AN OVERVIEW OF ENGINEERING CONCEPTS AND CURRENT DESIGN ALGORITHMS FOR PROBABILISTIC STRUCTURAL ANALYSIS

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ABSTRACT

The article begins by examining the fundamentals of traditional deterministic design philosophy. The initial section outlines the concepts of failure criteria and limit state functions, two traditional notions that are embedded in deterministic design philosophy. This is followed by a discussion regarding safety factors (a possible limit state function) and the common utilization of statistical concepts in deterministic engineering design approaches. Next, the fundamental aspects of a probabilistic failure analysis are explored, and it is shown that deterministic design concepts mentioned in the initial portion of the article are embedded in probabilistic design methods. For components fabricated from ceramic materials (and other similarly brittle materials) the probabilistic design approach yields the widely used Weibull analysis after suitable assumptions are incorporated. The authors point out that Weibull analysis provides the rare instance where closed form solutions are available for a probabilistic failure analysis. Since numerical methods are usually required to evaluate component reliabilities, a section on Monte Carlo methods is included to introduce the concept. The article concludes with a presentation of the technical aspects that support the numerical method known as fast probability integration (FPI). This includes a discussion of the Hasofer-Lind and Rackwitz-Fiessler approximations.

INTRODUCTION

Most parameters that are incorporated into engineering analyses have to a greater, or lesser extent, some level of uncertainty. In order to achieve a general accounting of the entire spectrum of values that design parameters exhibit (especially for those cases where one or more of the design parameters exhibits substantial scatter) a design engineer should utilize probabilistic methods. However, a reliability approach to engineering design demands that an engineer must tolerate a finite risk of unacceptable performance. This risk of unacceptable performance is identified as a component's probability of failure. The primary concern of the engineer is minimizing this risk in an economical manner. To accomplish this requires analytical tools that quantify uncertainty in a rational fashion. The tools for dealing with uncertainty in a rational fashion have been developed in a field of mathematics known as probability theory. Since entire texts are dedicated to this field, only those concepts that are applicable to the design of engineered components are presented here.

In order to meet the numerical needs that accompany a probabilistic analysis several research teams sponsored by NASA Lewis Research Center (LeRC) have focused on the development and application of reliability design algorithms. Two of these groups have produced program deliverables that include reliability models and computer software. Specifically, one program focuses on the engineering analysis of components fabricated from ceramic materials. A number of reliability models developed for ceramic materials (see Duffy et al., 1992, for an overview) have been incorporated into public domain computer algorithms such as the CARES (Ceramics Analysis and Reliability Evaluation of Structures), T/CARES (Toughened Ceramics Analysis and Reliability Evaluation of Structures) and C/CARES (Composite Ceramics Analysis and Reliability Evaluation of Structures). These computer algorithms are coupled to an

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assortment of commercially available general purpose finite element programs. The algorithms yield quasi-static component reliability. In addition the CARES family of software includes parameter estimation modules that allow the design engineer to evaluate the strength distribution parameters from failure data. It is assumed that for this type of reliability analysis the failure strength of the material can be characterized by either a two- or three-parameter Weibull distribution.

The second program that has developed probabilistic tools for design engineers is the Probabilistic Structural Analysis Methods (PSAM) program. This endeavor takes on a more global perspective and deals with the stochastic nature of design parameters in a general fashion. In a manner similar to the structural ceramics effort mentioned above, this program integrates probabilistic algorithms with structural analysis methods. The primary result is the NESSUS (Numerical Evaluation of Stochastic Structures Under Stress) computer software.

This article presents the underlying engineering concepts that support the technical aspects of both the CARES and PSAM programs. In addition, the authors outline the commonality between the programs by demonstrating the shared technical principles. Specific details regarding the CARES family of software algorithms and the NESSUS software are incorporated into the conference presentation that accompanies this article.

FAILURE CRITERION & LIMIT STATE FUNCTION

The success of a structural analysis hinges on the appropriate choice of design variables used to describe the overall thermo-mechanical behavior of a component. The design variables can include, but are not limited to, strength parameters, external loads, allowable deformations at predetermined locations in the component, cycles-to-failure, and material stiffness properties. After the engineer has determined what design variables are pertinent to a given class of design problems, they can be assembled in an N-dimensional vector. This vector of design parameters can be identified as

$$y_a = (y_1, y_2, \cdots y_N) \qquad (1)$$

Design variables can easily interact with one another, thus a functional relationship is needed to describe any interaction. This function is most commonly referred to as a failure criterion. Common examples include strength based criterion such as the maximum distortional energy criterion and the Mohr-Coulomb criterion; fatigue failure criterion are represented by Miner's rule; and fracture criterion include the critical strain energy release rate method and the stress intensity factor methods. Usually a failure criterion represents the first step in defining a limit state. If an operational state for a structural component falls within the boundaries of a limit state, the performance of the structural component is acceptable. An operational state for a component

that falls on the boundary of a limit state denotes failure. For the failure criterion just cited a delineation between acceptable performance and failure is made at a point in the component..

A failure criterion and a limit state function can be expressed by the general formulation

$$g = g(y_a) \qquad (2)$$

Note that g defines a surface in an N-dimensional design variable space. Once again this function must stipulate how each design variable interacts in producing failure. Here values of $g > 0$ indicate a *safe* structure, whereas values of $g \le 0$ correspond to a *failed* structure. The failure criterion (or a limit state function) can be defined by either a complete loss of load carrying capacity, or alternatively by a loss in serviceability. The conceptual distinction between a limit state function and a failure criterion is based on scale. It was indicated above that a failure criterion focuses on a point. Limit state functions focus on the component or structure. Often times there is no difference between the two since failure at a point constitutes failure of the component. In contrast, consider a structural component where plastic yield is a possible failure mode. If yielding (failure) at a point is described by a yield function, then this function represents the failure criterion for this particular mode of failure. Yet a structural component may not fail if yielding has occurred only at a point. In fact the component may continue to function safely until a sufficient number of plastic hinges have formed and the structure collapses. The formation of a sufficient number of plastic hinges is described mathematically by a limit state function. The reader is directed to the extensive literature that followed Drucker's initial work (1952) in establishing bounds on limit state functions for this type of failure analysis. However, the point is that for a plasticity analysis the failure criterion (i.e., the yield criterion) is different from the limit state function.

As a prelude to the discussion that follows later concerning Weibull analysis, a structural component fabricated from a ceramic material is treated as a weakest-link system. If one link in the chain fails, the entire chain fails. This assumption gives rise to a particular modeling approach in calculating component reliability. It also infers that failure at a point constitutes component failure. In this sense the failure criterion and the limit state function will be one in the same.

