Degradation in Finite-Harmonic Subcarrier Demodulation

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Previous estimates on the degradations due to a subcarrier loop assume a square-wave subcarrier. This article provides a closed-form expression for the degradations due to the subcarrier loop when a finite number of harmonics are used to demodulate the subcarrier, as in the case of the buffered telemetry demodulator. We compared the degradations using a square wave and using finite harmonics in the subcarrier demodulation and found that, for a low loop signal-to-noise ratio, using finite harmonics leads to a lower degradation. The analysis is under the assumption that the phase noise in the subcarrier (SC) loop has a Tikhonov distribution. This assumption is valid for first-order loops.

I. Introduction

In an imperfect subcarrier demodulation, the difference between the phase of the reference signal and that of the subcarrier of the received signal causes the signal power to degrade while the noise power remains the same. This degradation is measured as the ratio of the reduced symbol energy-to-noise density ratio (E_s/N_0), or symbol signal-to-noise ratio (SNR), to the symbol SNR of an ideal demodulation where the phase difference is zero. The degradations due to the subcarrier loop were previously computed assuming a square wave [3]. This assumption is inappropriate in the case where only a finite number of harmonics of the subcarrier are there to be demodulated, as in the buffered telemetry demodulator (BTD) [2]. This article provides a closed-form expression for computing the degradation due to a finite-harmonic subcarrier tracking loop. Numerically, we found that, for low loop SNR cases, we actually have less degradation using a finite number of harmonics than using "all" the harmonics, namely, the square wave. The degradation due solely to the subcarrier loop using four harmonics is 0.15 to 0.3 dB lower than that using a square wave for loop SNRs in the range of 14 to 30 dB.

At first glance, the above may seem to contradict the intuition that the more harmonics we use, the higher the SNR we should get. This intuition is correct when the loop SNR is high, that is, when the jitter of the phase difference (between the true and the reference phases) is low. At low loop SNRs, however, we have a different scenario.

To explain this, let us first take a look at how the subcarriers are demodulated. A square-wave subcarrier is demodulated by multiplying the received signal by a square-wave reference signal. When we only have a finite number of harmonics of the square-wave subcarrier, the current design for the BTD [2]

Acknowledgments

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demodulates the subcarrier by multiplying each received harmonic by its reference signal and combining the resulting harmonics with the weights of $1/n$, where n indicates the nth harmonic. In the case of square-wave subcarrier demodulation, we are implicitly combining the harmonics the same way, only now we have an infinite number of harmonics (see the Appendix for the proof). The reference signals are generated by using the phase of the fundamental frequency component or the first harmonic.

Therefore, if the first harmonic has a phase noise with a standard deviation of σ, then the nth harmonic will have a phase noise with a standard deviation of $n\sigma$, which implies that the nth harmonic will suffer a higher degradation than the first one. At low loop SNRs, the degradations in higher harmonics can be even higher than the SNR that they contribute. In such cases, higher harmonics should not be used in the subcarrier demodulation.

In the full spectrum combining case [1], more harmonics means that more data need to be transmitted to the combining location or stored locally. In the case of intercontinental arraying, where data transmission becomes expensive, suppressing higher harmonics becomes an important issue. Later in this article, we will show that, for a given loop SNR, there is an optimum number of harmonics that should be used, and in the region of the operating loop SNRs, these numbers are mostly finite.

To compare the degradations when using finite harmonics and a square wave, we first give an expression to compute the degradations using a square wave, assuming that the phase noise has a Tikhonov distribution. This assumption is valid for first-order loops only [1]. For higher loop SNRs, the degradation due to the phase noise with a Tikhonov distribution is very close to that due to a phase noise with a Gaussian distribution. In the range of the operating loop SNRs, the two distribution assumptions lead to similar results. We then give an expression of degradation for finite-harmonic subcarrier demodulation, assuming that the phase noise has a Tikhonov distribution.

