg                                                  |

### 5.3.1 Orbital Inclination

As discussed in Section 5.2.1, the satellite geometry has a large impact on the attitude errors at any one time. The orbital inclination is therefore likely to affect the system performance. One might suspect that fewer satellites would be in view over the poles. Indeed, fewer satellites are seen at higher elevations, but since the orbital planes of the GPS constellation are equally dispersed in longitude, many satellites are visible at low elevations with good azimuth distribution when over a pole. A comparison of Figure 5.1 and Figure 5.19 actually indicates that GPS satellite visibility is somewhat worse for the equatorial orbit than for the polar orbit. Four satellites in view is a more common occurrence in the equatorial orbit, occasionally dipping down to three, and only once are more than six satellites in view.



**Figure 5.19 Simulated GPS Satellite Visibility in Equatorial Orbit**

Naturally, this degradation in visibility takes its toll on the GDOP and ADOP, as well as the attitude uncertainty, though the effect is not a drastic one, as illustrated in Table 5.7. Table 5.8 shows the rms of the attitude error magnitude over the last 6000 s of each 10000 s run for different inclinations. The nominal results are shaded. There seems to be no significant effect of the orbital inclination on the attitude uncertainty based on these results.

**Table 5.7 Attitude Errors Over Final 6000 s in Equatorial Orbit**

| Component | Min RMS Error (deg) | Max RMS Error (deg) | Average RMS Error (deg) |
|-----------|---------------------|---------------------|-------------------------|
| Roll      | 0.466               | 0.556               | 0.519                   |
| Pitch     | 0.468               | 0.554               | 0.515                   |
| Yaw       | 0.287               | 0.425               | 0.344                   |

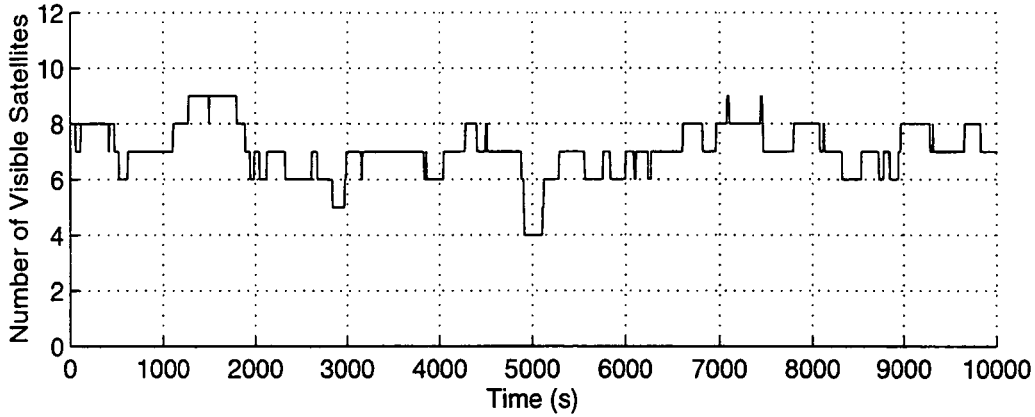
**Table 5.8 Average Attitude RMS Error Magnitudes vs Orbital Inclination**

| Inclination (deg) | Average RMS Error Magnitude (deg) |
|-------------------|-----------------------------------|
| 0.0               | 0.808                             |
| 28.5              | 0.785                             |
| 51.6              | 0.773                             |
| 90.0              | 0.769                             |

### 5.3.2 DOP Type

The figure of merit employed in rating the satellite geometry will obviously impact the attitude solution. The issue here is the effect of using GDOP, rather than the more appropriate ADOP, on performance. For the nominal case in which an 85 deg half-cone angle was used, the difference in errors between the two selection criteria was minimal since the number of visible satellites rarely exceeded the maximum number of six which can be tracked at one time. (Refer to Figure 5.1 for satellite visibility.) The largest difference in ADOP, at 1350 s, corresponded to a 0.1 deg, or 40%, difference in attitude error magnitude. To better see the effects of the selection criteria, the antenna visibility cone is now opened up with a 95 deg half-angle to provide a greater number of visible satellites than in the nominal case. Figure 5.20 shows the average

number of satellites in view for this case to be approximately seven, as opposed to five in the nominal run.



**Figure 5.20 Simulated GPS Satellite Visibility: 95° Half-Cone Angle**

In Figure 5.21, we now see more of a divergence between the best ADOP and the ADOP resulting from the satellites yielding the best GDOP. To translate this ADOP improvement into an attitude error improvement, define the difference in error magnitude to be:

$$\Delta\sigma = \sqrt{\sigma_{sel}^2 - \sigma_{best}^2}$$

where

$$\sigma_{sel} = ADOP_{sel} \sigma_v \text{ and } \sigma_{best} = ADOP_{best} \sigma_v$$

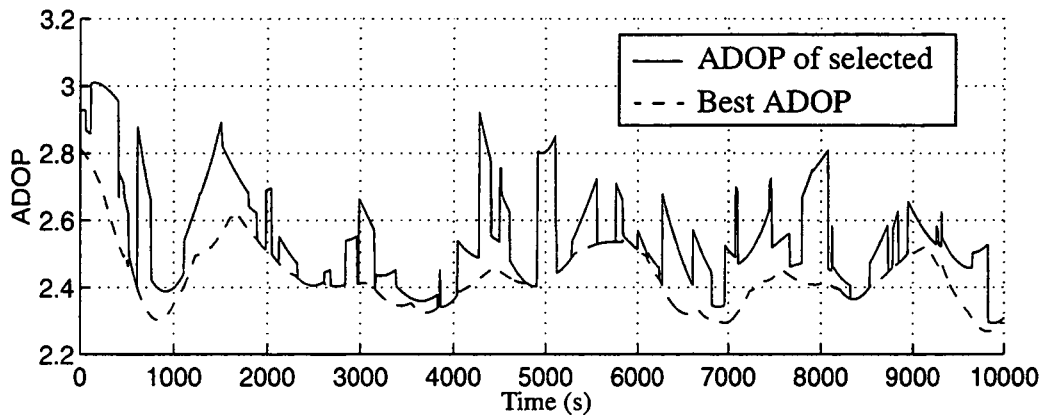
and the subscripts “sel” and “best” refer to the selected satellites based on the smallest GDOP, and the satellites giving the best ADOP, respectively. The per cent difference, plotted in Figure 5.22, is defined as:

$$Per\ Cent\ Diff = \frac{\Delta\sigma}{\sigma_{sel}} 100\%$$

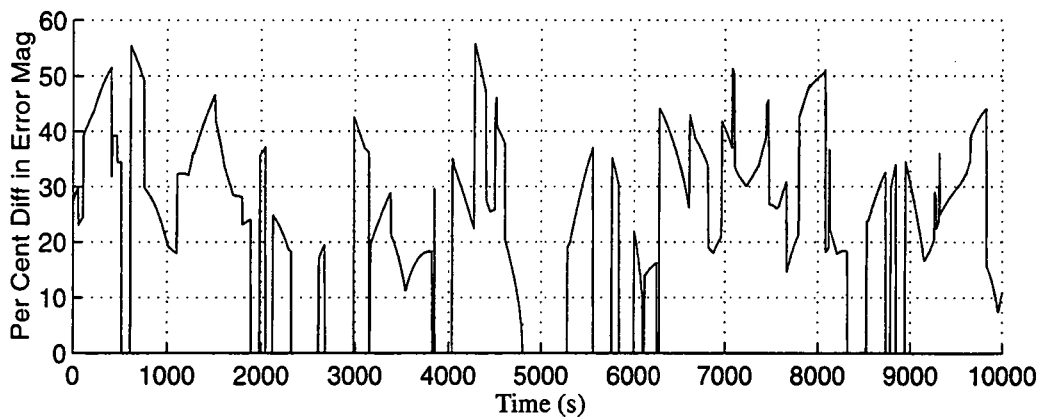
We see that, with this definition of improvement, using the ADOP criterion over GDOP results in a maximum decrease in rms error magnitude of 55% for this case.



