

NASA Contractor Report 198404
ICOMP-95-15

1N-34
66620
P. 23

On High-Order Radiation Boundary Conditions

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September 1995

(NASA-CR-198404) ON HIGH-ORDER
RADIATION BOUNDARY CONDITIONS
(NASA, Lewis Research Center) 23 p

N96-15991

Unclass

G3/34 0066620

Prepared for
Lewis Research Center
Under Cooperative Agreement NCC3-370



National Aeronautics and
Space Administration



ON HIGH-ORDER RADIATION BOUNDARY CONDITIONS

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Abstract. In this paper we develop the theory of high-order radiation boundary conditions for wave propagation problems. In particular, we study the convergence of sequences of time-local approximate conditions to the exact boundary condition, and subsequently estimate the error in the solutions obtained using these approximations. We show that for finite times the Padé approximants proposed by Engquist and Majda lead to exponential convergence if the solution is smooth, but that good long-time error estimates cannot hold for spatially local conditions. Applications in fluid dynamics are also discussed.

Key words. Radiation boundary conditions, integral equations, hyperbolic systems.

1. Introduction. Problems in wave propagation are generally posed on unbounded domains. Their numerical solution thus requires the introduction of an artificial boundary and the imposition of radiation boundary conditions there. Scores of authors have considered this problem, and a number of reasonably accurate procedures have been discovered. Nonetheless, in order to obtain some specified accuracy, it is still generally the practice to enlarge the domain - a process which may be inefficient and difficult to automate.

In this work we pursue a different approach - namely to fix the artificial boundary and to improve the accuracy by increasing the order of the approximate radiation conditions. From a practical point of view, we see that these high-order conditions can be easily implemented via the introduction of auxiliary functions on the boundary. From a theoretical point of view, estimates of convergence for fixed boundaries and increasing order are needed. We develop such estimates for the wave equation in a half-space by first finding a convenient representation of the exact radiation condition, which turns out to involve convolution in time with a Bessel kernel. Approximate conditions are similarly represented in terms of convolutions, and the error then depends on the difference between the exact and approximate kernels. Using approximations to an integral representation of the exact kernel, convergent time-local approximate conditions are derived. These include the spatially local Padé conditions proposed by Engquist and

* Supported, in part, by NSF Grant No. DMS-9304406. Part of the work was also carried out during a visit to the IMA, U. of Minnesota.

Majda [7,8]. For long time computations, on the other hand, it is shown that spatially nonlocal conditions are generally needed.

Generalizations to other problems are presented, including the linearized Euler equations as well as the wave equation with circular and spherical boundaries. Throughout we indicate some interesting theoretical and practical issues which remain unresolved.

2. The Wave Equation in a Half-Space.

2.1. Exact Boundary Conditions. We consider:

$$(2.1) \quad u_{tt} = c^2 \nabla^2 u + f, \quad t > 0, \quad x = (x_1, y) \in (0, \infty) \times \mathbb{R}^{n-1},$$

$$(2.2) \quad u(x, 0) = g(x), \quad B_0 u(0, y, t) = g_0(y, t).$$

We suppose, for some $L, \delta > 0$, that $f = g = 0$ and $c = 1$ for $x_1 \geq L - \delta$. Let $\hat{u}(x_1, k, s)$ be the Fourier-Laplace transform of u with respect to y and t . Then it is easily shown (e.g. [11]) that \hat{u} satisfies the exact boundary condition at $x_1 = L$:

$$(2.3) \quad \frac{\partial \hat{u}}{\partial x_1} + (s^2 + |k|^2)^{1/2} \hat{u} = 0.$$

(The branch of $(s^2 + |k|^2)^{1/2}$ is chosen so that it is analytic in the right half s -plane and has positive real part.) This exact condition is expressed in terms of u in the following way: Let \mathcal{F} denote Fourier transformation with respect to y and \mathcal{F}^{-1} be its inverse. Let:

$$(2.4) \quad \mathcal{K}(t) = \frac{J_1(t)}{t} = \frac{1}{\pi} \int_{-1}^1 \sqrt{1-w^2} \cos wt dw.$$

As shown in the appendix,

$$(2.5) \quad \hat{\mathcal{K}}(s) = (s^2 + 1)^{1/2} - s.$$

Using standard formulas from Laplace transform theory (e.g. [5]) we finally have the exact condition at $x_1 = L$:

$$(2.6) \quad \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial t} + \mathcal{F}^{-1}(|k|^2 \mathcal{K}(|k|t) * (\mathcal{F}u(x_1, \cdot, t))) = 0.$$

(Here, $*$ denotes convolution.)