II. Square-Wave Case

When the subcarrier is a square wave, the degradation due to the subcarrier loop has the form [1]

$$\overline{C^2_{sc_{sq}}} = 1 - \frac{4}{\pi}\overline{|\phi_{sc}|} + \frac{4}{\pi^2}\overline{\phi^2_{sc}} \tag{1}$$

where ϕ_{sc} is the phase noise in the square-wave subcarrier tracking loop. If the phase noise, ϕ_{sc}, is assumed to have a Gaussian distribution with zero mean and a variance of σ^2, then [1]

$$\overline{|\phi_{sc}|} = \sqrt{\frac{2}{\pi}}\sigma$$

$$\overline{\phi^2_{sc}} = \sigma^2$$

The degradation due to the subcarrier loop is [1]

$$\overline{C^2_{scG_{sq}}} = 1 - \sqrt{\frac{32}{\pi^3}}\sigma + \frac{4}{\pi^2}\sigma^2 \tag{2}$$

While the Gaussian assumption is accurate for high loop SNR cases, Tikhonov distribution is a better assumption for low loop SNR cases. Note that the Tikhonov assumption is valid for first-order loops. If the phase noise ϕ_{sc} in a Costas loop is assumed to have a Tikhonov distribution, then we can show that

$$\overline{|\phi_{sc}|} = \int\limits_{-\pi/2}^{\pi/2} \frac{\exp\left[(1/4)\rho_{sc}\cos 2\phi_{sc}\right]}{\pi I_0(\rho_{sc}/4)} |\phi_{sc}| d\phi_{sc}$$

$$= \frac{\pi}{4} + \frac{1}{\pi I_0(\rho_{sc}/4)} \sum_{k=1}^{\infty} I_k(\rho_{sc}/4) \frac{(-1)^k - 1}{k^2} \tag{3}$$

$$\overline{\phi_{sc}^2} = \int\limits_{-\pi/2}^{\pi/2} \frac{\exp\left[(1/4)\rho_{sc}\cos 2\phi_{sc}\right]}{\pi I_0(\rho_{sc}/4)} \phi_{sc}^2 \, d\phi_{sc}$$

$$= \frac{\pi^2}{12} + \frac{1}{I_0(\rho_{sc}/4)} \sum_{k=1}^{\infty} I_k(\rho_{sc}/4) \frac{(-1)^k}{k^2} \tag{4}$$

where I_k is the modified Bessel function of order k, and ρ_{sc} is the subcarrier-loop SNR, which can be computed using

$$\rho_{sc} = \frac{4}{\pi^2} \frac{1}{B_{sc}W_{sc}} \frac{P_d}{N_0} \left[1 + \frac{1}{2E_s/N_0}\right]^{-1} \tag{5}$$

Here B_{sc} denotes the one-sided subcarrier loop bandwidth, W_{sc} denotes the subcarrier window size [2], P_d/N_0 denotes the total data power over the one-sided noise density, and E_s/N_0 denotes the symbol energy-to-noise density ratio.

Note that Eqs. (3) and (4) are different from Eqs. (22) and (23) in [1] in that the former are for Costas loops and the latter are for phase-locked loops. Assuming a Tikhonov distribution, the degradation due to the subcarrier loop is

$$\overline{C_{sc_{sq}}^2} = \frac{1}{3} + \frac{4}{\pi^2} \frac{1}{I_0(\rho_{sc}/4)} \sum_{k=1}^{\infty} I_k(\rho_{sc}/4) \frac{1}{k^2} \tag{6}$$

III. Finite Number of Harmonics Case

When a finite number of harmonics are used to track and demodulate the subcarrier, as in the BTD, the signal amplitude has the form [2]

$$S_{sc} = \frac{8}{\pi^2} \sum_{m=0}^{L-1} \frac{\cos\left[(2m+1)\phi_{sc}\right]}{(2m+1)^2} \tag{7}$$

where L is the number of harmonics and ϕ_{sc} is the phase noise resulting from the subcarrier tracking loop. Clearly, when $\phi_{sc} = 0$, we have the ideal case,

$$S_{sc_{ideal}} = \frac{8}{\pi^2} \sum_{m=0}^{L-1} \frac{1}{(2m+1)^2} \tag{8}$$

Taking the ratio of Eq. (7) and Eq. (8), we obtain the signal-amplitude degradation,

$$C_{sc} = \frac{S_{sc}}{S_{sc_{ideal}}}$$

$$= \frac{\sum_{m=0}^{L-1} \left(\cos\left[(2m+1)\phi_{sc} \right] / (2m+1)^2 \right)}{\sum_{m=0}^{L-1} \left(1/(2m+1)^2 \right)} \tag{9}$$