The average difference over the 10000 s shown constitutes a sizable 23% improvement.



**Figure 5.21 ADOP's for 95° Half-Cone Angle Case**



**Figure 5.22 Per Cent Difference in Attitude Error Magnitude:  
GDOP vs ADOP**

The DOP criterion is only an indicator of the effects of instantaneous satellite geometry on the states of primary interest, assuming no previous knowledge. It is not a predictor of system performance, as implied by Table 5.9, because other states are involved and the covariance prior to the measurement is not diagonal and infinite. Though the states of primary interest in this application are the roll, pitch, and yaw, Table 5.9 actually shows the performance in attitude to be slightly better when using the GDOP figure-of-merit. It is possible to choose the set of satellites which minimizes

the attitude uncertainty given the existence of the other 46 states and the finite covariance, but this would defeat the purpose of using the relatively easily calculated figure-of-merit. Regardless, the selection criteria will become less of an issue as hardware technology improves, allowing more satellites to be tracked at one time.

**Table 5.9 Attitude Performance vs Selection Criterion**

| Selection Criterion | Average RMS Errors (deg) |       |       | RSS (deg) |
|---------------------|--------------------------|-------|-------|-----------|
|                     | Roll                     | Pitch | Yaw   |           |
| GDOP                | 0.483                    | 0.483 | 0.289 | 0.742     |
| ADOP                | 0.487                    | 0.485 | 0.300 | 0.750     |

### 5.3.3 Multipath

The error budget in Section 5.2.6 indicates that the multipath states have a large impact on the attitude error statistics for the nominal run. The attitude error sensitivity to these states is examined further here by varying separately the maximum uncertainty and the correlation between antennae about the nominal values,  $\sigma_{\xi} = 5$  mm and  $\rho_{\xi} = 0.5$ .

Table 5.10 shows the maximum roll, pitch, and yaw rms errors and the average rms errors over the last 6000 s of the run for  $\sigma_{\xi} = 0, 2, 5$ , and 10 mm. Not surprisingly, the performance substantially degrades with increasing multipath noise. Plots of the attitude errors (not provided here) show that the magnitude of the jumps at satellite switches also increase with multipath noise and are more pronounced in the yaw component.

Table 5.11 below shows the attitude error statistics for 0%, 50%, and 100% correlation of the multipath states between the four antennae to each satellite. Clearly, the attitude knowledge becomes better with multipath correlation. Note that the rms errors

**Table 5.10 Attitude RMS Errors vs Multipath Level**

| ADOP (rad/m) | Multipath Level, $\sigma_{\xi}$ (mm) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|--------------|--------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|              |                                      | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
|              | 0                                    | 0.325                | 0.323 | 0.204 | 0.293                    | 0.294 | 0.194 |
|              | 2                                    | 0.399                | 0.397 | 0.257 | 0.369                    | 0.367 | 0.221 |
|              | 5                                    | 0.536                | 0.533 | 0.415 | 0.496                    | 0.494 | 0.316 |
|              | 10                                   | 0.803                | 0.787 | 0.719 | 0.721                    | 0.722 | 0.523 |

for the 100% correlated case are the same as those for  $\sigma_{\xi} = 0$ , shown in Table 5.10. In the 100% correlated case, the filter is effectively only estimating six multipath states, one per satellite, as opposed to 24 states for the uncorrelated or partially correlated case. The multipath states for each antenna to one satellite are indistinguishable and drift out in the phase difference, and their uncertainties consequently remain at the maximum 5 mm value.

**Table 5.11 Attitude RMS Errors vs Multipath Correlation**

| Per Cent Diff In Error Mag (%) | Multipath Correlation (%) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|--------------------------------|---------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                |                           | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
|                                | 0.0                       | 0.637                | 0.632 | 0.539 | 0.587                    | 0.587 | 0.399 |
|                                | 50.0                      | 0.536                | 0.533 | 0.415 | 0.496                    | 0.494 | 0.316 |
|                                | 100.0                     | 0.325                | 0.323 | 0.204 | 0.293                    | 0.294 | 0.194 |

### 5.3.4 Lever Arm

The error budget for the nominal run reveals a strong sensitivity of the attitude uncertainty to the lever arm noise level. This sensitivity is confirmed in Table 5.12, which provides the attitude uncertainties for variations in this parameter. The same trends are seen: yaw is consistently more observable than the roll or pitch, and all three become less observable with poorer knowledge of relative antenna position.

**Table 5.12 Attitude RMS Errors vs Lever Arm Noise Level**

| Lever Arm Noise Level, $\sigma_l$ (mm) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|----------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                        | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 0                                      | 0.468                | 0.460 | 0.367 | 0.422                    | 0.418 | 0.256 |
| 2                                      | 0.536                | 0.533 | 0.415 | 0.496                    | 0.494 | 0.316 |
| 5                                      | 0.790                | 0.799 | 0.600 | 0.764                    | 0.771 | 0.529 |
| 10                                     | 1.308                | 1.348 | 0.998 | 1.289                    | 1.325 | 0.951 |

### 5.3.5 Line Bias

The results of varying the maximum line bias uncertainty, provided in Table 5.13, concur with the error budget tabulations: the yaw rms error is fairly insensitive to line bias errors. There is only a 7% increase in yaw variance from  $\sigma_\beta = 0$  to  $\sigma_\beta = 10$  mm, compared to a 300% increase in roll and pitch variances. Why the line bias error affects roll and pitch more than yaw is not entirely clear, but it may have to do with the satellite distribution in azimuth as mentioned in the nominal error budget discussion.

**Table 5.13 Attitude RMS Errors vs Line Bias Noise Level**

| Line Bias Noise Level, $\sigma_\beta$ (mm) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|--------------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                            | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 0                                          | 0.460                | 0.441 | 0.408 | 0.399                    | 0.399 | 0.315 |
| 2                                          | 0.536                | 0.533 | 0.415 | 0.496                    | 0.494 | 0.316 |
| 5                                          | 0.742                | 0.729 | 0.430 | 0.660                    | 0.652 | 0.321 |
| 10                                         | 0.991                | 0.956 | 0.450 | 0.826                    | 0.811 | 0.326 |

### 5.3.6 Torque

The attitude uncertainties about all three axes are affected very little by the torque noise level, as evidenced by Table 5.14. The nominal value used for the maximum

torque uncertainty is 5% of the gravity gradient torque acting on the spacecraft for pitch and roll angles of 5 deg. The greatest value, 0.01 Nm, is four orders of magnitude larger than the nominal, and yet the attitude variances increase by less than 9%. This noise in the torque does, however, affect the attitude error rate statistics. These results are reasonable since the torque, or acceleration, is integrated to give the velocity, and the phase difference is insensitive to both - they must be inferred from the attitude through correlations.

**Table 5.14 Attitude RMS Errors vs Torque Noise Level**

| Torque Noise Level, $\sigma_T$ (Nm) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|-------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                     | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 0                                   | 0.536                | 0.533 | 0.414 | 0.496                    | 0.495 | 0.317 |
| $5 \times 10^{-7}$                  | 0.536                | 0.533 | 0.415 | 0.496                    | 0.495 | 0.317 |
| $10^{-5}$                           | 0.541                | 0.535 | 0.418 | 0.500                    | 0.496 | 0.319 |
| $10^{-2}$                           | 0.550                | 0.543 | 0.437 | 0.509                    | 0.505 | 0.332 |

### 5.3.7 Phase Measurement Noise

The effect of the accuracy of the total phase measurement on the attitude performance is also investigated. The results for 1, 3, 10, and 50 deg uncertainties, corresponding to 0.5, 1.6, 5.3, and 26.4 mm, respectively, are given in Table 5.15. The attitude accuracy does not appear to degrade as much as one might expect from a seemingly large measurement uncertainty, as indicated in the 50 deg case. Simply based on an average ADOP of 2.5, the RSS error from one set of measurements, between 12 and 18 phase differences given the satellite visibility shown in Figure 5.1, would be in the neighborhood of 3.8 deg, but fortunately, a set of measurements is taken once every 6 s. The poor measurement accuracy is counteracted by the sheer number of measurements, since probability theory asserts that the uncertainty after  $n$

samples will be reduced by a factor of  $\sqrt{n}$  (with no process noise of course). This insensitivity is also reflected in the nominal error budget and Figure 5.18.

