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Tables of Properties of Airfoil Polynomials

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Abstract

This monograph provides an extensive list of formulas for airfoil polynomials. These polynomials provide convenient expansion functions for the description of the downwash and pressure distributions of linear theory for airfoils in both steady and unsteady subsonic flow.

Symbols

F	hypergeometric function
f	arbitrary function
J_n	Bessel function of first kind
K	kernel function of pressure-downwash integral equation
P_n	Legendre polynomial
$P_n^{1/2}, P_n^{-1/2}$	associated Legendre functions of first kind
$P_n^{(-1/2,\,1/2)}, P_n^{(1/2,\,-1/2)}$	Jacobi polynomials
p	arbitrary polynomial; airfoil pressure
$Q_n^{1/2}, Q_n^{-1/2}$	associated Legendre functions of second kind
T_n	Chebyshev polynomial of first kind
t_m, t_n	airfoil polynomial of first kind
U_n	Chebyshev polynomial of second kind
u_m, u_n	airfoil polynomial of second kind
w	airfoil downwash
x, y	real variables
z	complex variable
ξ	real variable

Introduction

The so-called airfoil polynomials are used as expansion functions to compute the pressure on an airfoil in steady or unsteady subsonic flow. In kernel function aerodynamics (e.g., ref. 1), the unknown pressure distribution on the airfoil is expressed as the solution of the integral equation

$$w(x) = \int_{-1}^{1} p(\xi) K(x-\xi)\, d\xi$$

in which the nondimensional variables are the known downwash w, the unknown lifting pressure p, and the kernel function K. The integral extends over the chord, which is normalized such that the interval $[-1, 1]$ extends from the leading edge to the trailing edge of the airfoil. The pressure has a square-root singularity at the leading edge and a square-root zero at the trailing edge. The kernel function involves an integral of Bessel functions and has both a first-order pole and logarithmic singularity at the point $x = \xi$ on the airfoil. The downwash, pressure, and kernel function are functions of the flow Mach number and the airfoil oscillation frequency in addition to the indicated dependence on the chordwise variables x or ξ.

The airfoil polynomials are particularly convenient expansion functions for the pressure and downwash. For example, the pressure may be represented by the expansion

$$p(\xi) = \sqrt{\frac{1-\xi}{1+\xi}} \sum_{n=0}^{N} u_n(\xi)$$

in terms of the airfoil polynomials of the second kind $u_n(\xi)$. These polynomials possess many nice properties such as the interdigitation of the zeros of the polynomials of the second kind u_n for the pressure and of the first kind t_n for the downwash. Applications of these polynomials to subsonic lifting theory may be found in references 1–3. Also, they may be used to describe the chordwise pressure distributions in three-dimensional wing theory. In the three-dimensional theory, the spanwise pressure is typically expanded in a series of Chebyshev polynomials of the second kind $U_n(y)$.

These polynomials are actually renormalized Jacobi polynomials as described, for example, in chapter 22 of reference 4. The present monograph roughly follows the organization of that chapter; however, it contains many additional formulas. Many of the formulas can be derived from the trigonometric definitions of the polynomials or from their expressions in terms of the more familiar Chebyshev polynomials.

Formulas are included that relate the airfoil polynomials to many of the standard mathematical functions. Of particular use are the expressions for the singular integrals occurring in linear theory aerodynamics. Tables I and II summarize the coefficients for $t_n(x)$ and $u_n(x)$, respectively.

Mathematical Properties

1. Definitions

$$t_n(x) = \frac{\cos\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\cos\left(\frac{1}{2}\arccos x\right)}$$

$$u_n(x) = \frac{\sin\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\sin\left(\frac{1}{2}\arccos x\right)}$$

2. Orthogonality Relations

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,t_n(x)t_m(x)\,dx = \begin{cases} 1 & (m=n) \\ 0 & (m \neq n) \end{cases}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,u_n(x)u_m(x)\,dx = \begin{cases} 1 & (m=n) \\ 0 & (m \neq n) \end{cases}$$

3. Recursion Formulas

$$t_{n+1}(x) - 2x t_n(x) + t_{n-1}(x) = 0$$

$$u_{n+1}(x) - 2x u_n(x) + u_{n-1}(x) = 0$$

$$t_{n+2}(x) - (4x^2 - 2) t_n(x) + t_{n-2}(x) = 0$$

$$u_{n+2}(x) - (4x^2 - 2) u_n(x) + u_{n-2}(x) = 0$$

4. Relations to Other Functions

4.1. Chebyshev polynomials.

$$t_n(x) = \sqrt{\frac{2}{1+x}}\, T_{2n+1}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$u_n(x) = U_{2n}\left(\sqrt{\frac{1+x}{2}}\right)$$

