BENCHMARK PROBLEMS AND SOLUTIONS
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The Scientific Committee, after careful consideration, adopted six categories of benchmark problems for the workshop. These problems do not cover all the important computational issues relevant to Computational Aeroacoustics (CAA). The deciding factor to limit the number of categories to six was the amount of effort needed to solve these problems. For reference purpose, the benchmark problems are provided below. They are followed by the exact or approximate analytical solutions. At present, an exact solution for the Category 6 problem is not available.

BENCHMARK PROBLEMS

Category 1

Problems to test the numerical dispersion and dissipation properties of a computation scheme (linear waves).

Use nondimensional variables with the following scales

$$\Delta x = \Delta r = \text{ length scale}$$

$$a_\infty \text{ (ambient sound speed) } = \text{ velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{ time scale}$$

$$\rho_\infty = \text{ density scale}$$

$$\rho_\infty a_\infty^2 = \text{ pressure scale}$$

1. Solve the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$t = 0 \quad u = 0.5 \exp\left[-(\ln 2)\left(\frac{x}{3}\right)^2\right]$$

Give numerical solution at $t = 100, 200, 300$ and 400 over $-20 \le x \le 450$. State the size of Δt used.

2. Solve the spherical wave problem

$$\frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial u}{\partial r} = 0$$

over the domain $5 \leq r \leq 450$, with initial condition $t = 0$, $u = 0$. The boundary condition at $r = 5$ is:

$$r = 5, \qquad u = \sin \omega t$$

(a) $\omega = \dfrac{\pi}{4}$

(b) $\omega = \dfrac{\pi}{3}$

Give the numerical solution at $t = 100, 200, 300$ and 400 for each case. (Do not recast the equation in a plane wave form.) State the size of Δt used.

Category 2

Problems to test the nonlinear wave propagation properties of a computational scheme.

Use dimensionless variables with the following scales

$$\Delta x = \text{ length scale}$$

$$a_\infty \text{ (ambient sound speed)} = \text{ velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{ time scale}$$

$$\rho_\infty = \text{ density scale}$$

$$\rho_\infty a_\infty^2 = \text{ pressure scale}$$

In both problems, the one-dimensional Euler equations are to be solved.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \gamma p \frac{\partial u}{\partial x} = 0$$

(You may use an equivalent form of the Euler equations.)

1. Solve the initial value problem

$$t = 0 \qquad u = 0.5 \exp \left[-(\ln 2) \left(\frac{x}{5} \right)^2 \right]$$

$$p = \frac{1}{\gamma} \left(1 + \frac{\gamma - 1}{2} u \right)^{\frac{2\gamma}{\gamma - 1}}$$

$$\rho = \left(1 + \frac{\gamma - 1}{2} u \right)^{\frac{2}{\gamma - 1}} ; \quad \gamma = 1.4$$

Use a computational domain $-50 \leq x \leq 350$. Give the spatial distribution of u, ρ, and p at $t = 10, 20, 30, 40, 50, 100, 150, 200$ and 300.

2. Solve the one-dimensional shock tube problem using the following initial conditions

$$t = 0 \qquad u = 0$$

$$p = \begin{cases} 4.4, & x < -2 \\ 2.7 + 1.7 \cos \left[\frac{(x+2)\pi}{4} \right], & -2 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

$$\rho = (\gamma p)^{\frac{1}{\gamma}}, \qquad \gamma = 1.4$$

Use a computational domain $-100 \leq x \leq 100$. Give the spatial distribution of p, ρ and u at $t = 40, 50, 60$ and 70.

Category 3

Problems to test the effectiveness of radiation boundary conditions, inflow and outflow boundary conditions and the isotropy property of the computation algorithm.