SAFETY FACTORS & DETERMINISTIC FAILURE ANALYSIS

To begin contrasting the difference between deterministic and probabilistic failure analyses the discussion in this section is focused on a specific failure mode, i.e., the exhaustion of strength capacity. A structural component can fail when it encounters an extreme load, or when a combination of loads reaches a critical collective magnitude, and the ability to withstand the applied load is exhausted. With the design algorithms presented in this article the engineer can easily

quantify the magnitude of the extreme load event leading to failure, and account for the frequency at which this extreme event occurs. In addition, the strength (or capacity) of the material and any variation in this design parameter can be quantified. A brief discussion regarding safety factors (as well as safety margins) follows. This discussion underscores the need to account for variability of design parameters in a coherent manner. These quantities are typically utilized in deterministic designs, and are easily incorporated into probabilistic designs.

In the field of structural mechanics it is customary to define safety factors (and sometimes safety margins) in order to ascertain how "close" a component is to failing. If L represents the load on a component, and R represents the resistance (or capacity) of the material, then the safety factor is defined as

$$S. F. = \frac{R}{L} \qquad (3)$$

Alternatively, the safety margin can be utilized, and this measure is defined as

$$S. M. = R - L \qquad (4)$$

Failure occurs when the safety factor falls below one, or when the safety margin falls below zero. These two expressions represent the simplest and most fundamental definition of a limit state. In the following section where the principles of probabilistic failure analysis are outlined, these two expressions are utilized to explain basic concepts. However, to employ either safety factors or safety margins the design engineer must quantify parameters R and L. Data must be collected and a single "most-likely" value must be assigned for each parameter. If the typical structural engineer has been exposed to statistical methods, these concepts were encountered in quantifying material properties from experimental data. Thus the design engineer is familiar with the concept of a central location parameter for experimental data defined by the sample mean, i.e.,

$$\bar{x} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} x_i \qquad (5)$$

A second parameter, the sample variance, serves as a measure of data dispersion. It is defined as

$$S^2 = \left(\frac{1}{N-1}\right) \sum_{i=1}^{N} [x_i - \bar{x}]^2 \qquad (6)$$

In the expressions for the sample mean and variance, x_i represents the ith observation in an experiment with N observations. Other sample descriptors exist (e.g., skewness and kurtosis); however, the mean and the variance are more widely recognized and understood. Usually the sample mean is used to identify a single "most-likely" value for a design parameter. The variance is most often utilized in a simple minded fashion to indicate how well the experiment is being performed, i.e., a small variance indicates good experimental technique. However, this attitude tends to minimize the fact that some design parameters inherently behave in a random fashion. Variation in experimental data can easily be a fundamental property of a particular design parameter, not a commentary on experimental technique.

Often the engineer is not required to determine values for design parameters directly from experiments. Values for resistance parameters can be obtained from handbooks or existing corporate data bases. Either source of information may concurrently list values for the standard deviation (defined as the square root of the sample variance) but this information is too often ignored in a deterministic failure analysis where the sample mean is used to represent the "most-likely" value of the design parameter. Increasing the mean value of the load parameter by a multiple (usually three) of standard deviations, and decreasing the mean value of resistance parameter by the same multiple of standard deviations is one way of including information regarding data dispersion in a safety factor design. This method, referred to as the three-sigma approach, yields the following definition for the factor of safety

$$S. F. = \frac{\bar{x}_R - 3 (S_R)^{1/2}}{\bar{x}_L + 3 (S_L)^{1/2}} \qquad (7)$$

Clearly this definition of the safety factor admits information concerning the data dispersion for both the load and resistance parameters.

However, if either of the design parameters exhibits a significant scatter, as evidenced by a relatively large sample standard deviation, then the design engineer must compensate in some manner to maintain a prescribed safety factor for a component. Thus, in an effort to maintain a given level for the safety factor, analyses predicated on equation (7) may easily lead to uneconomical designs. Utilizing probabilistic methods can readily compensate for parameter variation, enabling an engineer to further pursue a design that would be otherwise rejected based on traditional methods. In addition, for materials that exhibit size effects where the average strength decreases with specimen size (e.g., ceramic materials) the three-sigma approach leads to a fundamental problem in identifying what value to use for the resistance random variable.

FUNDAMENTALS OF PROBABILISTIC FAILURE ANALYSIS

Utilization of equation (7) represents an attempt to include more information regarding the true characteristics of

design parameters. Momentarily focusing on the resistance parameter, equation (7) implies that the resistance design parameter is inherently multi-valued. If this multi-valued resistance parameter assumes different values at random during strength-to-failure experiments, then the parameter should be treated as a random variable. This holds for the load design parameter as well. Specifically note that if the load and resistance design parameters are treated as random variables, then equation (7) does not include any information on the underlying distribution (e.g., normal, log-normal, Weibull, exponential, etc.) that characterizes the design parameter. As the discussion unfolds in this section the reader will see that reliability methods attempt to overcome this inadequacy.

Consider a component fabricated from a material with a resistance described by the random variable R. A single load, represented by the random variable L, is applied to the component. Both random variables are represented mathematically by distinctly different probability density functions (e.g., normal, log-normal, exponential, Weibull, Rayleigh, etc.). The load and resistance random variables can be described by the same type of probability density function as long as the distribution parameters are different.

However, in this discussion the distributions for the random variables are left unspecified. This is intentional in order to simplify and emphasize several issues. Making use of the concept of a safety factor, the probability of failure for a component where a single load is applied is given by the expression

$$P_f = Probability\,(R/L \leq 1) \qquad (8)$$

Alternatively, the probability of failure can be defined using the safety margin. Here

$$P_f = Probability\,(R - L \leq 0) \qquad (9)$$

For either definition, P_f is the product of two finite probabilities summed over all possible outcomes. Each probability is associated with an event and a random variable. The first event is defined by the random variable L taking on a value in the range

$$\left(x - \frac{dx}{2}\right) \leq L \leq \left(x + \frac{dx}{2}\right) \qquad (10)$$