2.2. Approximate Conditions. Although it may be possible to directly implement (2.6) using FFT's and fast convolutions, most work has been focussed on the development of approximate conditions involving differential operators in time and, usually, space. Local approximations in time correspond to rational approximations in s to $\hat{\mathcal{K}}$:

$$(2.7) \quad (s^2 + |k|^2)^{1/2} - s = |k| \left((z^2 + 1)^{1/2} - z \right) \approx |k| R(z), \quad z = s/|k|.$$

We take R to be a rational function of degree $(p, p+1)$, that is,

$$(2.8) \quad R(z) = \frac{P(z)}{Q(z)}, \quad \deg(P) = p, \quad \deg(Q) = p+1.$$

The approximate condition may be directly localized in time by applying the operator $\mathcal{F}^{-1}Q(|k|^{-1}\partial/\partial t)\mathcal{F}$. However, this leads to differential operators of high order as p is increased. To develop a more convenient framework for implementation, we make the additional assumption that the roots of Q are distinct. Then, R has a partial fraction expansion:

$$(2.9) \quad R(z) = \sum_{j=1}^{p+1} \frac{\alpha_j}{z - \rho_j}.$$

Let

$$(2.10) \quad \hat{h}_j = \frac{\alpha_j |k|}{z - \rho_j} \hat{u},$$

and let $\bar{v}(x_1, |k|, t)$ be the Fourier transform of v with respect to y . Here, v denotes the approximation to u computed on the bounded domain. We finally have the approximate boundary condition:

$$(2.11) \quad \frac{\partial \bar{v}}{\partial x_1} + \frac{\partial \bar{v}}{\partial t} + \sum_{j=1}^{p+1} \bar{h}_j = 0,$$

$$(2.12) \quad \left(\frac{\partial}{\partial t} - |k| \rho_j \right) \bar{h}_j = \alpha_j |k|^2 \bar{v}.$$

The advantage of this formulation is clear: the order is increased simply by increasing the number of terms in the sum. From the point of view of code development, this is very convenient.

We note that the conditions above are still nonlocal in space. For periodic problems this is no obstacle, as FFT's can be used. However, the nonlocality does preclude their use at more general boundaries. A glance at (2.12) reveals the condition for spatial locality: the poles, ρ_j , of $R(z)$ must come in conjugate, imaginary pairs, or be 0 and R itself must be an odd function of z . We may then assume that R has an expansion of the form:

$$(2.13) \quad R(z) = \sum_{j=1}^q \frac{\gamma_j z}{z^2 + \beta_j^2}.$$

This leads to the local implementation:

$$(2.14) \quad \frac{\partial v}{\partial x_1} + \frac{\partial v}{\partial t} + \sum_{j=1}^q \phi_j = 0,$$

$$(2.15) \quad \left(\frac{\partial^2}{\partial t^2} - \beta_j^2 \nabla_{\tan}^2 \right) \phi_j = -\gamma_j \nabla_{\tan}^2 \frac{\partial v}{\partial t}.$$

We would like to impose such a locality condition on $R(z)$, but it will be shown later that such approximations cannot lead to good error estimates uniformly in time or in tangential wave number.

In what follows, we will view

$$(2.16) \quad \mathcal{G}(t) = \mathcal{L}^{-1} R(s),$$

as an approximation to $\mathcal{K}(t)$. For reference we note that (2.9) corresponds to:

$$(2.17) \quad \mathcal{G}(t) = \sum_{j=1}^{p+1} \alpha_j e^{\rho_j t},$$

while (2.13) implies:

$$(2.18) \quad \mathcal{G}(t) = \sum_{j=1}^q \gamma_j \cos \beta_j t.$$

2.3. Error Estimates. We have seen that it is relatively straightforward to implement conditions of increasing order, at least in the half-space (or periodic) case. This leads to the question of convergence. Naturally, error estimates for approximate boundary conditions have been considered (e.g. [2],[8], [13],[20]). However, none of these consider convergence for a fixed problem in a fixed domain as the order of the conditions is increased.