Use dimensionless variables with the following scales

$$\Delta x = \text{length scale}$$

$$a_\infty \text{ (ambient sound speed)} = \text{velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{time scale}$$

$$\rho_\infty = \text{density scale}$$

$$\rho_\infty a_\infty^2 = \text{pressure scale}$$

In both problems, the linearized two-dimensional Euler equations on a uniform mean flow are to be solved.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} = 0$$

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where

$$\mathbf{U} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} M_x\rho + u \\ M_x u + p \\ M_x v \\ M_x p + u \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} M_y\rho + v \\ M_y u \\ M_y v + p \\ M_y p + v \end{bmatrix}$$

M_x and M_y are constant mean flow Mach number in the x and y direction, respectively. (You may use an equivalent form of the above equations.)

Use a computational domain $-100 \le x \le 100$, $-100 \le y \le 100$ embedded in free space.

1. Let $M_x = 0.5$, $M_y = 0$. Solve the initial value problem, $t = 0$.

$$p = \exp\left[-(\ln 2)\left(\frac{x^2 + y^2}{9}\right)\right]$$

$$\rho = \exp\left[-(\ln 2)\left(\frac{x^2 + y^2}{9}\right)\right] + 0.1\exp\left[-(\ln 2)\frac{(x-67)^2 + y^2}{25}\right]$$

$$u = 0.04y\exp\left[-(\ln 2)\frac{(x-67)^2 + y^2}{25}\right]$$

$$v = -0.04(x - 67)\exp\left[-(\ln 2)\frac{(x-67)^2 + y^2}{25}\right]$$

Give the distributions of p, ρ, u and v at $t = 30, 40, 50, 60, 70, 80, 100, 200$ and 600.

2. Let $M_x = M_y = 0.5\cos(\frac{\pi}{4})$. Solve the initial value problem, $t = 0$.

$$p = \exp\left[-(\ln 2)\left(\frac{x^2 + y^2}{9}\right)\right]$$

$$\rho = \exp\left[-(\ln 2)\left(\frac{x^2 + y^2}{9}\right)\right] + 0.1\exp\left[-(\ln 2)\frac{(x-67)^2 + (y-67)^2}{25}\right]$$

$$u = 0.04(y - 67)\exp\left[-(\ln 2)\frac{(x-67)^2 + (y-67)^2}{25}\right]$$

$$v = -0.04(x - 67)\exp\left[-(\ln 2)\frac{(x-67)^2 + (y-67)^2}{25}\right]$$

Note: The mean flow is in the direction of the diagonal of the computational domain.

Give the distributions of p, ρ, u and v at $t = 60, 70, 80, 90, 100, 200, 600$ and 1000.
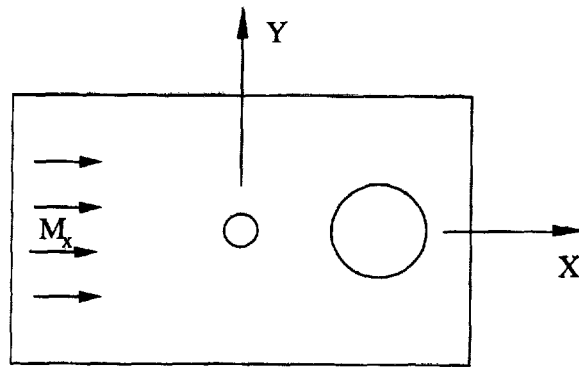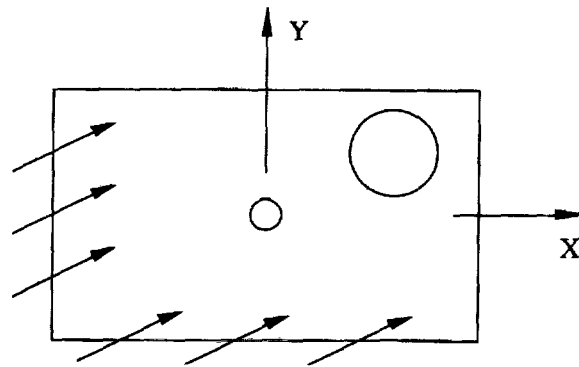
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Figure for Problem 1, Category 3

Figure for Problem 2, Category 3

Category 4

Problems to test the effectiveness of wall boundary conditions.