The probability associated with this event is the area under the probability density function for the load random variable over this interval, i.e.,

$$P_1 = f_L(x)\,dx \qquad (11)$$

The second event is associated with the probability that the random variable R is less than or equal to x. This is the area under the probability density function for the resistance random variable over the range from minus infinity (or an appropriate lower limit defined by the range of the resistance random variable) to x. This second probability is given by the cumulative distribution function evaluated at x, i.e.,

$$P_2 = F_R(x) \qquad (12)$$

With the probability of failure defined as the product of these two probabilities, summed over all possible values of x, then

$$P_f = \int_{-\infty}^{\infty} F_R(x)\,f_L(x)\,dx \qquad (13)$$

To interpret this integral expression, consider Figure 1. This figure contains a graph of an arbitrary probability density function (f_R) for the resistance random variable superimposed on the graph of an arbitrary probability density function (f_L) for the load random variable. Note that R and L must have the same dimensions to plot these two quantities on the same graph. A common misconception is that P_f is the area of overlap encompassed by the two probability density functions. Scrutiny of equation (13) leads to the appropriate conclusion that the probability of failure is the area under the composite function

$$\eta_{RL}(x) = F_R(x)\,f_L(x) \qquad (14)$$

Due to the complexities introduced by specifying $F_R(x)$ and $f_L(x)$ a closed form solution rarely exists for equation (13). One exception is the application of equation (13) to ceramic materials, which is discussed in the next section.

CERAMIC MATERIALS AND SYSTEM RELIABILITY

Even though variations in loads and strength can be readily accommodated by the concepts presented in the previous section, for components fabricated from ceramic materials it is the variation in material strength that dominates the design. Lack of ductility combined with flaws, defects, or inclusions that have various sizes and orientations leads to scatter in failure strength. Thus the strength associated with these ceramic materials reflects an intrinsic fracture toughness and a homogeneous distribution of flaws present in the materials. The analytical concepts presented in this section will accommodate this singular focus on strength variation.

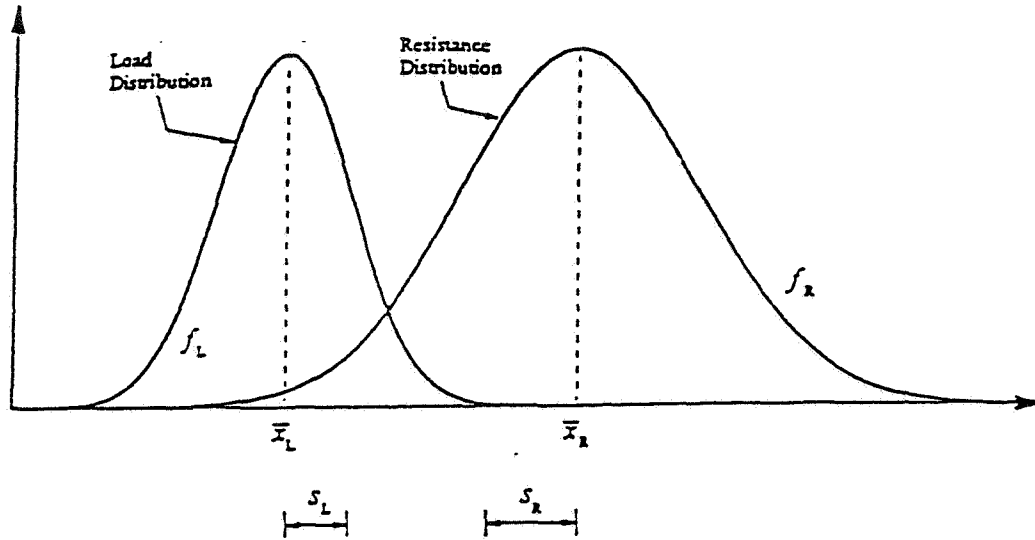Experimental data indicates that the continuous random

Figure 1 An arbitrary load - resistance interference graph

variable representing uniaxial tensile strength (a resistance design parameter) of monolithic ceramics is asymmetrical about the mean and will assume only positive values. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull distribution or a similarly skewed distribution. The three-parameter Weibull probability density function for a continuous random strength variable, denoted as Σ, is given by the expression

$$f_\Sigma(\sigma) = \left[\frac{\alpha}{\beta}\right]\left[\frac{\sigma-\gamma}{\beta}\right]^{(\alpha-1)}\exp\left[-\left[\frac{\sigma-\gamma}{\beta}\right]^\alpha\right] \quad (15)$$

for $\sigma > \gamma$, and

$$f_\Sigma(\sigma) = 0 \quad (16)$$

for $\sigma \leq \gamma$. In equation (15) α is the Weibull modulus (or the shape parameter), β is the Weibull scale parameter, and γ is a threshold parameter. If the value of the random variable is below the threshold parameter, the probability density function is zero. Often the value of the threshold parameter is taken to be zero. In component design this represents a conservative assumption, and yields the more widely used two-parameter Weibull formulation.

If the resistance design parameter is characterized by the Weibull distribution and the load design parameter is assumed deterministic, then the following probability density function

$$f_L(x) = \delta(x - x_o) \quad (17)$$

is utilized in equation (13) for the load random variable. Here δ is the Dirac delta function defined as

$$\delta(x - x_o) = \begin{cases} \infty & x = x_o \\ 0 & x \neq x_o \end{cases} \quad (18)$$

Note that the Dirac delta function satisfies the classical definition of a probability density functions. This function represents the scenario where the standard deviation of a distribution approaches zero in the limit, and the random variable takes on a central value (identified here as x_o). Insertion of equation (17) into equation (13) yields the following expression for the probability of failure:

$$P_f = \int \overline{F_R(x)} \delta(x - x_o) \, dx \quad (19)$$

However, with the Dirac delta function embedded in the integral

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expression, the probability of failure simplifies to

$$P_f = F_R(x_o) \qquad (20)$$

Thus the probability of failure is governed by the cumulative distribution function that characterizes the resistance random variable. This expression (with modification) is a fundamental concept associated with Weibull analysis.

Equation (20) yields the probability of failure (after an appropriate distribution has been specified for the random variable R) for a simple component with a single deterministic load which is identified as x_o. However, a unique property of ceramic materials is an apparent decreasing trend in strength with an increase in the size of the component. This is the so called size effect. As an example, consider that the simple component represents a uniaxial tensile specimen. Now suppose that two groups of these simple components exist. Each group is identical with the exception that the size of the specimens in the first group is uniformly smaller than the specimens in the second group. For ceramic materials the sample mean from the first group would be consistently and distinctly larger in a manner that can not be accounted for by randomness. Thus equation (20) must be transformed in some fashion to admit a size dependence. This is accomplished through the use of system reliability concepts. It should be understood that the expression given in equation (20) represents the probability of failure *for a uniform set of boundary conditions*. If the boundary conditions are modified in any fashion, or the geometry of the component changes, equation (20) is no longer valid. To account for size effects and deal with the probability of failure for a component in a general manner, the component should be treated as a system, and the focus must be directed on the probability of failure of the system.