Squaring Eq. (9) and taking the expectation, we have the signal power degradation,

$$\overline{C_{sc}^2} = \frac{1}{\left(\sum_{m=0}^{L-1} \left[1/(2m+1)^2 \right] \right)^2} \sum_{m=0}^{L-1} \sum_{n=0}^{L-1} \frac{\overline{\cos 2(m-n)\phi_{sc}} + \overline{\cos 2(m+n+1)\phi_{sc}}}{2(2m+1)^2(2n+1)^2} \tag{10}$$

The noise power after the subcarrier demodulation is not affected by the phase noise in the subcarrier loop. This can be observed from the noise power expressions in Eqs. (A-28) and (A-29) of [2]. This implies that the degradation in the symbol SNR is the same as the signal-power degradation as given in Eq. (10).

For first-order Costas loops, the phase noise ϕ_{sc} has a Tikhonov distribution:

$$p(\phi_{sc}) = \begin{cases} \dfrac{\exp[(1/4)\rho_{sc}\cos(2\phi_{sc})]}{\pi I_0(\rho_{sc}/4)}, & |\phi_{sc}| \leq \dfrac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Hence we have,

$$\overline{\cos(n\phi_{sc})} = \frac{I_{n/2}(\rho_{sc}/4)}{I_0(\rho_{sc}/4)} \tag{12}$$

where n is an even number, I_n is the modified Bessel function of order n, and ρ_{sc} is the subcarrier-loop SNR.

Plugging Eq. (12) in Eq. (10), we have

$$\overline{C_{sc}^2} = \frac{1}{\left(\sum_{m=0}^{L-1} \left[1/(2m+1)^2 \right] \right)^2} \frac{1}{I_0(\rho_{sc}/4)} \sum_{m=0}^{L-1} \sum_{n=0}^{L-1} \frac{I_{m-n}(\rho_{sc}/4) + I_{m+n+1}(\rho_{sc}/4)}{2(2m+1)^2(2n+1)^2} \tag{13}$$

As L approaches infinity, Eq. (13) becomes identical to Eq. (6) (see the Appendix for the proof).

For $L = 4$, we have the SNR degradation due to the subcarrier loop,

$$\overline{C_{sc}^2} = \frac{1}{\left(\sum_{m=0}^{L-1} [1/(2m+1)^2]\right)^2} \left[0.507181 + 0.616372\frac{I_1(\rho_{sc}/4)}{I_0(\rho_{sc}/4)} + 0.153379\frac{I_2(\rho_{sc}/4)}{I_0(\rho_{sc}/4)}\right.$$

$$+ 0.066581\frac{I_3(\rho_{sc}/4)}{I_0(\rho_{sc}/4)} + 0.0248526\frac{I_4(\rho_{sc}/4)}{I_0(\rho_{sc}/4)} + 0.00306757\frac{I_5(\rho_{sc}/4)}{I_0(\rho_{sc}/4)}$$

$$\left. +0.000816327\frac{I_6(\rho_{sc}/4)}{I_0(\rho_{sc}/4)} + 0.000208247\frac{I_7(\rho_{sc}/4)}{I_0(\rho_{sc}/4)}\right] \tag{14}$$