Use dimensionless variables with the following scales

$$\Delta x = \Delta r = \text{ length scale}$$

$$a_\infty \text{ (ambient sound speed)} = \text{ velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{ time scale}$$

$$\rho_\infty = \text{ density scale}$$

$$\rho_\infty a_\infty^2 = \text{ pressure scale}$$

1. Reflection of an acoustic pulse off a wall in the presence of a uniform flow in semi-infinite space.
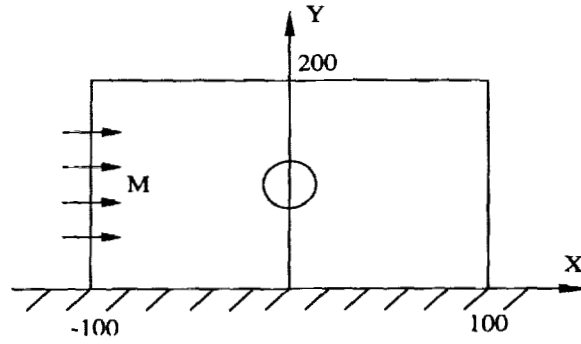
Figure for Problem 1, Category 4

Use a computational domain $-100 \leq x \leq 100, 0 \leq y \leq 200$. The wall is at $y = 0$. The linearized Euler equation in two dimensions are

$$
\frac{\partial}{\partial t}
\begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}
+ \frac{\partial}{\partial x}
\begin{bmatrix} M\rho + u \\ Mu + p \\ Mv \\ Mp + u \end{bmatrix}
+ \frac{\partial}{\partial y}
\begin{bmatrix} v \\ 0 \\ p \\ v \end{bmatrix} = 0
$$

where $M = 0.5$. The initial condition is

$$
t = 0, \qquad u = v = 0
$$

$$
p = \rho = \exp \left\{ -(\ln 2) \left[\frac{x^2 + (y - 25)^2}{25} \right] \right\}
$$

Give the pressure field at $t = 15, 30, 45, 60, 75, 100$ and 150.

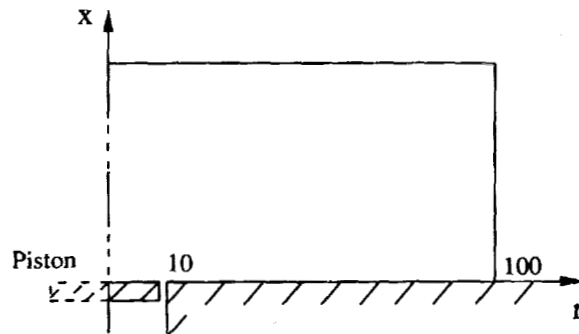2. Acoustic radiation from an oscillating circular piston in a wall

Figure for Problem 2, Category 4

Radius of piston $= 10$. Velocity of piston $u = 10^{-4} \sin(\frac{\pi t}{5})$. Use a computational domain $0 \leq x \leq 100, 0 \leq r \leq 100$. The wall and the piston are at $x = 0$. The cylindrical coordinate system is centered at the center of the piston. With axisymmetry, the linearized Euler

equations are

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} + \frac{\partial}{\partial r} \begin{bmatrix} v \\ 0 \\ p \\ v \end{bmatrix} + \begin{bmatrix} \frac{v}{r} \\ 0 \\ 0 \\ \frac{v}{r} \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} u \\ p \\ 0 \\ u \end{bmatrix} = 0$$

The initial conditions are:

$$t = 0 \qquad \rho = u = v = p = 0$$

Give the time harmonic pressure distribution at the beginning, $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of a period of piston oscillation.

Category 5

Problem to test the suitability of a numerical scheme for direct numerical simulation of very small amplitude acoustic waves superimposed on a non-uniform mean flows in a semi-infinite duct.