Typically, for a structural component with a varying stress field, the component is discretized, and the stress field is characterized using finite element methods. Since component failure may initiate in any of the discrete elements, it is easy to consider the discretized component from a systems viewpoint. A discretized component is a series system if it fails when one of the discrete elements fail. This concept gives rise to weakest-link reliability theories. A discretized component is a parallel system when failure of a single element does not necessarily cause the component to fail, since the remaining elements may sustain the load through redistribution. Parallel systems lead to what has been referred to in the literature as "bundle theories." These two types of systems represent the extremes of failure behavior and suggest more complex systems such as "r out of n" systems. Here a component (system) of n elements functions if at least r elements have not failed. However, the failure behavior of monolithic ceramic materials is brittle and catastrophic. This type of behavior fits within the description of a series system, thus ceramic materials are modeled as a weakest-link reliability system.

Now the focus is directed to the probability of failure

of a discrete element and how this failure relates to the overall probability of failure of the component. If the failure of an individual element is considered a statistical event, and if these events are independent, then the probability of failure of a discretized component that acts as a series system is given by

$$P_f = 1 - \prod_{i=1}^{N} (1 - \mathcal{P}_i) \qquad (21)$$

where N is the number of discrete finite elements for a given component. Here the probability of failure of the ith discrete element ($\mathcal{P}_i$) is given by the expression

$$\mathcal{P}_i = \psi \, \Delta V_i \qquad (22)$$

where V denotes volume and ψ is a failure function per unit volume of material. This introduces the requisite size scaling that is associated with ceramic materials. Adopting an argument used by Weibull (1939) where the norm of the ΔV_i's tends to zero in the limit as N goes to infinity, then the component probability of failure is given by the following expression

$$P_f = 1 - \exp\left(-\int_V \psi \, dV \right) \qquad (23)$$

What remains is the specification of the failure function ψ. The most basic formulation for ψ is given by the principle of independent action (PIA). For this reliability model

$$\psi = \left(\frac{\sigma_1}{\beta} \right)^m + \left(\frac{\sigma_2}{\beta} \right)^m + \left(\frac{\sigma_3}{\beta} \right)^m \qquad (24)$$

where σ_1, σ_2 and σ_3 are principle stresses. Equation (23) is the essence of Weibull analysis. The issue of other possible forms for ψ has been discussed in detail in articles by Duffy and Arnold (1990), Duffy and Manderscheid (1990), Thomas and Wetherhold (1991), and Duffy et al. (1993).

As a final note, equations (20) and (23) can be equated once a distribution function is specified for the resistance random variable. As was indicated earlier, the distribution of choice is the Weibull distribution. There is a fundamental reason for this choice that goes beyond the fact that the Weibull distribution usually provides a good fit to the data. Often times the log-normal distribution provides an adequate fit to failure data representing ceramic materials. However, the log-normal distribution precludes any accounting of size effects. The reader is directed to work by Hu (1995) for a detailed discussion on this matter. As it turns out, once a conscious choice is made to utilize the Weibull distribution, equations (20) and (23) provides a convenient formulation for parameter estimation. The details

for accomplishing this are provided in Duffy (1995).

The next issue the design engineer is confronted with concerns the numerical evaluation of equation (13) when a closed form solution is not readily available. The remainder of this article is dedicated to this important issue. However, before proceeding on to the next section the reader is reminded that probabilistic concepts were introduced by adopting a very simple failure criterion. In the sections that follow the failure criterion is left unspecified and the details of the numerical techniques are highlighted.

MONTE CARLO METHODS

In this section the authors expand the scope of the discussion beyond simple failure criterion represented by safety factors and safety margins. This discussion begins with the observation that most structural components are designed based on the results obtained from a finite element analysis. This analysis can incorporate a mechanical analysis, a thermal analysis, or both. In all cases the design engineer seeks to predict, and most times minimize, the stress field throughout the component in an economical fashion. The stress field is approximated by the stress state obtained from each discrete element. Once again the focus of the design algorithms presented here is on individual (discrete) elements. In general the reliability of an individual finite element is computed from the expression

$$R = Probability[g(y_e) > 0] \qquad (25)$$

when a failure criterion is used to define point failure. Note that the failure criterion is left unspecified thus equation (25) is a general, fundamental relationship. The discussion that follows outlines specific details that must be embedded in this relationship.

To evaluate equation (25) the design space must be defined. In addition, the relevant joint probability density function that represents the design variables must be established. As was indicated earlier if there are N random variables associated with a limit state function, then the design space is an N-dimensional space (a hyperspace) that represents the entire domain of possible values of the design variables. In order to transform a limit state function into a reliability model a joint probability density function must be utilized. This function establishes the relative frequency of occurrence for a specific combination of values (realizations) of the design random variables. Keep in mind that the limit state function is used to determine which area of the design space (a region that represents all possible outcomes of the design random variables) will result in a successful event. Thus according to equation (25) the *safe* domain of the design space should satisfy $g(y_e) > 0$. Obviously, the portion of the design space that satisfies $g(y_e) \leq 0$ is the *failure* domain for the finite element. Thus the reliability of a finite element is the integration of the joint probability density

function over the *safe* design space defined by the failure criterion. This *integration* takes the form

$$R = \int_{\delta_s} f(y_e) dy_e \qquad (26)$$

where $f(y_e)$ is the joint density function of the random variables, and δ_s is the *safe* domain of the design space. This concept is simplified to a two-dimensional design variable space depicted in Figure 2.

Unfortunately, the integral in equation (26) does not usually have a closed form solution. An exception to this was presented in the section where the principles of Weibull analysis are examined. Thus in general, numerical techniques must be utilized to evaluate the reliability of a finite element. Two numerical techniques are discussed in this article that provide approximate solutions of equation (26). They are the conventional Monte Carlo method, and the fast probability integration (FPI) method. Other methods exist (the reader is referred to Hu (1995) for an overview) but only these two methods are presented here due to limitations placed on the length of this article. The reader is referred to Wu (1994) for a more comprehensive development of the numerical techniques associated with the FPI method. In addition, Hu's thesis (1995) presents details regarding a Monte Carlo method with an extremely efficient sampling approach.