4.2. Jacobi polynomials.

$$t_n(x) = 2^{2n} \binom{2n}{n}^{-1} P_n^{(-1/2,\,1/2)}(x)$$

$$u_n(x) = 2^{2n} \binom{2n}{n}^{-1} P_n^{(1/2,\,-1/2)}(x)$$

4.3. Hypergeometric functions.

$$t_n(x) = F\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right)$$

$$u_n(x) = (2n+1) F\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

$$t_n(x) = (-1)^n (2n+1) F\left(-n, n+1; \frac{3}{2}; \frac{1+x}{2}\right)$$

$$u_n(x) = (-1)^n F\left(-n, n+1; \frac{1}{2}; \frac{1+x}{2}\right)$$

4.4. Legendre functions.

$$t_n(x) = \sqrt{\pi}\left(\frac{1-x}{1+x}\right)^{1/4} P_n^{1/2}(x)$$

$$u_n(x) = -\frac{2}{\sqrt{\pi}}\left(\frac{1+x}{1-x}\right)^{1/4} Q_n^{1/2}(x)$$

$$t_n(x) = -\left(n+\frac{1}{2}\right)\frac{2}{\sqrt{\pi}}\left(\frac{1-x}{1+x}\right)^{1/4} Q_n^{-1/2}(x)$$

$$u_n(x) = \left(n+\frac{1}{2}\right)\sqrt{\pi}\left(\frac{1+x}{1-x}\right)^{1/4} P_n^{-1/2}(x)$$

5. Generating Functions

$$\sum_{n=0}^{\infty} t_n(x)z^n = \frac{1-z}{1-2xz+z^2}$$

$$\sum_{n=0}^{\infty} u_n(x)z^n = \frac{1+z}{1-2xz+z^2}$$

6. Rodrigues' Formulas

$$t_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\sqrt{\frac{1-x}{1+x}}\frac{d^n}{dx^n}\left[\sqrt{\frac{1+x}{1-x}}(1-x^2)^n\right]$$

$$u_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\sqrt{\frac{1+x}{1-x}}\frac{d^n}{dx^n}\left[\sqrt{\frac{1-x}{1+x}}(1-x^2)^n\right]$$

7. Interrelations

$$u_n(x) = (-1)^n t_n(-x)$$

$$(1+x)t_n(x) = xu_n(x) - u_{n-1}(x)$$

$$(1+x)t_n(x) = -xu_n(x) + u_{n+1}(x)$$

$$(1-x)u_n(x) = -xt_n(x) + t_{n-1}(x)$$

$$(1-x)u_n(x) = xt_n(x) - t_{n+1}(x)$$

$$u_n(x)t_{n-1}(x) = t_n(x)u_{n-1}(x) + 2$$

$$t_n(x) + t_{n-1}(x) = u_n(x) - u_{n-1}(x)$$

$$(1+x)\left[-t_n(x) + t_{n-1}(x)\right] = (1-x)\left[u_n(x) + u_{n-1}(x)\right]$$

8. Explicit Expansions

$$t_n(x) = 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} (1+x)^{n-k}(1-x)^k$$

$$u_n(x) = 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (1+x)^{n-k}(1-x)^k$$

$$x^n = 2^{-n} \sum_{k=0}^{n} \binom{n}{\lfloor (n-k)/2 \rfloor} t_k(x)$$

$$x^n = 2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{\lfloor (n-k)/2 \rfloor} u_k(x)$$

9. Expressions for Negative Degree

$$t_{-n-1}(x) = t_n(x)$$

$$u_{-n-1}(x) = -u_n(x)$$

10. Differential Equations

$$(1-x^2)t_n''(x) + (1-2x)t_n'(x) + n(n-1)t_n(x) = 0$$

$$(1-x^2)u_n''(x) - (1+2x)u_n'(x) + n(n+1)u_n(x) = 0$$

$$\frac{d}{dx}\left[(1-x^2)\sqrt{\frac{1+x}{1-x}}t_n'(x)\right] + n(n+1)\sqrt{\frac{1+x}{1-x}}t_n(x) = 0$$

$$\frac{d}{dx}\left[(1-x^2)\sqrt{\frac{1-x}{1+x}}u_n'(x)\right] + n(n+1)\sqrt{\frac{1-x}{1+x}}u_n(x) = 0$$