Use nondimensional variables with the following scales

$$\Delta x = \text{length scale}$$

$$a_\infty \text{ (sound speed far upstream)} = \text{velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{time scale}$$

$$\rho_\infty \text{ (density of gas upstream)} = \text{density scale}$$

$$\rho_\infty a_\infty^2 = \text{pressure scale}$$

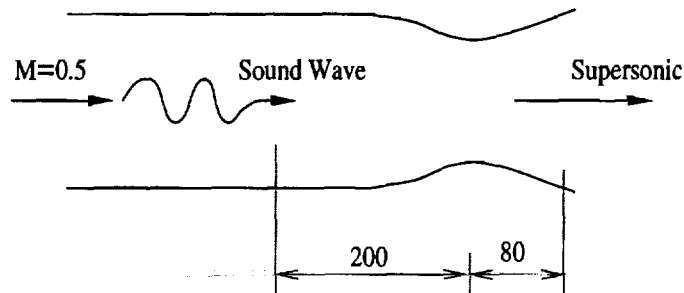A small amplitude sound wave is incident on a convergent-divergent nozzle as shown

Figure for Category 5 Problem

Use a computational domain $-200 \le x \le 80$. The area of the nozzle is given by

$$A(x) = \begin{cases} 134 & x \le -100 \\ 117 - 17\cos\left(\frac{\pi x}{100}\right) & -100 \le x \le 19 \\ 97.2 + 0.3x & 19 \le x \le 80 \end{cases}$$

The quasi-one-dimensional unsteady flow equations are

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho u A}{\partial x} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p A}{\partial t} + \frac{\partial p u A}{\partial x} + (\gamma - 1)p\frac{\partial u A}{\partial x} = 0$$

Far upstream $x \le -200$, there is an incoming acoustic wave. Together with the steady inflow, the velocity, pressure and density are given by

$$\begin{bmatrix} u \\ p \\ \rho \end{bmatrix} = \begin{bmatrix} M \\ \frac{1}{\gamma} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \varepsilon \sin\left[\omega\left(\frac{x}{1+M} - t\right)\right] \cdot$$

Take $\gamma = 1.4$, $M = 0.5$, $\varepsilon = 10^{-6}$, $\omega = 0.1\pi$, calculate the transmitted sound wave at the nozzle exit. Give $p(t) - \bar{p}$ over a time period; $\bar{p}$ is the time averaged pressure.

Category 6

Problems to test the ability of a numerical scheme to calculate aeroacoustic source.

Use dimensionless variables with the following scales

$$\Delta x = \text{ length scale}$$

$$a_\infty \text{ (ambient sound speed)} = \text{ velocity scale}$$

$$\frac{\Delta x}{a_\infty} = \text{ time scale}$$

$$\rho_\infty = \text{ density scale}$$

$$\rho_\infty a_\infty^2 = \text{ pressure scale}$$

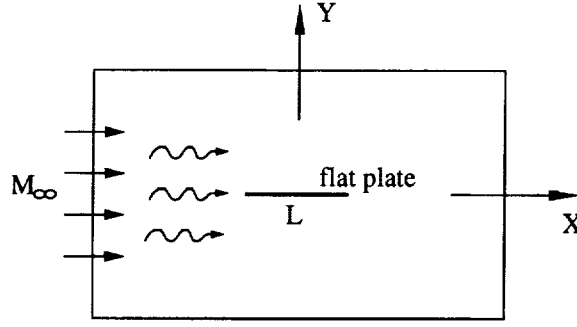1. Sound generation by gust-blade interaction (two-dimensional)

Figure for Category 6 Problem

Use a computational domain $-100 \leq x \leq 100$, $-100 \leq y \leq 100$. The blade is a flat plate of length L ($L = 30$) lying along the x-axis centered at the origin. There is a Mach 0.5 uniform mean flow in the x-direction. The mean flow carries a gust with velocity component in the y-direction given by

$$v = 0.1 \sin \left[\frac{\pi}{8} \left(\frac{x}{M_\infty} - t \right) \right], \quad M_\infty = 0.5$$

The linearized Euler equations are

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} M_\infty \rho + u \\ M_\infty u + p \\ M_\infty v \\ M_\infty p + u \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} v \\ 0 \\ p \\ v \end{bmatrix} = 0$$

Determine the intensity of radiated sound, $\overline{p}^2$, along the coordinate lines $x = \pm 95$ and $y = \pm 95$.