Let $e = u - v$ be the error. Then e satisfies:

$$(2.19) \quad e_{tt} = c^2 \nabla^2 e, \quad t > 0, \quad x = (x_1, y) \in (0, L) \times \mathbb{R}^{n-1} \equiv \Omega,$$

$$(2.20) \quad e(x, 0) = 0, \quad B_0 e(0, y, t) = 0,$$

$$(2.21) \quad \frac{\partial \bar{e}}{\partial x_1} + \frac{\partial \bar{e}}{\partial t} + |k|^2 \mathcal{G}(|k|t) * \bar{e} = |k|^2 \mathcal{E}(|k|t) * \bar{u}.$$

The error kernel is given by:

$$(2.22) \quad \mathcal{E}(\tau) = \mathcal{G}(\tau) - \frac{J_1(\tau)}{\tau}.$$

Estimates of e naturally require both the stability and consistency of the approximate boundary conditions. Stability is a consequence of the uniform Lopatinski condition:

$$(2.23) \quad s + (s^2 + |k|^2)^{1/2} + |k| R(z) \neq 0,$$

for

$$(2.24) \quad \Re(s) \geq 0, \quad k \in R^{n-1}, \quad (s, k) \neq (0, 0).$$

Then we have (e.g. Sakamoto [22, Ch. 3]):

$$(2.25) \quad \int_0^T \|e(\cdot, t)\|_{H_1[\Omega]}^2 dt \leq C^2 \int_0^T \|Eu(L, \cdot, t)\|_{H_0[R^{n-1}]}^2 dt,$$

where,

$$(2.26) \quad Ew = \mathcal{F}^{-1}|k|^2 \mathcal{E}(|k|t) * (\mathcal{F}w).$$

The error may now be bounded in terms of the error in the approximation, \mathcal{G} , to \mathcal{K} . In particular, suppose, for some $\mu \geq 0$ and $T \geq 1$:

$$(2.27) \quad \|\mathcal{E}\|_{L_1[(0, T)]} \leq \epsilon T^\mu.$$

Then, by Parseval's identity and standard estimates for convolutions,

$$(2.28) \quad \begin{aligned} \int_0^T \|Eu(L, \cdot, t)\|_{H_0[R^{n-1}]}^2 dt &= \int_0^T \int_{R^{n-1}} |k|^4 |(\mathcal{E}(|k|t) * \bar{u}(L, k, t))(t)|^2 dk dt \\ &\leq \int_0^T \int_{R^{n-1}} |k|^2 |\bar{u}(L, k, t)|^2 \|\mathcal{E}\|_{L_1[(0, |k|T)]}^2 dk dt \\ &\leq \epsilon^2 T^{2\mu} \int_0^T \|u(L, \cdot, t)\|_{H_{1+\mu}[R^{n-1}]}^2 dt. \end{aligned}$$

Substituting this into (2.25) we finally obtain:

$$(2.29) \quad \int_0^T \|e(\cdot, t)\|_{H_1[\Omega]}^2 dt \leq \epsilon^2 T^{2\mu} C^2 \int_0^T \|u(L, \cdot, t)\|_{H_{1+\mu}[R^{n-1}]}^2 dt.$$

This error estimate is best, both from the point of view of long time behavior and from the point of view of smoothness required of u , if $\mu = 0$. We note that such an estimate requires bounds on $\|\mathcal{E}\|_{L_1[(0, \infty)]}$. This cannot be attained for local conditions, as we have seen that they involve convolution kernels which are combinations of $\cos \beta_j t$ (2.18), and, hence, are not elements of $L_1[(0, \infty)]$. Time uniform estimates could be obtained using spatially nonlocal conditions, however. Some discussion of long-time behavior of spatially nonlocal boundary conditions appears in [6, 17]. In [15] we construct conditions using Laguerre and exponential expansions. Although the conditions so derived do lead to estimates with $\mu = 0$, convergence as the order of approximation was increased was slow at best. Below we introduce a new nonlocal approximation based on the direct approximation to an integral representation for $\mathcal{K}(t)$.