**Table 5.15 Attitude RMS Errors vs Phase Measurement Noise**

| Phase Meas<br>RMS Error,<br>$\sigma_{\nu_\phi}$ (deg) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|-------------------------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                                       | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 1                                                     | 0.529                | 0.525 | 0.407 | 0.490                    | 0.488 | 0.310 |
| 3                                                     | 0.536                | 0.533 | 0.415 | 0.496                    | 0.495 | 0.317 |
| 10                                                    | 0.570                | 0.572 | 0.453 | 0.527                    | 0.528 | 0.352 |
| 50                                                    | 0.825                | 0.855 | 0.773 | 0.765                    | 0.782 | 0.617 |

### 5.3.8 Attitude Rate Noise

As mentioned earlier, the filter will weight the measurements more heavily with larger process noise. Essentially, the filter does not have as much “memory,” and its knowledge of the state decreases to a greater extent between measurements. It therefore can not rely on previous measurements to help lower the variances. When the process noise is large, the best the filter can do is bring the variance down to a level dictated by the measurement accuracy and the instantaneous geometry, as indicated by the dilution of precision. White noise was added to the attitude rate error state to account for unmodelled torque noise. Table 5.16 shows, not surprisingly, the roll, pitch, and yaw uncertainties to increase with the white noise level. An interesting result does appear, however, in the relative level of uncertainty between the roll and pitch: with no noise, roll is more observable than pitch, and adding noise obscures the difference. Intuitively, the pitch should be better estimated since any rotation about the roll axis is unobservable to the second baseline, which lies along that axis. Recall from Section 3.2.2, however, that the roll and yaw are coupled in the equations of motion. The roll can therefore be inferred via this coupling. The additive process noise

obscures the coupling, effectively reducing the correlation between roll and yaw, and the roll thus becomes less observable.

**Table 5.16 Attitude RMS Errors vs Attitude Rate Noise Level**

| $Q_{\dot{\theta}}$ (deg/s) <sup>2</sup> /s | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|--------------------------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                                            | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 0                                          | 0.427                | 0.491 | 0.376 | 0.389                    | 0.445 | 0.304 |
| $10^{-7}$                                  | 0.536                | 0.533 | 0.415 | 0.496                    | 0.495 | 0.317 |
| $10^{-2}$                                  | 0.555                | 0.548 | 0.441 | 0.513                    | 0.509 | 0.334 |

### 5.3.9 Antenna Half-Cone Angle

Recall that satellites at lower elevations are more useful in estimating yaw, while satellites overhead provide better roll and pitch accuracy. This assertion is supported by Table 5.17, which gives the attitude uncertainties for three antenna half-cone angles. Since, for RADCAL, a satellite must be in view of all four antennae, restricting the visibility cone results in the loss of measurements to satellites at lower elevations. Note that the roll and pitch accuracies do degrade with smaller cone angle, and the yaw is still more observable. The yaw accuracy, however, suffers more than the other two components with tighter viewing restrictions - the average roll and pitch variances are only 27% greater in the 70 deg case than in the 95 deg case, whereas the average yaw variance is 185% greater.

**Table 5.17 Attitude RMS Errors vs Antenna Half-Cone Angle**

| Ant Half-Cone Angle (deg) | Max RMS Errors (deg) |       |       | Average RMS Errors (deg) |       |       |
|---------------------------|----------------------|-------|-------|--------------------------|-------|-------|
|                           | Roll                 | Pitch | Yaw   | Roll                     | Pitch | Yaw   |
| 70                        | 0.783                | 0.711 | 1.078 | 0.541                    | 0.544 | 0.488 |
| 85                        | 0.536                | 0.533 | 0.415 | 0.496                    | 0.495 | 0.317 |
| 95                        | 0.534                | 0.531 | 0.370 | 0.483                    | 0.483 | 0.289 |

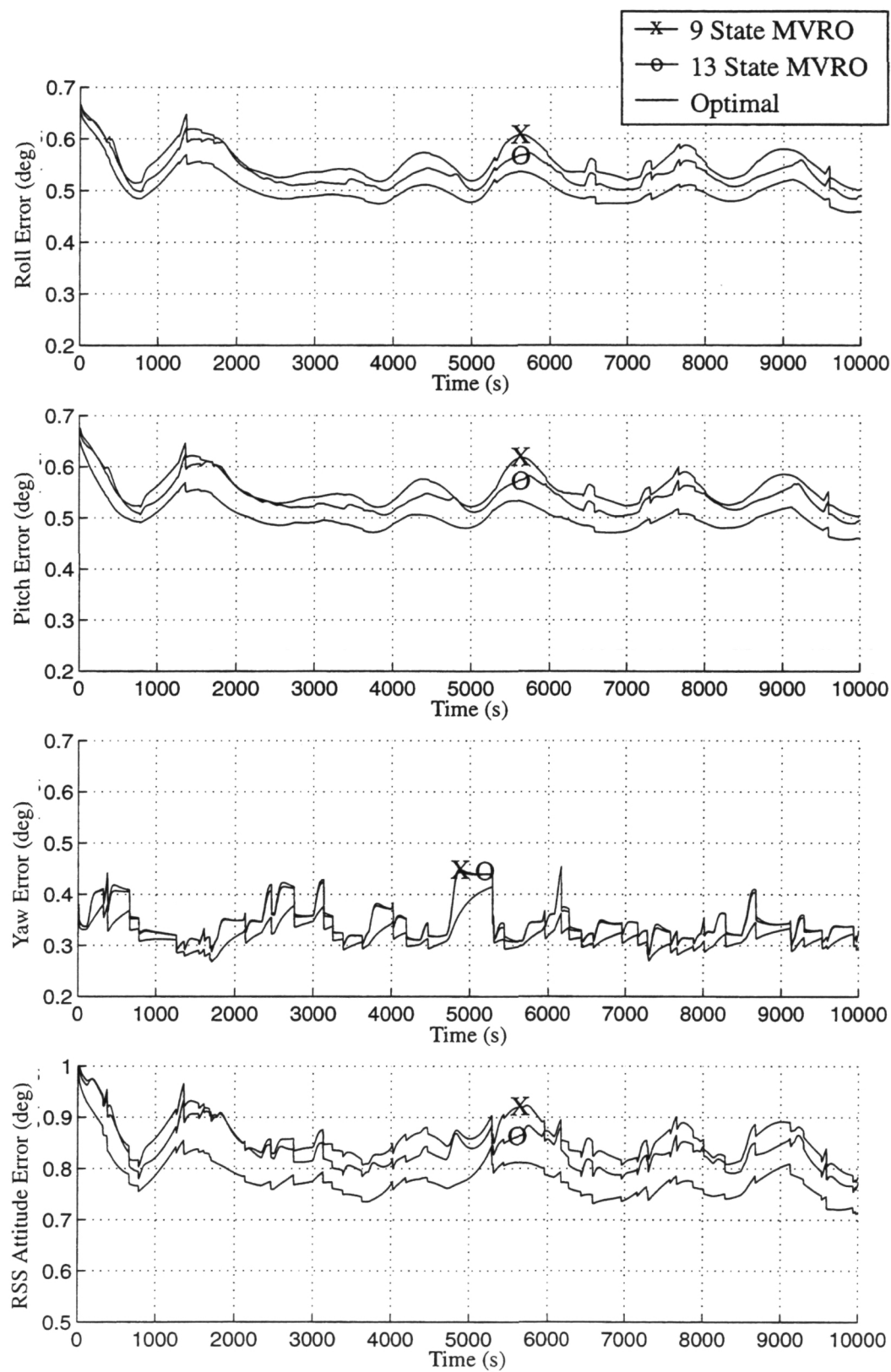
## 5.4 Optimal-Suboptimal Filters

It is desirable to reduce the order of the filter from 49 states to something more manageable for an onboard navigator. Considered here are two optimal-suboptimal filters (using consider state analysis): a 9 state filter which estimates attitude, attitude rate, and torque, and a 13 state filter which includes the 4 line biases. (If the double difference measurement is used instead of the single difference, the line bias states could potentially be removed from the filter since they are specific to the antennae and therefore cancel. The performance in using double differences, however, would be worse, since this is equivalent to assuming infinite line bias variance in the filter using single differences; the filter is not taking advantage of the limits on line bias uncertainty [12].) As previously discussed, the lever arm and multipath states (altogether,  $36 = 12 \text{ lever arm} + 24 \text{ multipath}$ ) appear to have a significant impact on system performance errors. The effect of their omission is examined by “considering” them as described in Section 4.1.3. The results presented here are the attitude errors from these so-called “optimal-suboptimal” filters.