SOLUTIONS

Category 1

Problem 1. The solution is

$$u(x,t) = 0.5 \exp \left[-(\ell n\, 2) \left(\frac{x-t}{3} \right)^2 \right]$$

Problem 2. The solution is

$$u(r,t) = \begin{cases} 0, & r > t + 5 \\ \frac{5}{r} [\sin \omega (t - r + 5)], & r \leq t + 5. \end{cases}$$

Problem 1. An approximate solution can be found by using the simple wave assumptions (Chapter 6, G.B. Whitham, "Linear and nonlinear waves"). These assumptions are

1. The flow is isentropic
2. The Riemann invariant $\frac{2a}{\gamma-1} - u = \frac{2}{\gamma-1}$, which starts from the uniform region ahead of the pulse, is valid everywhere.

With these assumptions, the Euler equations reduces to the nonlinear simple wave equation

$$\frac{\partial u}{\partial t} + \left(1 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = 0$$

This quasi-linear first-order equation can be solved by the method of characteristics. For the given initial conditions, a shock will form at the front of the pulse as the disturbance propagates to the right. The location of the shock may be found approximately by the use of Whitham's equal area rule.

Problem 2. The standard shock tube solution is a good approximate solution. The standard solution is available in most books on gas dynamics.

<center>Category 3</center>

Problem 1. Let $\alpha_1 = \frac{(\ell n 2)}{9}$, $\alpha_2 = \frac{(\ell n 2)}{25}$, $M = 0.5$, $\eta = [(x - Mt)^2 + y^2]^{\frac{1}{2}}$.

The solution is

$$u = \frac{(x - Mt)}{2\alpha_1 \eta} \int_0^\infty e^{\frac{-\xi^2}{4\alpha_1}} \sin(\xi t) J_1(\xi\eta)\xi \, d\xi + 0.04 y e^{-\alpha_2[(x-67-Mt)^2+y^2]}$$

$$v = \frac{y}{2\alpha_1 \eta} \int_0^\infty e^{\frac{-\xi^2}{4\alpha_1}} \sin(\xi t) J_1(\xi\eta)\xi \, d\xi - 0.04(x - 67 - Mt) e^{-\alpha_2[(x-67-Mt)^2+y^2]}$$

$$p = \frac{1}{2\alpha_1} \int_0^\infty e^{\frac{-\xi^2}{4\alpha_1}} \cos(\xi t) J_0(\xi\eta)\xi \, d\xi$$

$$\rho = p + 0.1 e^{-\alpha_2[(x-67-Mt)^2+y^2]}$$

where $J_0(\ )$ and $J_1(\ )$ are Bessel functions of order 0 and 1.

(Reference: C.K.W. Tam and J.C. Webb, "Dispersion-Relation-Preserving finite difference schemes for computational acoustics," J. Computational Phys., Vol. 107, pp. 262–281, 1993.)

Problem 2. The solution can be obtained from that of problem 1 by a coordinate transformation.

Category 4

Problem 1. Let $\alpha = \frac{(\ell n 2)}{25}$, $\eta = [(x - Mt)^2 + (y - 25)^2]^{\frac{1}{2}}$, $\zeta = [(x - Mt)^2 + (y + 25)^2]^{\frac{1}{2}}$.
The solution is

$$u = \frac{(x - Mt)}{2\alpha\eta} \int_0^\infty e^{-\frac{\xi^2}{4\alpha}} \sin(\xi t) J_1(\xi\eta)\xi \, d\xi + \frac{(x - Mt)}{2\alpha\zeta} \int_0^\infty e^{-\frac{\xi^2}{4\alpha}} \sin(\xi t) J_1(\xi\zeta)\xi \, d\xi$$

$$y = \frac{(y - 25)}{2\alpha\eta} \int_0^\infty e^{-\frac{\xi^2}{4\alpha}} \sin(\xi t) J_1(\xi\eta)\xi \, d\xi + \frac{(y - 25)}{2\alpha\zeta} \int_0^\infty e^{-\frac{\xi^2}{4\alpha}} \sin(\xi t) J_1(\xi\zeta)\xi \, d\xi$$

$$p = \rho = \frac{1}{2\alpha} \int_0^\infty e^{-\frac{\xi^2}{4\alpha}} \cos(\xi t)[J_0(\xi\eta) + J_0(\xi\zeta)]\xi \, d\xi$$