Our point of view leads to an interesting, and to our knowledge unsolved, problem in approximation theory. Define \mathcal{H}_p to be the set of all

real functions in $L_1[(0, \infty)]$ whose Laplace transform is a rational function of degree $(p, p+1)$. More directly, a function in \mathcal{H}_p takes the form:

$$(2.30) \quad \mathcal{G}(t) = \sum_{j=1}^{p_1} M_j(t) e^{-\gamma_j t} \cos(\beta_j t + \phi_j) + \sum_{j=1}^{p_2} N_j(t) e^{-\kappa_j t},$$

where γ_j and κ_j are positive and $M_j(t)$ and $N_j(t)$ are polynomials. Here p is given by:

$$(2.31) \quad p+1 = 2 \sum_{j=1}^{p_1} (\deg(M_j) + 1) + \sum_{j=1}^{p_2} (\deg(N_j) + 1).$$

Problem A: For fixed p characterize and find an algorithm to compute the best $L_1[(0, \infty)]$ approximation, $\mathcal{G} \in \mathcal{H}_p$, to \mathcal{K} or to other kernels. Estimate the behavior of the error as p is increased.

The solution of this problem would provide us with optimal approximations to convolutions via the solution of differential equations. We note that for bounded intervals and sums excluding trigonometric terms (i.e. $p_1 = 0$), a theory does exist. (See Braess [3, Ch. 6].)

2.4. Methods Derived via Quadrature. In general, approximate boundary conditions have been derived either by direct approximation to the symbol (e.g. [24]) or through the use of far-field asymptotics (e.g. [2, 16]). Here we show how a class of convergent local (in space and time) approximate conditions may be derived by approximating the integral representation of the exact kernel:

$$(2.32) \quad \frac{J_1(t)}{t} = \frac{1}{\pi} \int_{-1}^1 \sqrt{1-w^2} \cos wt dw.$$

The simplest example is the trapezoid rule:

$$(2.33) \quad \frac{J_1(t)}{t} \approx \frac{2}{(q+1)\pi} \sum_{j=1}^q \sqrt{1-w_j^2} \cos w_j t, \quad w_j = -1 + \frac{2j}{q+1}.$$

After Laplace transformation we find that:

$$(2.34) \quad R(z) = \frac{2}{(q+1)\pi} \sum_{j=1}^q \frac{\sqrt{1-w_j^2} z}{z^2 + w_j^2}.$$

(Of course, in implementations of the condition the number of terms in the sum can be halved using the evenness in w_j of the integrand.)

The well-posedness of these approximations follows directly from a check on the Lopatinski condition. (It also follows from general results

for local conditions in [12,23].) In order to derive error estimates, however, we must explicitly bound the stability constant, C in (2.29), as a function of q . Using Parseval's relation (e.g. [21]) we have:

$$(2.35) \quad C \leq c_0(T) \sup_{s=\gamma+i\eta, \ k \in \mathbb{R}^{n-1}} \left| \frac{(s^2 + |k|^2)^{1/2} + |k|}{s + (s^2 + |k|^2)^{1/2} + |k|R(s/|k|)} \right|.$$

Here $\gamma > 0$ is fixed and $c_0(T)$ depends on γ but is independent of the boundary condition. Dividing through by $|k|$ we see that our problem is reduced to the estimation of:

$$(2.36) \quad \sup_{\Re(z) \geq 0} |Q(z)|, \quad Q(z) = \frac{(z^2 + 1)^{1/2} + 1}{z + (z^2 + 1)^{1/2} + R(z)}.$$

Noting the form of $R(z)$, and recalling the choice of branch for the roots, we see that the Q is bounded as $z \rightarrow \infty$ independent of q . Therefore, by the maximum principle, we can restrict attention to the imaginary axis, $z = i\eta$. In this case the poles of R are located on the imaginary axis with $|\eta| < 1$ and $|R|$ is strictly decreasing in $|\eta|$ for $|\eta| \geq 1$. Therefore, for $|\eta| \geq 1$ we have:

$$(2.37) \quad |Q| \leq \frac{1 + \sqrt{\eta^2 - 1}}{|\eta| + \sqrt{\eta^2 - 1} - |R(i)|} \leq \frac{1}{1 - |R(i)|}.$$

Now

$$(2.38) \quad 1 - |R(i)| = 1 - \frac{2}{(q+1)\pi} \sum_{j=1}^q \frac{1}{\sqrt{1-w_j^2}}.$$