11. Hilbert and Related Transforms

11.1. Hilbert transforms.

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \frac{t_n(x)}{x-y}\, dx = u_n(y)$$

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{u_n(x)}{x-y}\, dx = -t_n(y)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{(1+x)t_n(x)}{x-y}\,dx = (1-y)u_n(y) \qquad (n \neq 1)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(1-x)u_n(x)}{x-y}\,dx = -(1+y)t_n(y) \qquad (n \neq 1)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{t_n^2(x)}{x-y}\,dx = \frac{1-y}{1+y}t_n(y)u_n(y) - \frac{2(2n+1)}{1+y}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{u_n^2(x)}{x-y}\,dx = -\frac{1+y}{1-y}t_n(y)u_n(y) + \frac{2(2n+1)}{1-y}$$

$$\int_{-1}^{1}\frac{t_n(x)}{x-y}\,dx = t_n(y)\ln\frac{1-y}{1+y} + 4\sum_{k=1,3,\ldots}^{n}\frac{t_{n-k}(y)}{k}$$

$$\int_{-1}^{1}\frac{u_n(x)}{x-y}\,dx = u_n(y)\ln\frac{1-y}{1+y} + 4\sum_{k=1,3,\ldots}^{n}\frac{u_{n-k}(y)}{k}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{t_n(x)p(x)}{x-y}\,dx = u_n(y)p(y)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{u_n(x)p(x)}{x-y}\,dx = -t_n(y)p(y)$$

where $p(x)$ is any polynomial of degree n or less. For complex z,

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{t_n(x)}{x-z}\,dx = u_n(z) - \sqrt{\frac{z+1}{z-1}}t_n(z)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{u_n(x)}{x-z}\,dx = -t_n(z) + \sqrt{\frac{z-1}{z+1}}u_n(z)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}(x-z)}\,dx = -\frac{1}{\sqrt{z^2-1}}$$

11.2. Finite part transforms.

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{t_n(x)}{(x-y)^2}\,dx = \frac{-(2n+1)t_n(y)+u_n(y)}{2(1-y)} = u_n'(y)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{u_n(x)}{(x-y)^2}\,dx = \frac{-(2n+1)u_n(y)+t_n(y)}{2(1+y)} = -t_n'(y)$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{(1+x)t_n(x)}{(x-y)^2}\,dx = \frac{-(2n+1)t_n(y)-u_n(y)}{2}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(1-x)u_n(x)}{(x-y)^2}\,dx = \frac{-(2n+1)u_n(y)-t_n(y)}{2}$$

11.3. Logarithmic transforms.

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}t_n(x)\ln|x-y|\,dx = \begin{cases} \dfrac{u_{n-1}(y)-u_n(y)}{2n}+\dfrac{u_n(y)-u_{n+1}(y)}{2(n+1)} & (n \neq 0) \\ -\ln 2 - y & (n=0) \end{cases}$$

$$= -\frac{T_{n+1}(y)}{n+1}-\frac{T_n(y)}{n} = \int u_n(y)\,dy$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}u_n(x)\ln|x-y|\,dx = \begin{cases} -\dfrac{t_{n-1}(y)+t_n(y)}{2n}+\dfrac{t_n(y)+t_{n+1}(y)}{2(n+1)} & (n \neq 0) \\ -\ln 2 + y & (n=0) \end{cases}$$

$$= \frac{T_{n+1}(y)}{n+1}-\frac{T_n(y)}{n} = \int t_n(y)\,dy$$

12. Derivatives

$$(1+x)t_n'(x) = \left(n+\frac{1}{2}\right)u_n(x) - \frac{1}{2}t_n(x)$$

$$(1-x)u_n'(x) = -\left(n+\frac{1}{2}\right)t_n(x) + \frac{1}{2}u_n(x)$$

$$(1-x^2)t_n'(x) = -\left(nx+\frac{1}{2}\right)t_n(x) + \left(n+\frac{1}{2}\right)t_{n-1}(x)$$

$$(1-x^2)u_n'(x) = -\left(nx-\frac{1}{2}\right)u_n(x) + \left(n+\frac{1}{2}\right)u_{n-1}(x)$$