Problem 2. Let $\varepsilon = 10^{-4}$, $R = 10$, $\omega = \frac{\pi}{5}$.
The solution is

$$p = \rho = \text{Re}\left[\varepsilon R\omega \int_0^\infty \frac{J_1(\xi R)}{(\xi^2 - \omega^2)^{\frac{1}{2}}} J_0(\xi r) e^{-(\xi^2 - \omega^2)^{\frac{1}{2}}x - i\omega t} d\xi\right]$$

$$u = \text{Im}\left[-\varepsilon R \int_0^\infty J_1(\xi R) J_0(\xi r) e^{-(\xi^2 - \omega^2)^{\frac{1}{2}}x - i\omega t} d\xi\right]$$

$$v = \text{Im}\left[-\varepsilon R \int_0^\infty \frac{J_1(\xi R)}{(\xi^2 - \omega^2)^{\frac{1}{2}}} \xi J_1(\xi r) e^{-(\xi^2 - \omega^2)^{\frac{1}{2}}x - i\omega t} d\xi\right]$$

where Re[ ] = the real part of and Im[ ] = the imaginary part of.

Note: $(\xi^2 - \omega^2)^{\frac{1}{2}} = -i|\xi^2 - \omega^2|^{\frac{1}{2}}$ for $\xi < \omega$.

A fairly accurate solution of this problem can be found by first determining the governing equations for the amplitude functions of the time-periodic disturbances inside the nozzle. These equations are ordinary differential equations but with variable coefficients. They can be integrated numerically.

Let the solution be separated into a mean and a time-periodic part in the form

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = \begin{bmatrix} \overline{\rho} \\ \overline{u} \\ \overline{p} \end{bmatrix} + \mathrm{Re}\left\{ \begin{bmatrix} \widehat{\rho}(x) \\ \widehat{u}(x) \\ \widehat{p}(x) \end{bmatrix} e^{-i\omega t} \right\}$$

The physical quantities of the mean flow at the nozzle throat will be denoted by a subscript $*$. With the area ratio A_*/A_1 known, where A_1 is the area of the uniform duct, ρ_* is first found by solving the equation

$$\rho_*^{\gamma+1}\left(\frac{A_*}{A_1}\right)^2 + \frac{2}{\gamma-1} = \frac{\gamma+1}{\gamma-1}\rho_*^{\gamma-1}$$

The other variables at the nozzle throat are given by

$$p_* = \frac{1}{\gamma}\rho_*^{\gamma}, \qquad u_* = \rho_*^{\frac{\gamma-1}{2}}$$

The mean flow solution is

$$\overline{\rho}\,\overline{u}\,A = \rho_* u_* A_*$$

$$\frac{\overline{p}}{p_*} = \left(\frac{\overline{\rho}}{\rho_*}\right)^{\gamma}$$

$$\frac{\overline{u}^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_*}{\rho_*^{\gamma}}\overline{\rho}^{\gamma-1} = \frac{u_*^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_*}{\rho_*}$$