To bound this we note:

$$(2.39) \quad \pi = \int_{-1}^1 \frac{dw}{\sqrt{1-w^2}}.$$

For $q = 2p$, even, we have:

$$(2.40) \quad \pi > 2 \sum_{j=1}^p \int_{w_{j-2/(q+1)}}^{w_j} \frac{dw}{\sqrt{1-w^2}}.$$

Generally we have:

$$(2.41) \quad \int_{w_{j-2/(q+1)}}^{w_j} \frac{dw}{\sqrt{1-w^2}} > \frac{2}{q+1} \frac{1}{\sqrt{1-w_j^2}},$$

and for $j = 1, q \geq 2$;

$$(2.42) \quad \int_{-1}^{w_1} \frac{dw}{\sqrt{1-w^2}} > \frac{\sqrt{2} - \sqrt{3/2}}{\sqrt{q+1}} + \frac{2}{q+1} \frac{1}{\sqrt{1-w_1^2}}.$$

Hence, for some constant, c , independent of q :

$$(2.43) \quad \frac{1}{1 - |R(i)|} \leq c\sqrt{q}.$$

For $q = 2p + 1$, odd,

$$(2.44) \quad \pi = 2 \sum_{j=1}^{p+1} \int_{w_{j-2/(q+1)}}^{w_j} \frac{dw}{\sqrt{1-w^2}},$$

and (2.43) is similarly established. For $|\eta| \leq 1$ we distinguish two cases, $|\eta| < |w_1|$ and $|\eta| > |w_1|$. For the former we have:

$$(2.45) \quad |Q| \leq \frac{2}{\sqrt{1-w_1^2}} \leq c\sqrt{q},$$

while for the latter it is easily shown that $|Q|$ is maximized at $|\eta| = 1$, which we have estimated above. Our final estimate is:

$$(2.46) \quad C \leq \bar{C}(T)\sqrt{q},$$

with \bar{C} independent of q .

To derive error estimates we first estimate the error in the quadrature formula for fixed t . Although the integrand doesn't possess a bounded derivative, its derivative is integrable. Therefore, employing the Peano Kernel representation of the error:

$$(2.47) \quad \left| \frac{J_1(t)}{t} - \frac{2}{(q+1)\pi} \sum_{j=1}^q \sqrt{1-w_j^2} \cos w_j t \right| \leq \frac{c_1 + 2c_2 t}{q}.$$

Here, the c_j are constants independent of t and q . Then we have:

$$(2.48) \quad \begin{aligned} \|\mathcal{E}_{\text{trap}}(t)\|_{L_1((0,T))} &= \int_0^T \left| \frac{J_1(t)}{t} - \frac{2}{(q+1)\pi} \sum_{j=1}^q \sqrt{1-w_j^2} \cos w_j t \right| dt \\ &\leq \frac{1}{q} (c_1 T + c_2 T^2). \end{aligned}$$

Substituting this into (2.29) and using (2.46) we finally obtain, for some $\bar{c}(T)$ independent of q :

$$(2.49) \quad \int_0^T \|e(\cdot, t)\|_{H_1(\Omega)}^2 dt \leq \bar{c} \frac{T^4}{q} \int_0^T \|u(L, \cdot, t)\|_{H_3(R^{n-1})}^2 dt.$$

We have shown, then, that the trapezoid rule produces a one-half order approximation to the true solution in the sense that the error decays at least as the square root of the reciprocal of the number of terms in the boundary condition.

A more accurate formula is obtained through the application of the Gaussian quadrature rule associated with the weight $\sqrt{1-w^2}$:

$$(2.50) \quad \frac{J_1(t)}{t} \approx \frac{1}{q+1} \sum_{i=1}^q \sin^2 \frac{j\pi}{q+1} \cos w_j t, \quad w_j = \cos \frac{j\pi}{q+1},$$

$$(2.51) \quad R(z) = \frac{1}{q+1} \sum_{j=1}^q \frac{(\sin^2 \frac{j\pi}{q+1})z}{z^2 + w_j^2}.$$

This approximation is discussed in [15]. Remarkably, it is shown to be equivalent to the stable Padé approximants introduced by Engquist and Majda [7,8]. The stability of this approximation is well-known. To estimate the stability constant, we repeat the analysis given above for the trapezoid approximation. As in that case, we must estimate $(1 - |R(i)|)^{-1}$. We find:

$$(2.52) \quad (1 - |R(i)|)^{-1} = \left(1 - \frac{1}{q+1} \sum_{j=1}^q \frac{\sin^2 \frac{j\pi}{q+1}}{1 - w_j^2} \right)^{-1} = q+1.$$

We also have:

$$(2.53) \quad \frac{1}{\sqrt{1-w_1^2}} = \frac{1}{\sin \frac{\pi}{q+1}} \leq c(q+1).$$

Hence we conclude:

$$(2.54) \quad C \leq \bar{C}(T)q,$$

with \bar{C} independent of q .