Figure 5.23 shows the time histories of the attitude rms errors and the magnitudes (the RSS of the roll, pitch and yaw errors) for the two optimal-suboptimal filters as well as the full optimal filter. The same initial conditions and process noise parameters are used in all three cases. Not surprisingly, the 9 state filter’s performance is the worst. The 13 state filter is seen to perform somewhat better, in general.

The results are summarized in Table 5.18, which shows the average attitude rms errors and their RSS’s over the last 6000 s of the 10000 s run. The results of the nominal error budget (Section 5.2.6) might lead one to believe that a filter which does not estimate the lever arm, line bias, and multipath states would perform quite poorly compared to the optimal filter. The table shows, however, that the attitude’s average





**Figure 5.23 Suboptimal Filter Performance**

mean square errors of the 9 state filter are less than 26% larger than the 49 state “truth” filter. The average attitude variances of the 13-state filter are less than 18% larger than those of the optimal filter. This seems to imply that the optimal filter does not do as good a job at estimating the lever arm, line bias and multipath states as might be thought.

**Table 5.18 Optimal-Suboptimal Filter Performance**

| Filter       | Average RMS Errors (deg) |       |       | RSS (deg) |
|--------------|--------------------------|-------|-------|-----------|
|              | Roll                     | Pitch | Yaw   |           |
| 9 State O-S  | 0.552                    | 0.555 | 0.346 | 0.856     |
| 13 State O-S | 0.528                    | 0.534 | 0.344 | 0.826     |
| Optimal      | 0.496                    | 0.495 | 0.317 | 0.769     |

The error budgets for the 9 state and 13 state suboptimal filters are provided in Table 5.19 and Table 5.20, respectively. A comparison with the error budget of the nominal run reveals that the IC contribution, due essentially to the multipath IC, is larger for the optimal-suboptimal filter. Note also that, for the 9 state filter, the line bias process noise contributes more to the attitude uncertainty, in an absolute as well as relative sense, than in the optimal case (e.g., the line bias contribution to roll error: 35.18%, 0.298 deg vs 23.16%, 0.221 deg). In adding the line bias states, their contributions are seen to decrease in Table 5.20.

One might expect the contributions from the states which are not estimated to be larger, in an absolute sense, than in the optimal filter. This is indeed the case in the line bias PN contribution for the 9 state filter. Note, however, that the lever arm PN contribution is only slightly larger, and the multipath PN contribution is actually somewhat smaller! The contributions still add to give larger final variances. A crucial point is the fact that the optimal filter minimizes the trace of the entire 49x49 covariance matrix,

**Table 5.19 Error Budget for 9 State Optimal-Suboptimal Filter**

| Contribution          | Per Cent of Total Variance |       |       | Final RMS Errors (deg) |       |       |
|-----------------------|----------------------------|-------|-------|------------------------|-------|-------|
|                       | Roll                       | Pitch | Yaw   | Roll                   | Pitch | Yaw   |
| 1) Initial Conditions | 4.08                       | 3.92  | 27.15 | 0.102                  | 0.100 | 0.172 |
| 2) Process Noise:     |                            |       |       |                        |       |       |
| Attitude Rate         | 1.11                       | 1.34  | 4.88  | 0.053                  | 0.058 | 0.073 |
| Torque                | 0.00                       | 0.00  | 0.12  | 0.002                  | 0.003 | 0.011 |
| Lever Arm             | 26.60                      | 26.56 | 33.86 | 0.260                  | 0.260 | 0.192 |
| Line Bias             | 35.18                      | 35.98 | 1.54  | 0.298                  | 0.303 | 0.041 |
| Multipath             | 32.08                      | 31.32 | 30.85 | 0.285                  | 0.283 | 0.183 |
| 3) Measurement Noise  | 0.95                       | 0.87  | 1.58  | 0.049                  | 0.047 | 0.041 |
| Totals                | 100.0                      | 100.0 | 99.98 | 0.503                  | 0.505 | 0.329 |

**Table 5.20 Error Budget for 13 State Optimal-Suboptimal Filter**

| Contribution          | Per Cent of Total Variance |       |       | Final RMS Errors (deg) |       |       |
|-----------------------|----------------------------|-------|-------|------------------------|-------|-------|
|                       | Roll                       | Pitch | Yaw   | Roll                   | Pitch | Yaw   |
| 1) Initial Conditions | 6.54                       | 6.84  | 28.15 | 0.126                  | 0.130 | 0.174 |
| 2) Process Noise:     |                            |       |       |                        |       |       |
| Attitude Rate         | 1.30                       | 1.21  | 3.96  | 0.056                  | 0.055 | 0.065 |
| Torque                | 0.00                       | 0.00  | 0.09  | 0.003                  | 0.002 | 0.010 |
| Lever Arm             | 28.01                      | 27.80 | 34.11 | 0.260                  | 0.261 | 0.192 |
| Line Bias             | 17.92                      | 17.87 | 0.69  | 0.208                  | 0.210 | 0.027 |
| Multipath             | 42.45                      | 42.86 | 31.20 | 0.320                  | 0.325 | 0.183 |
| 3) Measurement Noise  | 1.56                       | 1.51  | 1.65  | 0.061                  | 0.061 | 0.042 |
| Totals                | 97.79                      | 98.09 | 99.85 | 0.491                  | 0.496 | 0.328 |

not the roll, pitch, and yaw variances. Therefore, it is possible for a suboptimal filter to estimate some individual states better than the optimal filter, but overall, the performance will be worse.

Based on this cursory analysis, the 13 state filter appears to be a good choice for an onboard system. It provides substantially smaller computer burden with little significant reduction in attitude accuracy. Of course, the performance shown here is the ideal overall attitude performance of the reduced-order filters, because the filters considered the omitted states (but did not estimate them). The actual reduced-order filter would require "tuning" to approach this "best" performance. Filter tuning can involve such methods as underweighting and adjusting the process noise. Better attitude accuracy, with little additional computer burden, may be also be achieved by lumping the omitted states into one new state.



## Chapter 6

### Conclusions

The goals of this thesis were to determine the attitude accuracy achievable in processing GPS phase difference measurements with a Kalman filter and to understand the effects of the major error sources on that accuracy. The performance of a GPS interferometric attitude determination system was investigated via a linear covariance analysis. Error sources were identified and the significant contributors were modelled. The optimal system's sensitivity to these error sources was examined through the error budget and by varying system parameters. The performance of two optimal-suboptimal filters was also investigated as a first step toward designing an onboard filter. A summary of the results and suggestions for future work are provided here.

#### 6.1 Summary of Results

For all runs investigated, the yaw component of the attitude consistently exhibited markedly different behavior from roll and pitch. In fact, the yaw was repeatedly better estimated than roll and pitch, due to the availability of lower-elevation satellites which provide larger yaw-sensitivity of the phase difference measurement. This sensitivity was observed in varying the antenna field-of-view: the yaw suffered more than the roll and pitch with decreasing cone angle. Under nominal conditions, the average rms errors for roll, pitch, and yaw were 0.496, 0.495, and 0.317 deg, respectively. In the presence of attitude rate noise, roll and pitch were observable to the same degree, while in its absence, roll was more observable because no noise obscured its coupling with the better estimated yaw component. Yaw was found to be fairly insensitive to uncertainties in the electrical path lengths to the antennae, compared to the other two

components. Lastly, yaw showed more correlation with the multipath states, as evidenced by the more pronounced changes (compared to roll and yaw) in its error with satellite switches.