The subcarrier-loop SNR, ρ_{sc}, can be computed using the following equations:[1]

$$\rho_{sc} = \frac{\alpha\beta^2}{\gamma B_{sc}}\frac{P_d}{N_0}\left(\alpha + \frac{1}{2E_s/N_0}\right)^{-1}$$

where

$$\alpha = \frac{8}{\pi^2}\sum_{m=0}^{L-1}\frac{1}{(2m+1)^2}$$

$$\beta = \frac{8}{\pi^2}\sum_{n=0}^{L-1}w_n$$

$$\gamma = \frac{8}{\pi^2}\sum_{n=0}^{L-1}w_n^2$$

and

$$w_n = \frac{\sin[(2n+1)(\pi/2)W_{sc}]}{2n+1}$$

For different loop SNRs, the degradations $\overline{C_{scGsq}^2}$, $\overline{C_{sc_{sq}}^2}$, and $\overline{C_{sc}^2}$ in Eqs. (2), (6), and (14), respectively, are plotted in Fig. 1. Figure 2 shows the achievable subcarrier-loop SNR for both square wave and four harmonics for $P_d/N_0 = 15$ dB-Hz at a symbol rate of 100 sym/s with a suppressed carrier. The window sizes in the subcarrier loops for the square wave and the four harmonics are $W_{sc} = 1/4$ and $W_{sc} = 1/16$, respectively. For the above parameters, the achievable subcarrier-loop SNRs are almost the same.

IV. Optimum Number of Harmonics

To make a fair comparison among the square wave and different numbers of harmonics in the subcarrier, we should compare the losses due to all three loops (carrier, subcarrier, and symbol) and the harmonic cutoffs, since the harmonic cutoffs also affect the carrier and symbol loop SNRs. The degradation due

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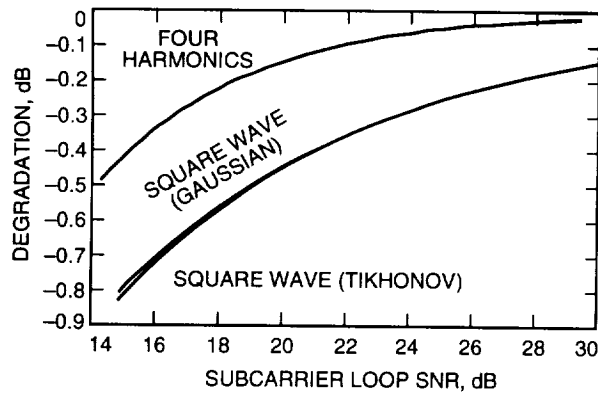
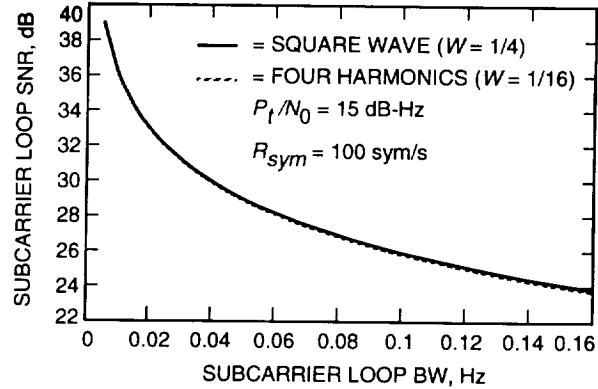**Fig. 1.** Degradation due to the subcarrier loop versus loop SNR.

Fig. 2. Subcarrier loop SNR versus loop bandwidth.

to the suppressed-carrier loop can be found in [3], while the degradation due to the symbol loop can be found in [1]. Finally, the degradation due to the harmonic cutoffs can be found in [4].

With the number of harmonics limited to less than or equal to four, we compare the degradations in all loops, including the loss due to using a finite number of harmonics. For a particular set of parameters, the comparison is shown in Fig. 3. It can be observed that for a subcarrier-loop SNR below 16 dB, adding the fourth harmonic does not increase symbol SNR. On the other hand, the loop may lose lock for a subcarrier-loop SNR below 16 dB, so the region of operation has to be greater than 16 dB. For this region, using four harmonics will lead to a lower degradation than will using fewer harmonics.