The conventional Monte Carlo simulation is conceptually simple, very general, and relatively straightforward to implement. Thus it is commonly used to numerically estimate the probability of failure when a closed form solution to equation (26) is unavailable. In general the probability of failure of a structural component can be expressed as

$$P_f = \int_{\delta_f} f(y_e) dy_e \qquad (27)$$

where δ_f is the failure domain that satisfies the expression

$$g(y_e) \leq 0 \qquad (28)$$

Equation (27) is an alternative expression to equation (26) since

$$P_f = 1 - R \qquad (29)$$

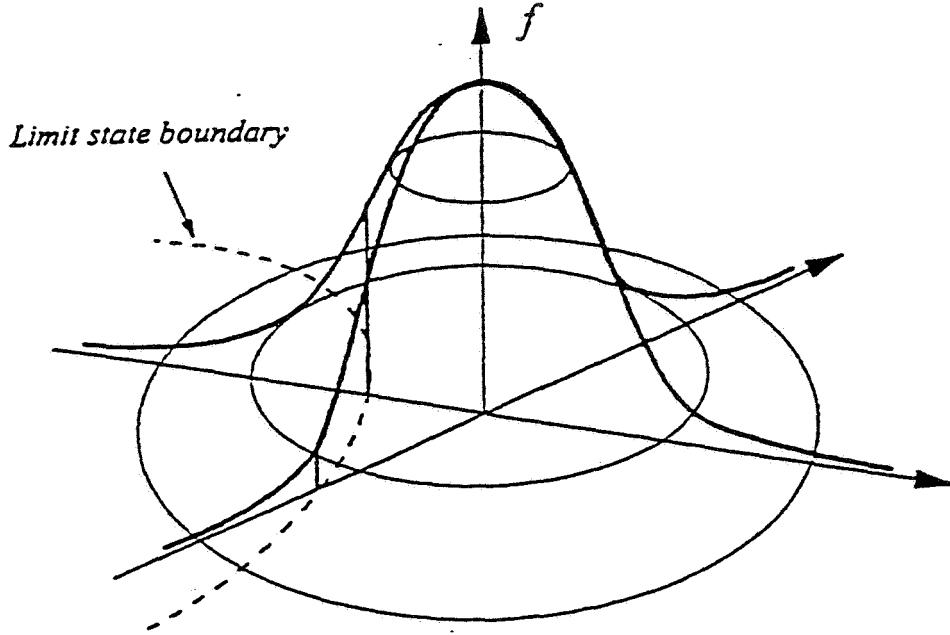Now define an indicator function I such that

Limit state boundary

Figure 2 Two-dimensional joint PDF with limit state

$$I = \begin{cases} 1 & g(y_e) \leq 0 \\ 0 & g(y_e) > 0 \end{cases} \qquad (30)$$

$$\mu_x = \int\limits_{-\infty}^{\infty} x f(x)\, dx \qquad (33)$$

This indicator function can be included in the integral defined by equation (27) if the integration range is expanded to include the range of the design space, i.e.,

$$P_f = \int\limits_{l_s \to l_f} I f(y_e)\, dy_e \qquad (31)$$

The integral on the right side of this expression defines the expectation of the indicator function, i.e.,

$$E[I] = \int\limits_{l_s \to l_f} I f(y_e)\, dy_e \qquad (32)$$

Recall from statistics that the definition of the mean (μ) of a random variable is the expectation of the variable. Thus

Also recall that the mean associated with a random variable can be estimated from a sample taken from the population that is being characterized by the distribution function $f(x)$. The estimated value of the mean is given by the simple expression

$$\bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j \qquad (34)$$

Where x_j is the jth observation in a random sample taken from the population. In a similar fashion the probability of failure (P_f) represents the expected value, of the indicator function. Thus equation (31) can be expressed as

$$P_f = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{j=1}^{N} I_j \right\} \qquad (35)$$

Here it is implied that a random sample of *successes* ($I=1$) and *failures* ($I=0$) has been generated. Thus I_j is the jth evaluation of the limit state function where the random observations have

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been generated from the cumulative distribution function F_{Y_a}.

The simulation method defined by equation (35) is referred to as a conventional Monte Carlo simulation. The objective is to generate a sufficiently large set of observations (i.e., large N) in order to reproduce the statistical characteristics of the underlying population that the observations are taken from. The concept of the conventional Monte Carlo method is shown in Figure 3, where the solid circles represent a *success*, i.e., $I=1$ and $g(y_a) \leq 0$. The open circles are observations that do not pass the failure criterion. Here $I=0$ and $g(y_a) > 0$. While this approach may not be the most efficient numerical technique, eventually it will converge to the correct solution, i.e., the solution approaches P_f in the limit as N approaches infinity.

THE FAST PROBABILITY INTEGRATION METHOD

This section presents the details of obtaining component reliabilities from fast probability integration (FPI) methods. These details are presented in terms of the simplified failure criterion defined by safety factors or safety margins. This is done to merely clarify technical concepts. At the end of the section the details are provided that allows the application of this method to arbitrary failure criterion.

Thus the probability of failure for a structural component can be expressed as

$$P_f = Probability(R - L \leq 0) \tag{36}$$

where R is the resistance random variable and L is the load random variable (both of which were stipulated as design variables). Define the safety margin as

$$M = R - L \tag{37}$$

The expectation of the safety margin is

$$\mu_M = \mu_R - \mu_L \tag{38}$$

where μ_M is the mean of M, μ_R is the mean of R and μ_L is the mean of L. Similarly, the variance of the safety margin is given by the expression

$$\delta_M^2 = \delta_R^2 + \delta_L^2 - 2cov[R,L] \tag{39}$$

where "*cov*" represents the covariance function, δ_R^2 is the variance of R, and δ_L^2 is the variance of L. However, if R and L are independent random variables, then

$$cov[R, L] = 0 \tag{40}$$

and

$$\delta_M^2 = \delta_R^2 + \delta_L^2 \tag{41}$$

Making use of the definition of the safety margin, the probability of failure can be expressed as

$$P_f = Probability(-\infty \leq M \leq 0) \tag{42}$$

If R and L are normal random variables, then M (which is also a limit state function) becomes a linear combination of two normally distributed random variables. Thus M is a normally distributed random variable. Making use of the standard normal CDF (Φ), the probability of failure is given by the expression

$$P_f = \Phi\left(-\frac{\mu_M}{\delta_M}\right) \tag{43}$$

Substitution yields

$$P_f = \Phi\left[\frac{\mu_L - \mu_R}{(\delta_R^2 + \delta_L^2)^{1/2}}\right] \tag{44}$$