After close inspection, it was found that the results of the simulation are consistent with the multipath model, which assumed that a new satellite is uncorrelated with the spacecraft's state. In truth, the multipath, though complex in nature and difficult to characterize, behaves in a predictable way and depends on the local reflective environment and the direction of signal incidence. In other words, the multipath from a newly introduced satellite should, in fact, be correlated with the attitude.

Based on the results of the error budget and the off-nominal runs, it was seen that good knowledge of relative antenna locations, path delays, and the multipath environment is required for high attitude accuracy. The error budget of the nominal run showed that roughly 70% of the roll, pitch, and yaw variances was due to noise in the lever arm and multipath states. (The lever arm flexures, however, are probably more easily controlled, i.e., more easily calibrated out, than multipath.) The noise in the line biases were seen to contribute about 23% to the roll and pitch variances, while contributing a meager 1% to the yaw variance. These results were confirmed in varying the error levels of these states. Performance improved significantly in all three components when the lever arm flexures and the multipath levels were reduced. Better accuracy was also achieved with greater multipath correlation between antennae. And though yaw accuracy remained largely unchanged, roll and pitch were better estimated with smaller line bias uncertainty.

Several factors which were thought to play a significant role in attitude performance but in fact did not were the torque error level, the measurement uncertainty, orbital inclination, and satellite selection algorithm. High accuracy was maintained

with large torque uncertainties. When the torque error level,  $\sigma_T$ , was increased by four orders of magnitude from its nominal value, the attitude variances increased by less than 9%. The performance did not considerably degrade with seemingly poor measurements. Yaw was seen to experience the largest deterioration; even so, its average variance increased by only 23% with a change in total phase measurement uncertainty from the nominal 3 deg to 10 deg. These insensitivities were also reflected in the error budget. The accuracy actually improved slightly with larger orbital inclinations. Finally, virtually no difference in performance was observed between selection criteria.

In summary, the results of the nominal and off-nominal cases indicate that attitude accuracy is limited by:

- Lever arm errors
- Line bias errors
- Multipath errors and correlation
- Antenna field-of-view

while it is fairly insensitive to:

- Satellite selection criterion (GDOP vs ADOP)
- Orbital inclination
- Torque errors
- Measurement uncertainty

The statistical performance of two suboptimal filters, using consider states, was also investigated. The first estimated 9 states, consisting of the attitude errors, their rates, and the torque error, and the second included the 4 line bias error states. In considering the apparently major contributors to attitude errors, the suboptimal filtering results did not show as large a deterioration in performance as the error budget and off-nominal runs implied. The average attitude variances in the 9 and 13 state filters were



only 26% and 18% larger than the 49 state optimal filter. This result implies that the optimal filter is not estimating the lever arm, line bias and multipath error states as well as might have been originally thought. Consistent with the nominal error budget, however, were the relative improvements in the attitude variances between the 9 and 13 state filters. Based on the optimal error budget, the yaw component is insensitive to line bias errors, and indeed, the yaw accuracy improved very little in adding those 4 states while the roll and pitch variances decreased by 10%. It is important to remember in comparing results of suboptimal and optimal filters, however, that the optimal filter minimizes the trace of the entire covariance matrix, not the attitude variances. It is therefore possible, though unlikely, that a suboptimal filter could be computed to give better attitude performance than the optimal filter. But based on this cursory analysis, the 13 state optimal-suboptimal filter represents a good compromise between accuracy and computational burden.

## **6.2 Suggestions for Future Work**

A logical first step in continuing the investigation of GPS interferometry applied to the attitude determination problem considered here would be to implement models which better characterize the error sources. As previously mentioned, the lever arm and line bias error models assumed in this study attempt to mimic the effect of thermal variations. An improvement in their characterizations would be to take into account the correlations which arise from the physical phenomenon assumed to drive these errors. The lever arm flexures are likely to behave similarly (as opposed to their assumed independence) when undergoing heating, for example. Another measure which would yield more realistic performance is to include inertia matrix errors in the state. This is because body axes and principal axes are always somewhat different. The

multipath model could be improved by accounting for its correlation with the attitude when a new satellite is introduced. One possible amendment is to maintain the correlations with the multipath states associated with each satellite. This could be achieved by including the multipath states associated with all 24 operational GPS satellites (thus increasing the state size from 49 to 121!). Finally, though the attitude performance was found to be largely insensitive to torque errors, the effects of solar radiation, atmospheric drag, and other sources of torques could be modelled more accurately. Also, a more in-depth analysis would include an investigation of the performance in off-LVLH nominal attitudes.

The next step in investigating system performance would be to implement a Monte Carlo simulation. The Monte Carlo simulation involves determining the statistical performance empirically through multiple runs and actual state estimation. This would be useful in verifying the validity of the covariance analysis results. Nonlinear effects such as integer ambiguities and cycle slips could be included to enhance the fidelity of the model.

Another area of further research is in suboptimal filtering. The performances of two optimal-suboptimal filters were summarily examined in this study. A more in-depth examination would involve the analyses of filters with other combinations of states and their tunings. Various methods of filter tuning could be explored, such as adjusting the process noise levels and underweighting the measurements. Another tuning technique is to approximate the observed optimal-suboptimal gains with analytic functions of time, typically exponentials and piecewise constants [10]. The result would be a filter suitable for onboard navigation.

Finally, actual flight data can be filtered, since that is the ultimate goal for the aforementioned exercises. As part of this, the whole area of initial attitude determina-

tion for an onboard system with multipath errors needs to be addressed. RADCAL provides one platform for investigation, and in the near future, other opportunities will arise in the increasing number of applications of GPS-based attitude determination systems.

## Appendix A

### Error Budget Implementation Issues

The major steps in the tabulation of the error budget are shown in the simulation flow chart in Figure 4.4. The measurement update and propagation steps for the error budget differ, however, from those of the total covariance. These steps are described in greater detail here.

#### A.1 Measurement Update

At the measurement time, the best satellites are first selected. Just as the multipath elements of the total covariance matrix must be reinitialized when a satellite has dropped out of view, so must the contributions if the error budget is desired. To effect this reinitialization in the contributions, the state transition matrix,  $\Phi$ , is reset by inserting a (4x4) identity matrix in the block corresponding to the lost satellite and zeroing the associated off-diagonal terms. The process noise constituent is further decomposed into the contributions from the noise on the attitude rate, torque, lever arm, line bias and multipath states:

$$\mathbf{P}_Q = \mathbf{P}_{Q_\theta} + \mathbf{P}_{Q_T} + \mathbf{P}_{Q_l} + \mathbf{P}_{Q_\beta} + \mathbf{P}_{Q_\xi}$$

The process and measurement noise contributions are reinitialized by simply zeroing out the entire block row and block column corresponding to the appropriate channel.

With the satellites selected and the multipath states in each component updated, the measurement sensitivity and thus the weighting, optimal or otherwise, is calculated. The measurement is then incorporated into the covariance components via equations (4.22).