$$(1-x^2)t_n'(x) = \left(nx+x-\frac{1}{2}\right)t_n(x) - \left(n+\frac{1}{2}\right)t_{n+1}(x)$$

$$(1-x^2)u_n'(x) = \left(nx+x+\frac{1}{2}\right)u_n(x) - \left(n+\frac{1}{2}\right)u_{n+1}(x)$$

$$(1-x)\left[t_{n-1}'(x)+t_n'(x)\right] = n[t_{n-1}(x)-t_n(x)]$$

$$t_n'(x) = \frac{1}{2}\sum_{k=0}^{n-1}\Big[(2n+1) - (-1)^{n-k}(2k+1)\Big]t_k(x)$$

$$u_n'(x) = \frac{1}{2}\sum_{k=0}^{n-1}\Big[(2k+1) - (-1)^{n-k}(2n+1)\Big]u_k(x)$$

$$t_n'(x) = -\sum_{k=0}^{n-1}(-1)^{n-k}[n(n+1) - k(k+1)]u_k(x)$$

$$u_n'(x) = \sum_{k=0}^{n-1}[n(n+1) - k(k+1)]t_k(x)$$

$$t_n'(x) + t_{n-1}'(x) = 2n\sum_{k=0}^{n-1} t_k(x)$$

$$u_n'(x) - u_{n-1}'(x) = -2n\sum_{k=0}^{n-1}(-1)^{n-k}u_k(x)$$

13. Indefinite Integrals

$$\int\sqrt{\frac{1+x}{1-x}}t_n(x)\,dx = \frac{\sqrt{1-x^2}}{2n(n+1)}[t_n(x) - (2n+1)u_n(x)]$$

$$= \begin{cases} -\sqrt{1-x^2}\left[\dfrac{U_n(x)}{n+1} + \dfrac{U_{n-1}(x)}{n}\right] & (n>0) \\ -\arccos x - \sqrt{1-x^2} & (n=0) \end{cases}$$

$$\int\sqrt{\frac{1-x}{1+x}}u_n(x)\,dx = \frac{\sqrt{1-x^2}}{2n(n+1)}[-u_n(x) + (2n+1)t_n(x)]$$

$$= \begin{cases} \sqrt{1-x^2}\left[\dfrac{U_n(x)}{n+1} - \dfrac{U_{n-1}(x)}{n}\right] & (n>0) \\ -\arccos x + \sqrt{1-x^2} & (n=0) \end{cases}$$

$$\int t_n(x)\,dx = -\frac{t_{n-1}(x) + t_n(x)}{2n} + \frac{t_n(x) + t_{n+1}(x)}{2(n+1)} = -\frac{T_n(x)}{n} + \frac{T_{n+1}(x)}{n+1}$$

$$\int u_n(x)\,dx = \frac{u_{n-1}(x) - u_n(x)}{2n} + \frac{u_n(x) - u_{n+1}(x)}{2(n+1)} = -\frac{T_n(x)}{n} - \frac{T_{n+1}(x)}{n+1}$$

$$\int\sqrt{\frac{1+x}{1-x}}\frac{1}{x-y}\,dx = \arccos x - \sqrt{\frac{1+y}{1-y}}\left(\ln\left|\sqrt{\frac{1-x}{1+x}} - \sqrt{\frac{1-y}{1+y}}\right| - \ln\left|\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right|\right)$$

14. Cross Expansions

$$u_n(x) = 2\sum_{m=0}^{n} t_m(x) - t_n(x)$$

$$t_n(x) = 2\sum_{m=0}^{n} (-1)^{n-m} u_m(x) - u_n(x)$$

15. Multiplication Formulas

In all cases $n \geq m$.

$$2t_n(x)u_m(x) = u_{n+m}(x) + t_{n+m}(x) - u_{n-m}(x) + t_{n-m}(x)$$

$$2u_n(x)t_m(x) = u_{n+m}(x) + t_{n+m}(x) + u_{n-m}(x) - t_{n-m}(x)$$

$$2t_n(x)t_m(x) = t_{n+m}(x) + t_{n-m}(x) + \frac{1-x}{1+x}[-u_{n+m}(x) + u_{n-m}(x)]$$

$$2(1+x)t_n(x)t_m(x) = t_{n+m}(x) + t_{n+m+1}(x) + t_{n-m}(x) + t_{n-m-1}$$

$$2u_n(x)u_m(x) = u_{n+m}(x) + u_{n-m}(x) + \frac{1+x}{1-x}[-t_{n+m}(x) + t_{n-m}(x)]$$