The linearized governing equations for the amplitude functions $\widehat{\rho}$, $\widehat{u}$ and $\widehat{p}$ are

$$\frac{d\widehat{u}}{dx} = \frac{1}{(\overline{\rho}\,\overline{u}^2 - \gamma\overline{p})}\left[-\overline{u}^2\frac{d\overline{u}}{dx}\widehat{\rho} + \left(i\omega\overline{\rho}\,\overline{u} - \overline{\rho}\,\overline{u}\frac{d\overline{u}}{dx} + \frac{d\overline{p}}{dx} - \frac{\gamma\overline{p}}{A}\frac{dA}{dx}\right)\widehat{u} - \left(i\omega - \frac{\gamma}{A}\frac{d(\overline{u}\,\overline{A})}{dx}\right)\widehat{p} \right]$$

$$\frac{d\widehat{p}}{dx} = \frac{1}{(\overline{\rho}\,\overline{u}^2 - \gamma\,\overline{p})}\left[\gamma\overline{p}\,\overline{u}\frac{d\overline{u}}{dx}\widehat{\rho} + \left(-i\omega\gamma\overline{p}\,\overline{\rho} + \gamma\overline{p}\,\overline{\rho}\frac{d\overline{u}}{dx} - \overline{\rho}\,\overline{u}\frac{d\overline{p}}{dx} - \frac{\gamma\overline{p}\,\overline{\rho}\,\overline{u}}{A}\frac{dA}{dx}\right)\widehat{u} \right.$$

$$\left. + \left(i\omega\overline{\rho}\,\overline{u} - \frac{\gamma\overline{\rho}\,\overline{u}}{A}\frac{d(\overline{u}\,A)}{dx}\right)\widehat{p} \right] \tag{1}$$

$$\frac{d\widehat{\rho}}{dx} = -\frac{\overline{\rho}}{\overline{u}}\frac{d\widehat{u}}{dx} + \left(i\omega A - \frac{d(\overline{u}\,A)}{dx}\right)\frac{1}{\overline{u}\,A}\widehat{\rho} - \frac{1}{\overline{u}\,A}\frac{d(\overline{\rho}\,A)}{dx}\widehat{u}$$

In the uniform region of the duct, the solution of (1) that matches the incoming acoustic wave is

$$
\begin{bmatrix} \widehat{\rho} \\ \widehat{u} \\ \widehat{p} \end{bmatrix} = -i\varepsilon \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{\frac{i\omega x}{1+M}} + c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{\frac{i\omega x}{1-M}} \tag{2}
$$

In (2), the second term represents the reflected acoustic wave. The unknown amplitude c is to be determined later.

Equations (1) have a regular singular point at the nozzle throat ($x = 0$). Near the throat, there are two non-singular series solutions. The first two terms of these solutions are

$$
\widehat{\rho} = \left\{ \frac{i\omega\rho_*}{u_*^2 [\frac{1}{(\gamma+1)A_*}(\frac{d^2A}{dx^2})_*]^{\frac{1}{2}}} - \frac{2\rho_*}{u_*} \right\} u_0 + \left\{ \frac{-i\omega}{u_*^3 [\frac{1}{(\gamma+1)A_*}(\frac{d^2A}{dx^2})_*]^{\frac{1}{2}}} + \frac{\gamma}{u_*^2} \right\} p_0 + \rho_1 x + \ldots
$$

$$
\widehat{u} = u_0 + u_1 x + \ldots \tag{3}
$$

$$
\widehat{p} = p_0 + p_1 x + \ldots
$$

where u_0 and p_0 are arbitrary constants. ρ_1, u_1 and p_1 are functions of u_0 and p_0.

A numerical solution of (1) can be constructed by starting the solution slightly upstream of the nozzle throat at $x = -\delta$ ($\delta << 1$) using (3) as the starting solution. (For small δ, the terms of the series involving δ and powers of δ may be neglected.) The numerical integration proceeds upstream until the uniform duct region is reached. At this point, the numerical solution must match solution (2). This provides three algebraic equations for the three unknowns p_0, u_0 and c. Once these constants are found, the solution upstream of the nozzle throat is known.

For the solution downstream of the nozzle throat one can start integrating (1) numerically at a point just downstream, say at $x = \delta$. Again (3) is used as the starting solution. The numerical integration proceeds downstream until the nozzle exit is reached. With p_0, u_0 already found, the amplitude functions are now completely determined along the entire length of the nozzle.

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