Without any limitation on the number of harmonics, we computed the degradations due to all three loops and to the harmonic cutoffs. For the same set of parameters, we plotted the degradation versus the subcarrier-loop SNR for different numbers of harmonics, as shown in Fig. 4. We found the optimum numbers of harmonics for three regions of subcarrier-loop SNR and tabulated them in Table 1. By the optimum number of harmonics, we mean that, using more harmonics than the optimum will result in a higher degradation in symbol SNR. Note that this table only applies to the set of parameters listed in Fig. 4.
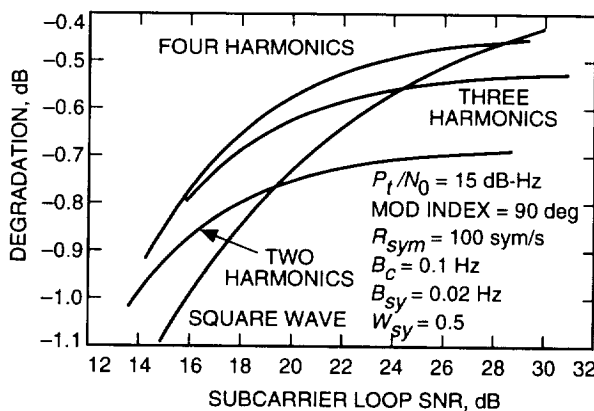

Fig. 3. Comparison using two, three, and four harmonics in terms of degradations due to all loops and harmonic cutoffs.
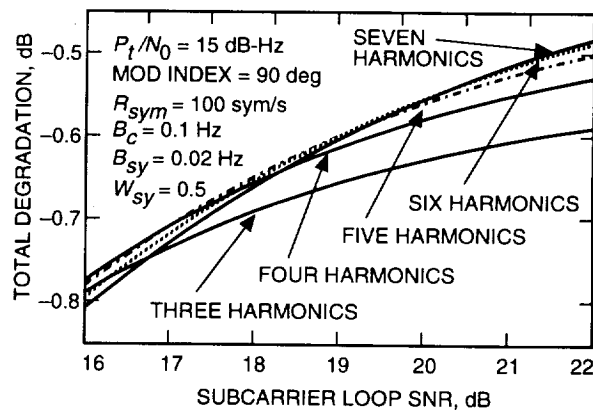

Fig. 4. Optimum number of harmonics in terms of degradations due to all loops and harmonic cutoffs.

C-2

Table 1. Optimum number of harmonics.

SC loop SNR, dB	Optimum number of harmonics
16.0 to 17.0	4
17.0 to 18.8	5
18.8 to 20.3	6

V. Conclusion

In this article, we presented a closed-form expression to compute the degradation due to the subcarrier loop when only a finite number of harmonics are used to demodulate the subcarrier. This expression assumes that the phase noise has a Tikhonov distribution, which is valid for first-order loops. Using this expression, we computed the degradations in the subcarrier loop for different numbers of harmonics in the subcarrier and found that, in certain regions of the subcarrier-loop SNRs, using a finite number of harmonics leads to a lower degradation in symbol SNR than does using all harmonics or a square wave.

Acknowledgments

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Appendix

As the Number of Harmonics Approaches Infinity

To prove that Eq. (13) approaches Eq. (6) as the number of harmonics approaches infinity, it suffices to prove that

$$\frac{1}{\left(\sum_{m=0}^{\infty}[1/(2m+1)^2]\right)^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\cos 2(m-n)\phi + \cos 2(m+n+1)\phi}{2(2m+1)^2(2n+1)^2} = \frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k\phi)}{k^2} \quad \text{(A-1)}$$

Expanding the left side of the above equation and ignoring the coefficient before the summations, which has the value $(8/\pi^2)^2$, we have

$$\text{left side} = \sum_{m=0}^{\infty}\frac{1}{2(2m+1)^4} + \sum_{m=0}^{\infty}\sum_{n=0,n\neq m}^{\infty}\frac{\cos(2(m-n)\phi)}{2(2m+1)^2(2n+1)^2} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\cos[2(m+n+1)\phi]}{2(2m+1)^2(2n+1)^2} \quad \text{(A-2)}$$

The first term of "left side" is

$$\sum_{m=0}^{\infty}\frac{1}{2(2m+1)^4} = \frac{1}{3}\left(\frac{\pi^2}{8}\right)^2 \quad \text{(A-3)}$$