$$2(1-x)u_n(x)u_m(x) = u_{n+m}(x) - u_{n+m+1}(x) + u_{n-m}(x) - u_{n-m-1}(x)$$

16. Christoffel-Darboux Formulas

$$\sum_{m=0}^{n} u_m(x)u_m(y) = \frac{u_{n+1}(x)u_n(y) - u_n(x)u_{n+1}(y)}{2(x-y)}$$

$$\sum_{m=0}^{n} t_m(x)t_m(y) = \frac{t_{n+1}(x)t_n(y) - t_n(x)t_{n+1}(y)}{2(x-y)}$$

$$\sum_{m=0}^{n} t_m(x)u_m(y) = \frac{2 + t_{n+1}(x)u_n(y) - t_n(x)u_{n+1}(y)}{2(x-y)}$$

$$\sum_{m=0}^{n} u_m^2(x) = \frac{1}{2}[u'_{n+1}(x)u_n(x) - u'_n(x)u_{n+1}(x)]$$

$$\sum_{m=0}^{n} t_m^2(x) = \frac{1}{2}[t'_{n+1}(x)t_n(x) - t'_n(x)t_{n+1}(x)]$$

$$\sum_{m=0}^{n} t_m(x)u_m(x) = \frac{1}{2}\left[t'_{n+1}(x)u_n(x) - t'_n(x)u_{n+1}(x)\right]$$

$$= \frac{1}{2}\left[u'_{n+1}(x)t_n(x) - u'_n(x)t_{n+1}(x)\right]$$

$$\frac{1}{x-y} = \sum_{m=0}^{\infty} t_m(x)u_m(y)$$

17. Special Values

$$t_n(-1) = (-1)^n(2n+1)$$

$$t_n(0) = \pm 1 = \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$$

$$t_n(1) = 1$$

$$u_n(-1) = (-1)^n$$

$$u_n(0) = \pm 1 = \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}$$

$$u_n(1) = 2n+1$$

18. Zeros

$$t_n(x_i) = 0 \qquad \text{where} \quad x_i = -\cos\frac{2i}{2n+1}\pi \qquad (i = 1, 2, \ldots, n)$$

$$u_n(x_i) = 0 \qquad \text{where} \quad x_i = -\cos\frac{2i-1}{2n+1}\pi \qquad (i = 1, 2, \ldots, n)$$

19. Miscellaneous Definite Integrals

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\, t_n(x)u_m(x)\, dx = \begin{cases} 1 & (m = n = 0 \ \text{ or } \ m = n+1) \\ -1 & (m = n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, u_n(x)u_m(x)\, dx = 4\min(m,n) + 2 - \delta_{mn}$$

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\, t_n(x)t_m(x)\, dx = (-1)^{n-m}[4\min(m,n) + 2 - \delta_{mn}]$$

$$\int_{-1}^{1} t_n(x)u_m(x)\,dx = \frac{1+(-1)^{n+m}}{n+m+1} - \frac{1-(-1)^{n-m}}{n-m}$$

$$\int_{-1}^{1} t_n(x)u_n(x)\,dx = \frac{2}{2n+1}$$

$$\int_{-1}^{1} t_n(x)\,dx = \frac{1+(-1)^{n}}{n+1} - \frac{1-(-1)^{n}}{n}$$

$$\int_{-1}^{1} u_n(x)\,dx = \frac{1+(-1)^{n}}{n+1} + \frac{1-(-1)^{n}}{n}$$

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}(x-y)^n\,dx = \sum_{k=0}^{n}(-1)^n 2^{-k}\binom{n}{k}\binom{k}{k/2}y^{n-k} \qquad (n \geq 0)$$

$$\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1-xy+\sqrt{1-x^2}\sqrt{1-y^2}}{1-xy-\sqrt{1-x^2}\sqrt{1-y^2}} t_n(x)\,dx =$$

$$\begin{cases} \sqrt{\dfrac{1-y}{1+y}}\left[\dfrac{u_n(y)+u_{n+1}(y)}{n+1} - \dfrac{u_{n-1}(y)+u_n(y)}{n}\right] & (n > 0) \\ 2(1+y)\sqrt{\dfrac{1-y}{1+y}} & (n = 0) \end{cases}$$