Now define the reliability index β such that

$$\beta = \frac{\mu_R - \mu_L}{(\delta_R^2 + \delta_L^2)^{1/2}} \tag{45}$$

then

$$P_f = \Phi(-\beta) \tag{46}$$

This is equivalent to the integral expression given in equation (19), i.e.,

$$P_f = \int_{\theta_f} f(R,L)\, dR\, dL \tag{47}$$

Thus equation (46) represents a "fast integration" of equation (47), hence the origin of term "fast probability integration" (FPI) for the approach that utilizes the reliability index. The expression
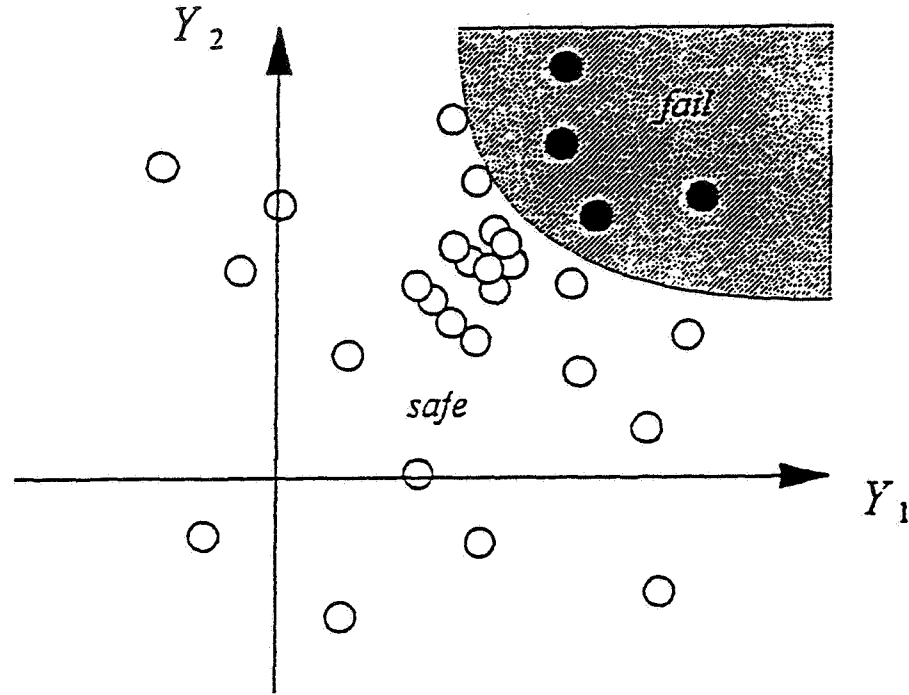
Figure 3 Conventional Monte Carlo methods

for the reliability index was derived based on the special case where the safety margin serves as the limit state function. More generally, other expressions for β can be derived by adopting different limit state functions. However, in general, the straight-forward relationship between the reliability index and the probability of failure expressed above no longer holds. If the limit state function is a non-linear function of the design variables or the design variables are not normally distributed, then equation (46) will not hold. Yet for either case there are approximations that can be utilized which yield good results, provided the limit state functions and the design variables are suitably restricted.

Before the discussion on how the relationship between the probability of failure and the reliability index can be approximated, a geometric interpretation of the reliability index is given using the simple definition of the safety margin expressed in equation (38). If R and L are normally distributed, they can be transformed to standard normal variables. By definition the transformed resistance variable is

$$R' = \frac{R - \mu_R}{\delta_R} \qquad (48)$$

and the transformed load variable is

$$L' = \frac{L - \mu_L}{\delta_L} \qquad (49)$$

Thus the resistance variable can be expressed as

$$R = R'\delta_R + \mu_R \qquad (51)$$

and the load variable can be expressed as

$$L = L'\delta_L + \mu_L \qquad (51)$$

In terms of the transformed random variables the safety margin becomes

$$M = \delta_R R' - \delta_L L' + (\mu_R - \mu_L) \qquad (52)$$

Now the reliability index can be interpreted as the shortest distance from the origin in the transformed variable space to the failure surface, which is defined by M. This is depicted in Figure 4 where the failure surface associated with this particular safety margin is shown in both the original and the transformed design variable space. The point on the failure surface that is
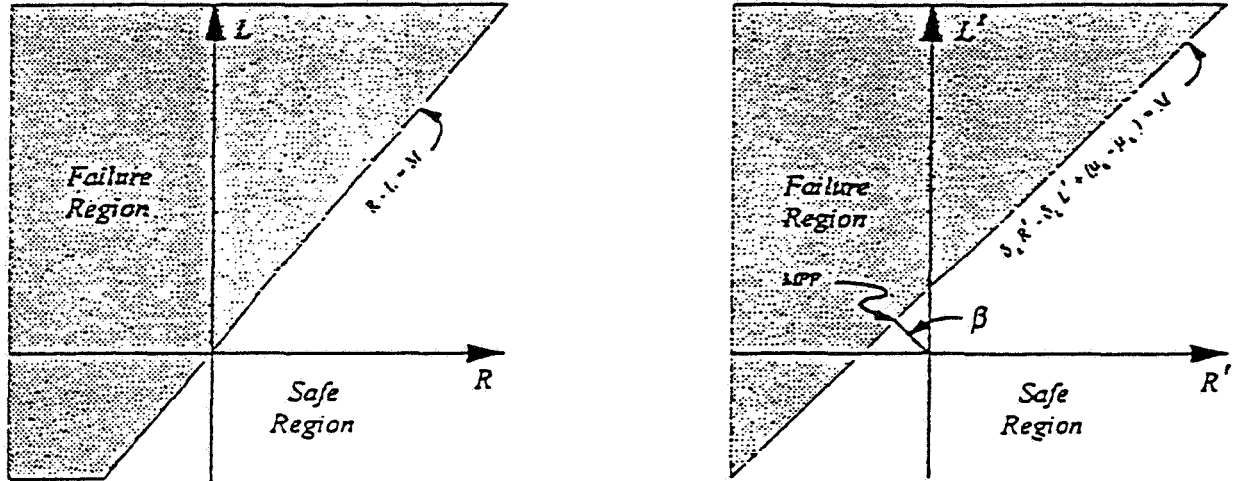
Figure 4 a) Failure surface in the standard variable space. b) Failure surface and MPP in the transformed variable space

nearest to the origin is referred to as the most probable point (MPP) in structural reliability literature. The reader can easily verify that equation (45) can be derived from the geometry presented in Figure 4. This last figure is important. If

- the limit state function can be linearized, and

- the design variables can be transformed to standard normal variables,

then β can be determined using analytic geometry concepts. Once β has been found, the probability of failure is calculated directly using equation (46).