For the second term of "left side," let $k = m - n$. For a fixed n, k runs from $-n$ to infinity. The second term becomes

$$\sum_{m=0}^{\infty}\sum_{n=0,n\neq m}^{\infty}\frac{\cos(2(m-n)\phi)}{2(2m+1)^2(2n+1)^2} = \sum_{n=0}^{\infty}\sum_{k=-n,k\neq 0}^{\infty}\frac{\cos 2k\phi}{2(2n+1+2k)^2(2n+1)^2}$$

$$= \sum_{k=1}^{\infty}\frac{\cos 2k\phi}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1+2k)^2(2n+1)^2} + \sum_{n=0}^{\infty}\sum_{k=1}^{n}\frac{\cos 2k\phi}{2(2n+1-2k)^2}$$

$$\text{(A-4)}$$

The inner sum of the first term in the above equation is

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1+2k)^2(2n+1)^2} = \frac{1}{(2k)^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1+2k)^2} + \frac{4}{(2k)^3}\sum_{n=0}^{\infty}\frac{1}{2n+1+2k}$$

$$+ \frac{1}{(2k)^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \frac{4}{(2k)^3}\sum_{n=0}^{\infty}\frac{1}{2n+1}$$

$$= \frac{1}{(2k)^2}\left(\frac{\pi^2}{4} - \sum_{q=0}^{k-1}\frac{1}{2q+1}\right) - \frac{4}{(2k)^3}\sum_{q=0}^{k-1}\frac{1}{2q+1} \quad \text{(A-5)}$$

For the third term of "left side," let $p = m + n + 1$. Then, for a fixed m, p runs from $m + 1$ to infinity. The third term becomes

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\cos(2(m + n + 1)\phi)}{2(2m + 1)^2(2n + 1)^2} = \sum_{m=0}^{\infty} \sum_{p=m+1}^{\infty} \frac{\cos 2p\phi}{2(2m + 1)^2(2m + 1 - 2p)^2}$$

$$= \sum_{p=1}^{\infty} \frac{\cos 2p\phi}{2} \sum_{m=0}^{\infty} \frac{1}{(2m + 1 - 2p)^2(2m + 1)^2} - \sum_{m=0}^{\infty} \sum_{p=1}^{m} \frac{\cos 2p\phi}{2(2m + 1 - 2p)^2}$$

$$\text{(A-6)}$$

The inner sum of the first term in the above equation is

$$\sum_{m=0}^{\infty} \frac{1}{(2m + 1 - 2p)^2(2m + 1)^2} = \frac{1}{(2p)^2} \sum_{m=0}^{\infty} \frac{1}{(2m + 1 - 2p)^2} - \frac{4}{(2k)^3} \sum_{m=0}^{\infty} \frac{1}{2m + 1 - 2p}$$

$$+ \frac{1}{(2p)^2} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^2} + \frac{4}{(2k)^3} \sum_{m=0}^{\infty} \frac{1}{2m + 1}$$

$$= \frac{1}{(2p)^2} \left(\frac{\pi^2}{4} + \sum_{q=0}^{p-1} \frac{1}{2q + 1} \right) + \frac{4}{(2p)^3} \sum_{q=0}^{p-1} \frac{1}{2q + 1} \qquad \text{(A-7)}$$

Substitute Eq.(A-5) in Eq. (A-4), and Eq. (A-7) in Eq. (A-6), and then, adding the results, we have the sum of the second and third terms of "left side":

$$\sum_{m=0}^{\infty} \sum_{n=0,n\neq m}^{\infty} \frac{\cos(2(m - n)\phi)}{2(2m + 1)^2(2n + 1)^2} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\cos(2(m + n + 1)\phi)}{2(2m + 1)^2(2n + 1)^2} = \sum_{k=1}^{\infty} \frac{\cos 2k\phi}{(2k)^2} \frac{\pi^2}{4} \qquad \text{(A-8)}$$

Finally, adding the first term to the above, and multiplying the coefficient $(8/\pi^2)^2$, we have "left side" equal to

$$\text{left side} = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos 2k\phi}{k^2} \qquad \text{(A-9)}$$