## A.2 Propagation

In propagating the total covariance matrix, recall that the matrix Riccati equation is integrated using a fourth-order Runge-Kutta scheme. For the error budget, however, we opt for the discrete form of the propagation which employs the state transition matrix as opposed to the dynamics matrix, and the process noise covariance instead of its power spectral density. The incremental state transition matrix,  $\phi$ , can be calculated by simply integrating the dynamics matrix,  $\mathbf{F}$ , over the time step. This is done using the Runge-Kutta integrator that is already in place. The process noise covariance,  $\mathbf{N}$ , is not as easily attained for the error budget calculations given in equations (4.20).  $\mathbf{N}$  is decomposed into its components for each state to propagate each of the aforementioned process noise constituents. An approximation to it is found by expanding Equation (4.19), rewritten here:

$$\mathbf{N} = \int_{t_{k-1}}^{t_k} \phi(t_k, \tau) \mathbf{Q} \phi^T(t_k, \tau) d\tau$$

in a truncated Taylor series. The derivative of  $\mathbf{N}$  is found by applying Leibnitz's rule to the above equation [20]:

$$\begin{aligned} \dot{\mathbf{N}} = & \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t_k} [\phi(t_k, \tau) \mathbf{Q} \phi^T(t_k, \tau)] d\tau + [\phi(t_k, t_k) \mathbf{Q} \phi^T(t_k, t_k)] \frac{\partial t_k}{\partial t_k} \\ & + [\phi(t_k, t_{k-1}) \mathbf{Q} \phi^T(t_k, t_{k-1})] \frac{\partial t_{k-1}}{\partial t_k} \end{aligned} \quad (\text{A.1})$$

for infinitesimal  $\Delta t = t_k - t_{k-1}$ . Noting that  $\frac{\partial t_{k-1}}{\partial t_k} = 0$  and that  $\phi(t_k, t_k) = \mathbf{I}$ , and taking  $\mathbf{Q}$  to be constant over  $\Delta t$ , the time derivative of  $\mathbf{N}$  reduces to:

$$\dot{\mathbf{N}} = \mathbf{Q} + \mathbf{F} \int_{t_{k-1}}^{t_k} \phi(t_k, \tau) \mathbf{Q} \phi^T(t_k, \tau) d\tau + \int_{t_{k-1}}^{t_k} \phi(t_k, \tau) \mathbf{Q} \phi^T(t_k, \tau) d\tau \mathbf{F}^T \quad (\text{A.2})$$

$$= \mathbf{Q} + \mathbf{F} \mathbf{N} + \mathbf{N} \mathbf{F}^T \quad (\text{A.3})$$

Assuming  $\mathbf{F}$  to be constant over  $\Delta t$ , the second and third derivatives of  $\mathbf{N}$  are obtained:

$$\dot{\mathbf{N}} = \mathbf{F}\dot{\mathbf{N}} + \dot{\mathbf{N}}\mathbf{F}^T \quad (\text{A.4})$$

$$\ddot{\mathbf{N}} = \mathbf{F}\ddot{\mathbf{N}} + \ddot{\mathbf{N}}\mathbf{F}^T \quad (\text{A.5})$$

These expressions for the derivatives are inserted into the Taylor series, which is:

$$\mathbf{N} = \mathbf{N}_0 + \dot{\mathbf{N}}_0\Delta t + \frac{1}{2}\ddot{\mathbf{N}}_0\Delta t^2 + \frac{1}{3!}\ddot{\mathbf{N}}_0\Delta t^3 + \dots \quad (\text{A.6})$$

where  $\mathbf{N}_0$  is the nominal noise covariance, i.e., the covariance at time  $t_{k-1}$ ; thus,  $\mathbf{N}_0 = 0$  and  $\dot{\mathbf{N}}_0 = \mathbf{Q}$ . Since the process noise contributions to the decomposed total covariance matrix are each handled separately, the first derivative of the 49x49 noise covariance of process,  $x$ , is:

$$\dot{\mathbf{N}}_{0_x} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Q}_x & \mathbf{0} \end{bmatrix}_{49 \times 49}$$

where  $\mathbf{Q}_x$  is the appropriately dimensioned noise PSD associated with process  $x$ . The higher derivatives are then computed using the assumed constant dynamics matrix,  $\mathbf{F}$  in equations (A.4) and (A.5) and inserted into the series in (A.6), truncated to four terms. (Four terms were deemed sufficient, since over 10000 s the maximum difference in the trace of  $\mathbf{P}$  between the two methods of propagation - integrating the Ricatti equation and using the state transition formulation - was less than 0.05%.) The state transition formulation is then used in propagating each noise component:

$$\mathbf{P}_{Q_x}^+ = \Phi \mathbf{P}_{Q_x} \Phi^T + \mathbf{N}_x$$

### A.3 Final Tabulation

At the end of the run, the covariance breakdown is now tabulated. The breakdown in full is:

$$\mathbf{P}_{Total} = \Phi \mathbf{P}_0 \Phi^T + \mathbf{P}_{Q_\theta} + \mathbf{P}_{Q_T} + \mathbf{P}_{Q_l} + \mathbf{P}_{Q_\beta} + \mathbf{P}_{Q_\xi} + \mathbf{P}_M$$

Extracted from each piece of the total covariance are the first three diagonal terms - the contributions to the attitude variances. The attitude variances in the first term on the right arise from all states' initial conditions. This piece can be decomposed even further to see the contributions from the initial conditions of each group of states at no extra cost:

$$\Phi \mathbf{P}_0 \Phi^T = \Phi [\mathbf{P}_{\theta_0} + \mathbf{P}_{\dot{\theta}_0} + \mathbf{P}_{\tau_0} + \mathbf{P}_{l_0} + \mathbf{P}_{\beta_0} + \mathbf{P}_{\xi_0}] \Phi^T$$

In performing the error budget, a total of eight 49x49 matrices - the total covariance, the state transition matrix, the five process noise covariance matrices, and the measurement noise contribution - must be propagated and updated with measurements. Though the number of operations can be reduced somewhat by utilizing the fact that the dynamics and process noise matrices are sparse, the error budget is still a costly undertaking, but a very illustrative one.

## **Appendix B**

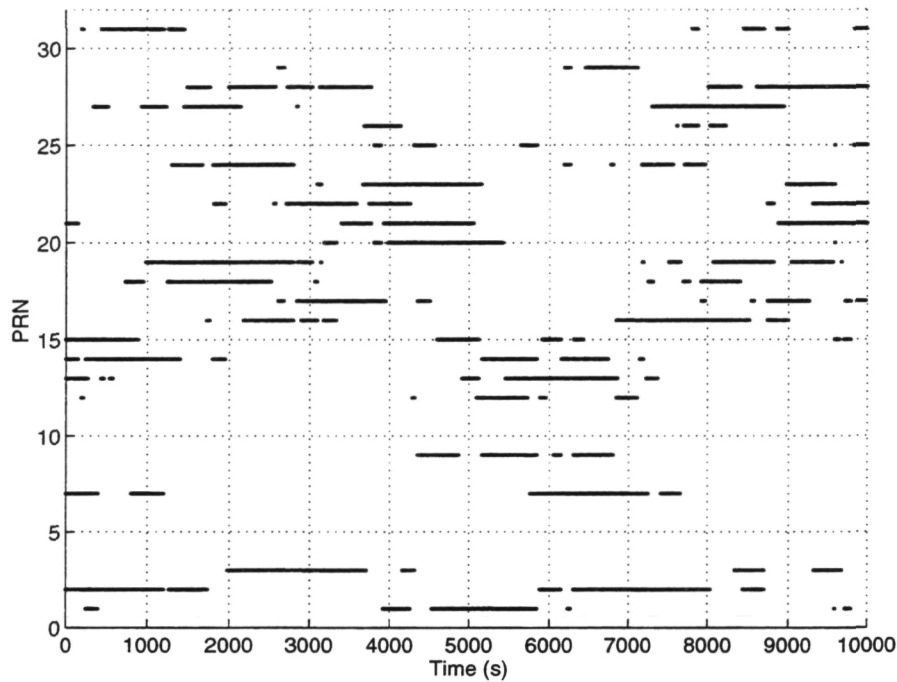
### **Discussion of RADCAL Flight Data**

Several hours of data from RADCAL's TANS Quadrex receiver was obtained from Penny Saunders at Johnson Space Center. The data is in binary form and is parsed into readable ASCII format using a program called TANSPOST, also provided by JSC. In addition to furnishing the raw differential phase data, TANSPOST gives RADCAL position fixes, velocities, latitudes, longitudes, and altitudes, satellite selection information (PDOP, HDOP, VDOP, TDOP), and receiver health information. The TANS receiver computes position solutions from the GPS Coarse/Acquisition (C/A) code, subject to selective availability (SA). Virtually no accuracy is lost in the attitude solution due to SA, however, since the lines-of-sight are insensitive to the position errors induced by the intentional degradation.