20. Relations Involving $T_n(x)$

$$t_n(x) + t_{n-1}(x) = u_n(x) - u_{n-1}(x) = 2T_n(x) \qquad (n \neq 0)$$

$$t_n(x) = \frac{T_n(x) + T_{n+1}(x)}{1+x}$$

$$u_n(x) = \frac{T_n(x) - T_{n+1}(x)}{1-x}$$

$$t_n(x)u_n(x) - t_{n-1}(x)u_{n-1}(x) = 2T_{2n}(x) \qquad (n \neq 0)$$

$$t_n(x) = \sqrt{\frac{2}{1+x}}T_{2n+1}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$t_n(x) = T_n(x) - (1-x)U_{n-1}(x)$$

$$u_n(x) = T_n(x) + (1+x)U_{n-1}(x)$$

21. Relations Involving $U_n(x)$

$$t_n(x) = U_n(x) - U_{n-1}(x)$$

$$u_n(x) = U_n(x) + U_{n-1}(x)$$

$$t_n(x) + u_n(x) = -t_{n+1}(x) + u_{n+1}(x) = 2U_n(x)$$

$$t_n(x)u_n(x) = U_{2n}(x)$$

$$u_n(x) = U_{2n}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$t_n(x) = T_n(x) - (1-x)U_{n-1}(x)$$

$$u_n(x) = T_n(x) + (1+x)U_{n-1}(x)$$

$$\frac{t_n(x) - t_{n+1}(x)}{1-x} = \frac{u_n(x) + u_{n+1}(x)}{1+x} = 2U_n(x)$$

22. Integrals Involving Legendre Polynomials

$$t_n(x) = \frac{2n+1}{2\sqrt{1+x}} \int_{-1}^{x} \frac{P_n(y)}{\sqrt{x-y}}\, dy$$

$$u_n(x) = \frac{2n+1}{2\sqrt{1-x}} \int_{x}^{1} \frac{P_n(y)}{\sqrt{y-x}}\, dy$$

$$P_n(x) = \frac{1}{\pi} \int_{x}^{1} \frac{t_n(y)}{\sqrt{1-y}\sqrt{y-x}}\, dy$$

$$P_n(x) = \frac{1}{\pi} \int_{-1}^{x} \frac{u_n(y)}{\sqrt{1+y}\sqrt{x-y}}\, dy$$

23. Fourier Series

$$f(x) = \sum_{n=0}^{\infty} t_n(x) \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+y}{1-y}}\, t_n(y) f(y)\, dy$$

$$f(x) = \sum_{n=0}^{\infty} u_n(x) \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-y}{1+y}}\, u_n(y) f(y)\, dy$$

$$f(x) = \sqrt{\frac{1+x}{1-x}} \sum_{n=0}^{\infty} t_n(x) \frac{1}{\pi} \int_{-1}^{1} t_n(y) f(y)\, dy$$

$$f(x) = \sqrt{\frac{1-x}{1+x}} \sum_{n=0}^{\infty} u_n(x) \frac{1}{\pi} \int_{-1}^{1} u_n(y) f(y)\, dy$$

24. Finite Fourier Transforms

$$\frac{1}{\pi}\int_{-1}^{1} e^{ixy}\sqrt{\frac{1+x}{1-x}}t_n(x)\,dx = i^n J_n(y) + i^{n+1} J_{n+1}(y)$$

$$\frac{1}{\pi}\int_{-1}^{1} e^{ixy}\sqrt{\frac{1-x}{1+x}}u_n(x)\,dx = i^n J_n(y) - i^{n+1} J_{n+1}(y)$$

$$\frac{1}{\pi}\int_{-1}^{1} e^{ixy}\sqrt{1-x^2}t_n(x)\,dx = \frac{i^{n+1}n}{y}J_n(y) + \frac{i^n(n+1)}{y}J_{n+1}(y)$$

$$\frac{1}{\pi}\int_{-1}^{1} e^{ixy}\sqrt{1-x^2}u_n(x)\,dx = \frac{i^{n-1}n}{y}J_n(y) + \frac{i^n(n+1)}{y}J_{n+1}(y)$$

Numerical Methods

1. Gaussian Quadrature

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}f(x)\,dx = \frac{2}{2n+1}\sum_{i=1}^{n}(1+x_i)f(x_i) \qquad \text{where} \quad x_i = -\cos\frac{2i}{2n+1}\pi$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)\,dx = \frac{2}{2n+1}\sum_{i=1}^{n}(1-x_i)f(x_i) \qquad \text{where} \quad x_i = -\cos\frac{2i-1}{2n+1}\pi$$