Finally, before discussing approximate methods associated with linearizing the limit state function and normalizing the design variables, a brief discussion is necessary concerning the extension of equation (46) to N design variables. The preceding discussion focused on two independent, normally distributed, random variables. Equation (47) holds for N independent, normally distributed random variables, if the limit state function (g) is a linear function of the random variables. Under these circumstances

$$\beta = \frac{\mu_g}{\delta_g} \qquad (53)$$

where

$$g = 1 - \sum_{i=1}^{N} a_i y_i \qquad (54)$$

$$\mu_g = E[g] \qquad (55)$$

and

$$\delta_g^2 = VAR[g] \qquad (56)$$

Note that a_i is the coefficient of the i'th term of the limit state function and depends on the particular limit state function utilized.

THE HASOFER-LIND APPROXIMATION

Hasofer and Lind (1974) proposed a technique (identified here as the H-L method) that approximates the failure surface for those cases where the limit state function is not a linear combination of the design variables. The failure surface (a hyper-surface in the N-dimensional design variable space) is approximated by a hyper-plane tangent to the failure surface at

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the MPP (see Figure 4). This approximation is accomplished by utilizing the first term of a Taylor series expansion of the limit state function at the MPP. Thus

$$g(z_a) = g(z_a^*) + \sum_{i=1}^{N} \left(\frac{\partial g}{\partial z_i}\right)_* (z_i - z_i^*) \qquad (57)$$

where z_a is the vector of standard normal variables which are related to the design variables in the following manner

$$z_a = \frac{(y_a - \mu_{Y_a})}{\delta_{Y_a}} \qquad (59)$$

Here z_a^* is the vector representing the location of the MPP, and the asterisk associated with the partial derivative indicates the vector and the associated derivatives are being evaluated at the MPP.

Since the limit state function is approximated by the first term of a Taylor series expansion, the H-L method is referred to as a "first order" method. Keep in mind that the H-L approximation will be *exact* if the design variables are normally distributed, and the true limit state function is linear. The reader should question how good the approximation is if the actual limit state function is not linear (a hyper-plane) in the transformed standard normal variable space. The joint probability density function tends to decay exponentially with a relative increase in distance from the mean (i.e., the "peak" of the joint PDF in Figure 2). For large values of β (i.e., low probability of failure) the main contribution to the probability integral, i.e., equation (28), usually comes from regions near the MPP, since the relevant functional values of the joint PDF will assume their largest values in the near vicinity of the MPP. Therefore, provided that the actual limit state surface is well-behaved and does not exhibit significant deviations from the tangent hyper-plane approximation in the neighborhood of the MPP, a reasonably accurate estimate of the actual probability of failure (P_f) can still be obtained from equation (46) by this first order approximation.

Since β represents the shortest distance from the origin to the failure surface in standard normal variable space, and the location of the MPP is not known a priori, a search algorithm must be employed. An optimization method making use of Lagrange multipliers is utilized here. The following steps represent the details of this search algorithm:

(1) Assume initial values for the normal design variables y_i^*, and transform these values to standard normal values z_i^* using equation (58).

(2) Transform the limit state function $g(y_a)$ to $g(z_a)$ using equation (57). This requires the evaluation of the

partial derivatives at the corresponding values of z_i^*.

(3) Evaluate the Lagrange multipliers

$$\alpha_i^* = \left\{ \frac{\left(\frac{\partial g}{\partial z_i}\right)}{\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial z_i}\right)^2\right]^{1/2}} \right\}_* \qquad (60)$$

(4) Assemble the vector z_i^* using

$$z_i^* = -\alpha_i^* \beta \qquad (60)$$

where β is unknown at this point.

(5) Substitute z_i^* into the following expression

$$g(z_a) = 0 \qquad (61)$$

and solve for β. Note that this last expression is a scalar valued function. Hence one equation is solved for one unknown (β).

(6) With β known, update values of z_i^* and repeat steps (3) to (5) until a suitable convergence criterion is met. This convergence criterion can be easily related to the change in β from one iteration to the next.

The geometric interpretation of the algorithm above is shown in Figure 5. Note that the rate of convergence for the H-L algorithm will depend on the following

- the nature of the true limit state function,

- the starting point,

- the characteristics of the random variables, and

- the correlation between the random variables.

These issues have been discussed thoroughly in the open literature and will not be revisited here.

THE RACKWITZ-FIESSLER APPROXIMATION

If the random variables that are utilized in the limit state function are not normally distributed, a second approximation must be employed. Based on a concept suggested by Paloheimo and Hannus (1974), Rackwitz and Fiessler (1978) proposed a modification of the FPI method to account for design variables
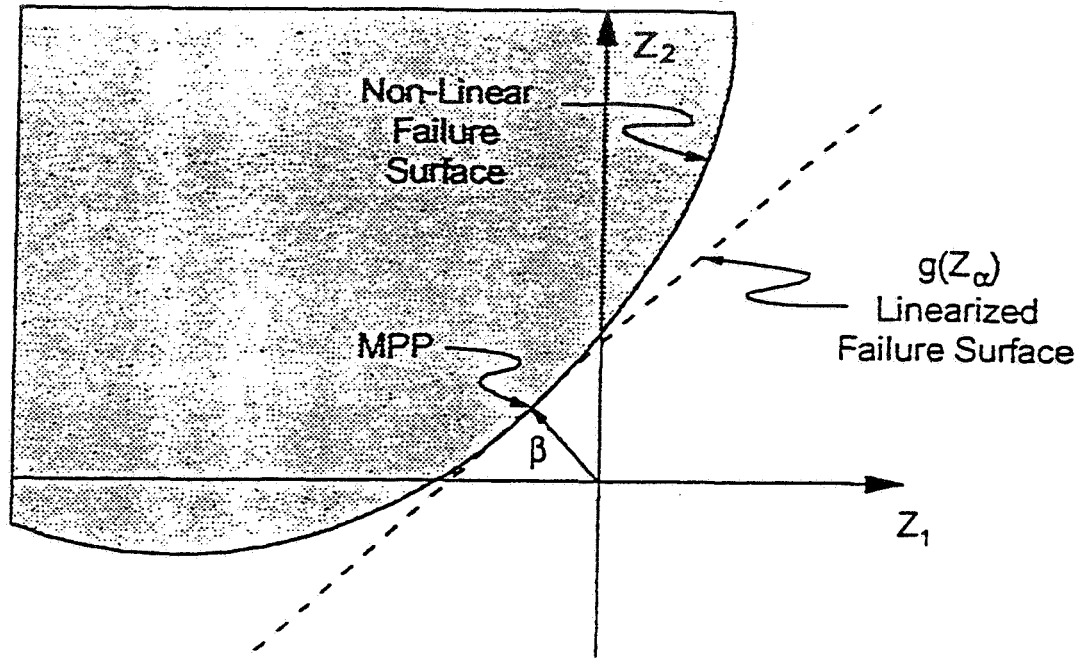
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Figure 5 Schematic of the Hasofer-Lind approximation where a non-linear failure surface is approximated by a linear tangent hyper-plane

with non-normal distributions. The technique (referred to here as the R-F method) converts non-normal random variables into standard normal variables by first equating the CDFs of the standard normal and non-normal distributions, i.e.,

$$\Phi\left(\frac{y_i^* - \mu_{Y_i}^N}{\delta_{Y_i}^N}\right) = F_{Y_i}(y_i^*) \qquad (63)$$