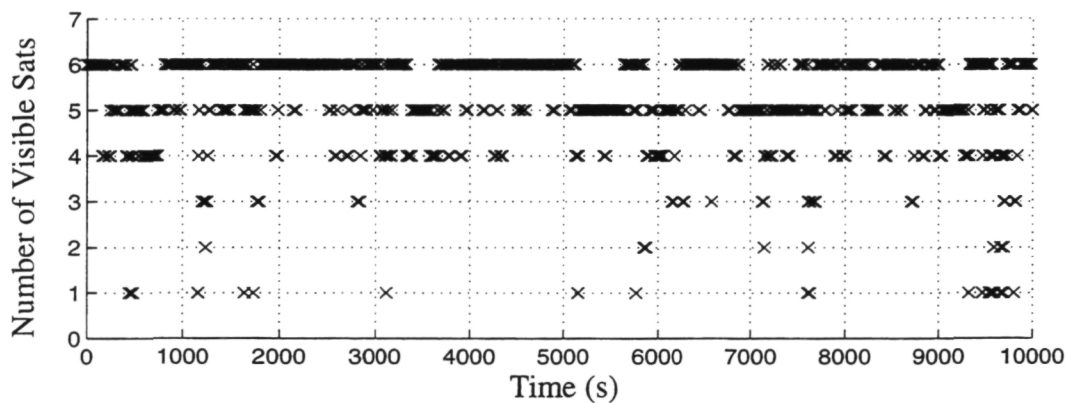
The raw phase difference measurement files contain phase differences only to satellites in view of all four antennae, even though a measurement could have been processed if only two antennae view a particular satellite. The RADCAL system is nevertheless still overdetermined given the 18 measurements to the three baselines. The data files obtained for this investigation corresponds to GPS week 708, day 6, or August 7, 1993. Figure B.1 shows the PRN's which are tracked for a 10000 s interval, and Figure B.2 shows the number of satellites in RADCAL's best view for the same interval. Note that the maximum number is six because there are only six channels per antenna, and the receiver requires that a satellite must be in view of all four antennae to take phase differences. The figure shows that at least four satellites will be in view 90% of the time. The 85 deg antenna half-cone angle in the simulation was chosen to



give approximately this visibility. The RADCAL satellite is over the North and South poles at  $t = 2600$  s, and  $5600$  s, respectively.



**Figure B.1 GPS Satellite Visibility for RADCAL**



**Figure B.2 Number of Satellites in RADCAL's Best View**

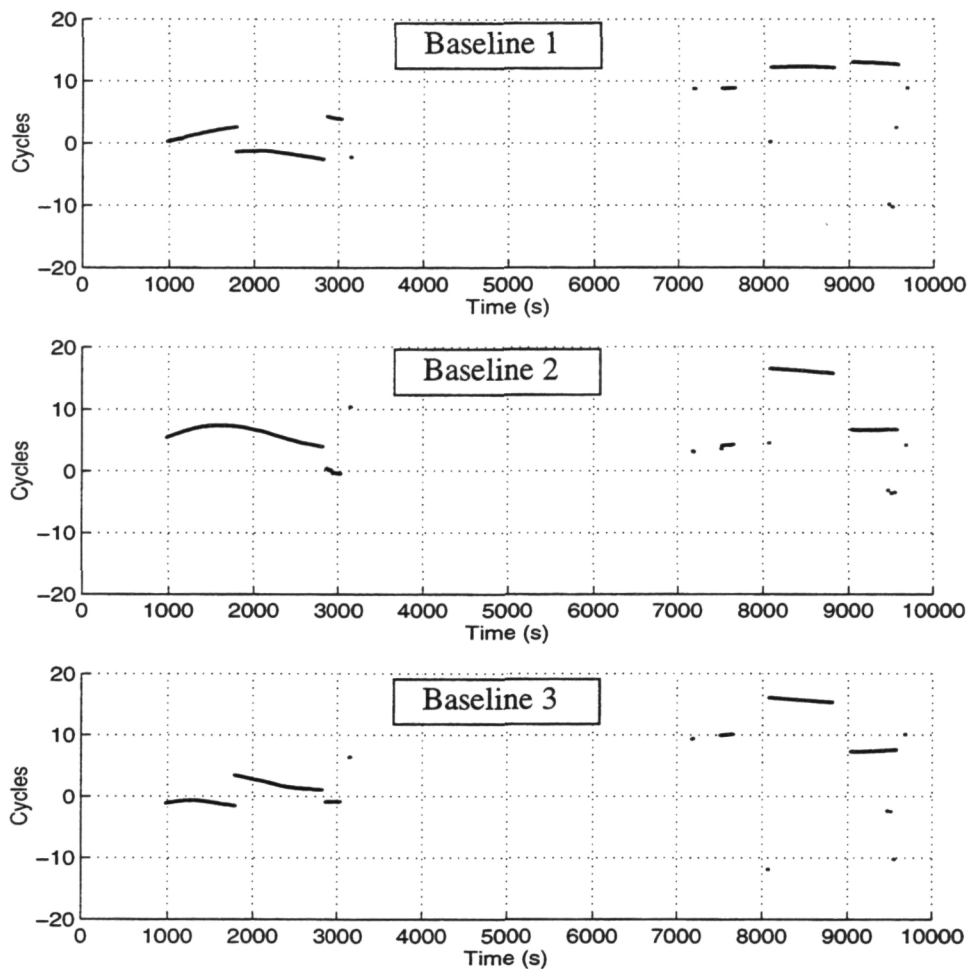
The phase differences to a satellite are referred to a master antenna, so denoted in the raw data file. The master antenna can switch for each satellite based on the signal-

to-noise ratio (SNR). It is therefore useful to designate a constant pseudo-master antenna, and recompute the phase differences to refer them to this pseudo-master. (Note the antenna indices in Figure 1.2. Antenna 1 is the pseudo-master.) For example, if antenna 2 is the master, we can refer the phase differences to antenna 1 by:

$$\begin{aligned}\Delta\phi_{1 \rightarrow 2} &= -\Delta\phi_{2 \rightarrow 1} \\ \Delta\phi_{1 \rightarrow 3} &= \Delta\phi_{2 \rightarrow 3} - \Delta\phi_{2 \rightarrow 1} \\ \Delta\phi_{1 \rightarrow 4} &= \Delta\phi_{2 \rightarrow 4} - \Delta\phi_{2 \rightarrow 1}\end{aligned}$$

where  $\Delta\phi_{i \rightarrow j}$  is the phase difference across the baseline from master,  $i$ , to slave,  $j$ . The actual phase differences to PRN 19 for each of the three baselines over a 10000 s interval of time were calculated and are shown in Figure B.3. The outages and cycle slips are painfully apparent. In the actual RADCAL flight data, we see phase differences which can be greater in magnitude than the baseline length. This results when the receiver loses lock and reacquires, and then differences the erroneous phases. It is possible then to have a phase difference of thousands of cycles, for different receivers [12]. Fortunately, this case involves only one receiver on a small platform, keeping the phase predictions during reacquisition comparable. It is also possible that some pre-processing on the phase differences was performed, either by the receiver or in TANS-POST. Except for the unusually long interval from 1000 to 2800 s in baseline 2 in which no slips occur, these plots are typical of the phase differences for other satellites.

It may be desirable to pre-process the phase differences even further by removing the cycle slips and outlying points. Of course, an onboard filter would have to deal with these situations in real time. The cycle slips are ideally integer jumps, but the RADCAL data can exhibit non-integer jumps. To remove the slip and patch the phase



**Figure B.3 RADCAL Phase Difference Measurements to PRN 19**

difference segments together, the time interval between measurements over which the slip occurs must be defined. Depending on the time interval, the dynamics between measurements can cause the phase difference to change enough to cause a slip to have an ambiguous number of cycles.