2. Hermite Quadrature

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)\,dx$$

$$= \frac{2}{2n+1}\sum_{i=1}^{n}\left[(1+x_i)f(x_i) - 2\left(1-x_i^2\right)f'(x_i)\right] \qquad \text{where} \quad x_i = -\cos\frac{2i}{2n+1}\pi$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}f(x)\,dx$$

$$= \frac{2}{2n+1}\sum_{i=1}^{n}\left[(1-x_i)f(x_i) + 2\left(1-x_i^2\right)f'(x_i)\right] \qquad \text{where} \quad x_i = -\cos\frac{2i+1}{2n+1}\pi$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}u_m(x)f(x)\,dx$$

$$= \frac{2}{2n+1}\sum_{i=1}^{n}(1+x_i)\left[(2m+1)t_m(x_i)f(x_i) - 2(1-x_i)u_m(x_i)f'(x_i)\right] \qquad \text{where} \quad x_i = -\cos\frac{2i}{2n+1}\pi$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}t_m(x)f(x)\,dx$$

$$=\frac{2}{2n+1}\sum_{i=1}^{n}(1-x_i)\left[(2m+1)u_m(x_i)f(x_i)+2(1+x_i)t_m(x_i)f'(x_i)\right] \qquad \text{where} \quad x_i=-\cos\frac{2i+1}{2n+1}\pi$$

Concluding Remarks

An extensive list of more than 150 formulas for the so-called airfoil polynomials has been given. These orthogonal polynomials are convenient for describing the chordwise pressure distribution in linear theory aerodynamics. Of particular use are the expressions for the singular integrals occurring in the linear theory.

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September 30, 1994

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Table I. Coefficients for $t_n(x)$

[$t_n(x) = \sum_{m=0}^{n} c_m x^m$ and $x^n = b_n^{-1} \sum_{m=0}^{n} d_m t_m(x)$; values in table correspond to coefficients as follows: c_m are given horizontally below the diagonally stepped line, d_m are given vertically above the diagonally stepped line, and b_n as indicated for a given x^n; e.g., $t_4(x) = 1 + 4x - 12x^2 - 8x^3 + 16x^4$ and $x^4 = \frac{1}{16}(6t_0 + 4t_1 + 4t_2 + t_3 + t_4)$]

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
t_0	1 1	1	2	3	6	10	20	35	70
t_1	−1	1 2	1	3	4	10	15	35	56
t_2	−1	−2	1 4	1	4	5	15	21	56
t_3	1	−4	−4	1 8	1	5	6	21	28
t_4	1	4	−12	−8	1 16	1	6	7	28
t_5	−1	6	12	−32	−16	1 32	1	7	8
t_6	−1	−6	24	32	−80	−32	1 64	1	8
t_7	1	−8	−24	80	80	−192	−64	1 128	1
t_8	1	8	−40	−80	240	192	−448	−128	1 256
b_n	1	2	4	8	16	32	64	128	256

Table II. Coefficients for $u_n(x)$

[$u_n(x) = \sum_{m=0}^{n} c_m x^m$ and $x^n = b_n^{-1} \sum_{m=0}^{n} d_m u_m(x)$; values in table correspond to coefficients as follows: c_m are given horizontally below the diagonally stepped line, d_m are given vertically above the diagonally stepped line, and b_n as indicated for a given x^n; e.g., $u_4(x) = 1 - 4x - 12x^2 + 8x^3 + 16x^4$ and $x^4 = \frac{1}{16}(6u_0 - 4u_1 + 4u_2 - u_3 + u_4)$]

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
u_0	1 / 1	−1	2	−3	6	−10	20	−35	70
u_1	1	1 / 2	−1	3	−4	10	−15	35	−56
u_2	−1	2	1 / 4	−1	4	−5	15	−21	56
u_3	−1	−4	4	1 / 8	−1	5	−6	21	−28
u_4	1	−4	−12	8	1 / 16	−1	6	−7	28
u_5	1	6	−12	−32	16	1 / 32	−1	7	−8
u_6	−1	6	24	−32	−80	32	1 / 64	−1	8
u_7	−1	−8	24	80	−80	−192	64	1 / 128	−1
u_8	1	−8	−40	80	240	−192	−448	128	1 / 256
b_n	1	2	4	8	16	32	64	128	256

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