Here F_{Y_i} represents the non-normal cumulative distribution function (e.g., the two-parameter Weibull distribution) and Φ is the standard normal cumulative distribution function. In addition the PDFs of the standard normal and non-normal distributions are equated leading to the expression

$$\frac{1}{\delta_{Y_i}^N} \phi\left(\frac{y_i^* - \mu_{Y_i}^N}{\delta_{Y_i}^N}\right) = f_{Y_i}(y_i^*) \qquad (64)$$

Here f_{Y_i} represents the non-normal probability density function and ϕ is the standard normal probability density function. These last two expressions must be evaluated at every approximated MPP. The character N signifies the normal distribution in both

expressions. Thus, the equivalent normal mean ($\mu_{Y_i}^N$) and equivalent normal standard deviation ($\delta_{Y_i}^N$) of non-normal variables can be derived from equations (62) and (63). Specifically

$$\mu_{Y_i}^N = y_i^* - \delta_{Y_i}^N \Phi^{-1}[F_{Y_i}(y_i^*)] \qquad (65)$$

and

$$\delta_{Y_i}^N = \frac{\phi[\Phi^{-1}[F_{Y_i}(y_i^*)]]}{f_{Y_i}(y_i^*)} \qquad (66)$$

SUMMARY

An overview is given of engineering concepts and computational algorithms which have been developed enabling probabilistic design approaches to structural analysis. Probabilistic design approaches are shown to have evolved as a natural extension of traditional deterministic design approaches.

The well established Weibull analysis approach, commonly utilized for the design of components fabricated of

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brittle materials such as ceramics, is shown to be a special case of the more general probabilistic design problem formulation. Substantial developments have occurred to extend the Weibull analysis approach to more complex failure mechanisms and for the approximate numerical solution of more realistic component design problems using finite element analysis techniques. These efforts have produced comprehensive design tools, such as is embodied in the CARES family of software developed at NASA Lewis Research Center.

In the more general case, much of the development effort has focused on efficient numerical algorithms to achieve accurate approximate solutions of probabilistic design problems involving complex and nonlinear failure or limit state functions and design parameters described by non-normal distributions. Several specific algorithms are described which are extensions of the fast probability integration approaches originally developed by Hasofer and Lind, and Rackwitz and Feissler, as well as extensions of Monte Carlo simulation approaches allowing for more selective sampling. Again, these efforts have produced comprehensive design tools, such as is embodied in the NESSUS family of software developed at Southwest Research Institute under the sponsorship of NASA Lewis Research Center.

The various methods presented provide a quantitative basis to account for design uncertainties inherent to physical systems. The ultimate benefit of probabilistic design approaches is a more rational basis for making design decisions that balance component or system efficiency with reliability or safety. This benefit is especially important in the design of high-performance and/or life-critical systems.

REFERENCES

Drucker, D.C., Greenberg, W. and Greenberg, H.J., "Extended Limit Design Theorems for Continuous Media," *Quart. Appl. Math.*, Vol. 9, pp. 381-389, 1952.

Duffy, S.F. and Arnold, S.M., "Noninteractive Macroscopic Statistical Failure Theory for Whisker Reinforced Ceramic Composites," *Journal of Composite Materials*, Vol. 24, No. 3, pp. 293-308, 1990.

Duffy, S.F. and Manderscheid, J.M., "Noninteractive Macroscopic Reliability Model for Ceramic Matrix Composites with Orthotropic Material Symmetry," *Journal of Engineering for Gas Turbines and Power*, Vol. 112, No. 4, pp. 507-511, 1990.

Duffy, S.F., Chulya, A. and Gyekenyesi, J.P., "Structural Design Methodologies for Ceramic Based Material Systems," Chapter 14 in Flight-Vehicle Materials, Structures and Dynamics Technologies - Assessment and Future Directions, eds. A.K. Noor and S.L. Venneri, The American Society of Mechanical Engineers, pp. 265-285, 1992.

Duffy, S.F., Palko, J.L. and Gyekenyesi, J.P., "Structural Reliability Analysis of Laminated CMC Components," *Journal of Engineering for Gas Turbines and Power*, Vol. 115, No. 1, pp. 103-108, 1993.

Duffy, S.F., "Issues Relating to Reliability Based Design Component Design," to appear in Handbook on Whisker and Particulate Reinforced Ceramic Matrix Composites, K. Bowman, S.K. El-Rahaiby, and J. Wachtman eds., to be published by DoD Ceramics Information Analysis Center.

Hasofer, A.M., and Lind, N.C., "An Exact and Invariant First Order Reliability Format," *Journal of the Engineering Mechanics Division*, Vol. 100, No. EM1, pp. 111-121, 1974.

Hu, J., "Modeling Size Effects and Numerical Techniques in Structural Reliability Analysis," *Masters Thesis*, Cleveland State University, 1995.

Paloheimo, E. and Hannus, M., "Structural Design Based on Weighted Fractiles," *Journal of the Structural Division*, Vol. 100, No. ST7, pp. 661-676, 1974.

Rackwitz, R. and Fiessler, B., "Structural Reliability Under Combined Random Load Sequences," *Journal of Composite Structures*, Vol. 9, pp. 489-494, 1978.

Thomas, D.J. and Wetherhold, R.C., "Reliability Analysis of Laminates with Load Sharing," *Journal of Composite Materials*, Vol. 25, pp. 1459-1475, 1991.

Weibull, W., "A Statistical Theory of the Strength of Materials," *Ingeniors Vetenskaps Akademien Handlinger*, No. 151, 1939.

Wu, Y.-T., "Computational Methods for Efficient Structural Reliability and Reliability Sensitivity Analysis," *AIAA Journal*, Vol. 32, No. 8, pp. 1717-1723, 1994.