The noise on the phase differences can be characterized by curve fitting, via the method of least squares, a linear segment and noting the mean square error. A simple second-order polynomial fitted to the linear segment of the phase difference for baseline 2 in the above plot between  $t = 2000$  s and  $t = 2800$  s gives a change in phase difference of  $0.0291 \pm 0.0131$  cycles ( $5.5 \pm 2.5$  mm) over the nominal 6 s time

interval. If the slip occurs over a short enough interval, the jump may be identified as an integer. For longer intervals, the gap may be bridged by using a simple curve fit to extrapolate the phase differences on either side to the center of the interval [12]. Hopefully, the jump there is sufficiently close to an integer. For long outages which preclude this extrapolation technique, the integer must be resolved as in the initialization process, but by this time, the attitude should be known well enough to easily determine the ambiguity.

The first attempt made at attitude initialization used a simple, brute-force approach. The first minute of data, over which no cycle slips occurred and the dynamics are essentially constant, was used. The single phase differences were differenced again in time; the integer ambiguities (one per satellite per baseline) and constant part of the line biases (one per baseline) therefore dropped out. The phase differences were predicted based on an assumed initial attitude and were differenced in time, like the real data. The best attitude should minimize the following cost function:

$$J(\theta) = \sum_{i=2}^m \sum_{j=1}^3 \left[ \left( \Delta\phi_j^i - \Delta\phi_j^1 \right) - \left( \hat{\Delta\phi}_j^i - \hat{\Delta\phi}_j^1 \right) \right]^2$$

where  $i$  indexes over the epochs,  $j$  indexes over the three baselines, and the caret denotes the predicted value. The above cost function was computed for the range of initial attitudes in which roll and pitch were between -20 and 20 deg and yaw was between 0 and 360 deg, at 5 deg increments ( $9 \times 9 \times 72 = 5832$  initial guesses). This search did not result in a clear-cut initial attitude, due most likely to the noise and the multipath on the actual phase differences.

A second, more sophisticated, attempt at initialization was made [24]. In this technique, the integers were varied to arrive at possible solutions. For each baseline, two between-satellite double difference were formed, and the constraint of the baseline

length was used to deterministically solve for each baseline vector in terms of the differences of the lines-of-sight and their cross-product, to within a sign ambiguity. Cycling through the allowable integers for each baseline gives multiple solutions, the number of which can be reduced by constraining the relative baseline geometry. Unreasonable solutions, such as one indicating the craft to be upside-down, can be ruled out as well. This still yielded several solutions. The same procedure was performed in using a different satellite at the same epoch, but unfortunately, the two sets of initial attitudes had no solutions in common. Another alternative would be to use Knight's method [14], which would result in a single most likely solution, though that solution may not be correct due to RADCAL's multipath environment. Some sort of batch estimation which accounts for the dynamics, as suggested by Axelrad and Ward [2, 28], is most likely required to solve for the initial attitude.

## References

- [1] Axelrad, P. and B. Chesley, "Performance Testing of a GPS Based Attitude Determination," Proceedings of the AIAA Guidance, Navigation, and Control Conference, Monterey, CA, Aug. 9-11, 1993. Report No. AIAA-93-3787-CP.
- [2] Axelrad, P., P. MacDoran, et al, "GPS Based Attitude Determination: FY'93-94 Technical Report I to the Naval Research Laboratory," Colorado Center for Astrodynamics Research, University of Colorado, Boulder, May 16, 1994.
- [3] Brown, A. K., *Interferometric Attitude Determination Using the Global Position System*, S.M. Thesis, Department of Aeronautics and Astronautics, MIT, Aug. 1981.
- [4] Brown, R. A., "Instantaneous GPS Attitude Determination," *IEEE AES Magazine*, June 1992.
- [5] Brown, R. G. and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, John Wiley & Sons, Inc., New York, NY, 1983.
- [6] Cohen, C. E., *Attitude Determination Using GPS*, Ph.D. Dissertation, Department of Aeronautics and Astronautics, Stanford University, Dec. 1992.
- [7] Cohen, C. E. and B. W. Parkinson, "Mitigating Multipath Error in GPS Based Attitude Determination", AAS 91-024.
- [8] Cohen, C. E., E. G. Lightsey, B. W. Parkinson, and W. A. Feess, "Space Flight Tests of Attitude Determination Using GPS," *International Journal of Satellite Communications*, Vol. 12, No. 5, Sept.-Oct. 1994, pp. 427-433.
- [9] Dula, A. S., "International Space Station Alpha: Global Position System Attitude and Navigation Experiment (GANE): Presentation to the OACT/PAMS Operations Working Group," Johnson Space Center, Sept. 29, 1994.
- [10] Gelb, A. (Ed.), *Applied Optimal Estimation*, The MIT Press, Cambridge, MA, 1974.
- [11] Hatch, R., "Ambiguity Resolution in the Fast Lane," *Proceedings of the Second International Technical Meeting of the Institute of Navigation's Satellite Division*, Colorado Springs, CO, 1989.
- [12] Herring, T., Private communications, Oct. 1994 - May 1995.
- [13] Hughes, P. C., *Spacecraft Attitude Dynamics*, John Wiley & Sons, Inc., New York, NY, 1986.
- [14] Knight, D., "A New Method of Instantaneous Ambiguity Resolution," *Proceedings of ION GPS-94: 7th International Technical Meeting of the Satellite Division of the Institute of Navigation*, Salt Lake City, Utah, Sept. 20-23, 1994.
- [15] Knight, J. and R. Hatch, "Attitude Determination Via GPS," *Proceedings of the KIS Symposium*, Banff, Canada, Sept. 11, 1990.
- [16] Langley, R. B., "The GPS Observables," *GPS World*, April 1993.

- [17] Leick, A., *GPS Satellite Surveying*, John Wiley & Sons, Inc., New York, NY, 1990.
- [18] Lightsey, E. G., C. E. Cohen, and B. W. Parkinson, "Application of GPS Attitude Determination to Gravity Gradient Stabilized Spacecraft," AIAA-93-3788-CP.
- [19] McMillan, J. C., "Interferometric GPS Attitude: A Stochastic Error Model," Defence Research Establishment Ottawa, Report No. 1168, Feb. 1993.
- [20] Neirinckx, P. J., *Autonomous Onboard Guidance and Navigation Performance for Earth to Mars Transfer Missions*, S.M. Thesis, Department of Aeronautics and Astronautics, MIT, Jan. 1992.
- [21] Rath, J. and P. Ward, "Attitude Estimation Using GPS," *Proceedings of the National Technical Institute of Navigation*, Jan. 23-26, 1984.
- [22] Setterlund, R. H., "New Insights into Minimum-Variance Reduced-Order Filters," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 6, Nov. - Dec. 1988.
- [23] Shepperd, S. W., *Efficient Covariance Update*, CSDL, unpublished notes.
- [24] Shepperd, S. W., Private communications, Oct. 1994 - May 1995.
- [25] Strang, G., *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, 1986.
- [26] Sullivan, W. I., *Performance Analysis of an Integrated Attitude Determination System*, S.M. Thesis, Department of Aeronautics and Astronautics, MIT, May 1994.
- [27] Van Graas, F. and M. S. Braasch, "Real-Time Attitude and Heading Determination Using GPS," *GPS World*, March 1992.
- [28] Ward, L., "Spacecraft Attitude Estimation using GPS: Methodology and Results," Ph.D. Dissertation Prospectus, Department of Aerospace Engineering Sciences, University of Colorado, Boulder, Aug. 12, 1994.
- [29] Wells, D., et al, *Guide to GPS Positioning*, University of New Brunswick Graphics Services, Fredericton, N.B., Canada, 1986.
- [30] Wertz, J. R. (Ed.), *Spacecraft Attitude Determination and Control*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1978.
- [31] "Codeless GPS Attitude Determination System (CGADS)," ITHACO Technical Services, Lanham, MD 20706, Aug. 12, 1992.