Using the error formula for the quadrature rule we find [15]:

$$(2.55) \quad \|\mathcal{E}_{\text{gauss}}(t)\|_{L_1(0,T)} \leq c \frac{(T/2)^{2q+1}}{(2q+1)!}.$$

From this, (2.29) and (2.54) we prove:

$$(2.56) \quad \int_0^T \|e(\cdot, t)\|_{H_1(\Omega)}^2 dt \leq c \frac{q^2 (T/2)^{4q+2}}{((2q+1)!)^2} \int_0^T \|u(L, \cdot, t)\|_{H_{2q+2}[R^{n-1}]}^2 dt.$$

Here we see exponential convergence in q for smooth w , as with spectral approximations to the solutions of differential equations, so that the method might reasonably be called infinite order.

Clearly, these estimates are nonuniform in T , as must be the case for spatially local boundary conditions. (See (2.18).) In [14,15] we presented a number of methods for approximating the convolution kernel by decaying functions, including exponential interpolation, exponential least squares

and approximations by Laguerre functions. None of these seemed entirely satisfactory: the interpolation could not be easily extended to high order, while the other approximations converged very slowly (if at all) in $L_1(0, \infty)$. (We note that the proposed conditions have not yet been tested.)

In order to use quadrature to construct long-time approximations we must derive an integral representation of $J_1(t)/t$ involving an integrand which decays in t . This may be accomplished by treating (2.4) as a complex integral and deforming the contour. For example, let:

$$(2.57) \quad z = w + i(1 - w^2)P(w), \quad P > 0, \quad w \in [-1, 1].$$

Then:

$$(2.58) \quad \begin{aligned} \frac{J_1(t)}{t} &= \frac{1}{\pi} \Re \left(\int_C (1 - z^2)^{1/2} e^{izt} dz \right) \\ &= \frac{1}{2\pi} \int_{-1}^1 \sqrt{1 - w^2} D(w, t) dw, \end{aligned}$$

$$(2.59) \quad D(w, t) = (f_1(w) \cos wt + f_2(w) \sin wt) e^{-(1-w^2)P(w)t},$$

$$(2.60) \quad f_1(w) = g(w) + (2wP(w) - (1 - w^2)P'(w))h(w),$$

$$(2.61) \quad f_2(w) = (2wP(w) - (1 - w^2)P'(w))g(w) - h(w),$$

$$(2.62) \quad g(w) = G(w) + (1 - w^2)P^2(w)/G(w),$$

$$(2.63) \quad G(w) = \sqrt{1 + \sqrt{1 + (1 + w)^2 P^2(w)}} \sqrt{1 + \sqrt{1 + (1 - w)^2 P^2(w)}},$$

$$(2.64) \quad h(w) = (1 - w)P(w)H(w) - (1 + w)P(w)/H(w),$$

$$(2.65) \quad H(w) = \frac{\sqrt{1 + \sqrt{1 + (1 + w)^2 P^2(w)}}}{\sqrt{1 + \sqrt{1 + (1 - w)^2 P^2(w)}}}.$$

Since the function $D(w, t)$ is smooth on the interval of integration, it is still reasonable to use the Gaussian quadrature scheme for the weight $\sqrt{1 - w^2}$. This yields:

$$(2.66) \quad \frac{J_1(t)}{t} \approx \frac{1}{2(q+1)} \sum_{i=1}^q \sin^2 \frac{j\pi}{q+1} D(w_j, t),$$

$$(2.67) \quad R(z) = \frac{1}{2(q+1)} \sum_{j=1}^q r_j(z),$$

$$(2.68) \quad r_j(z) = \frac{\sin^2 \frac{j\pi}{q+1} (f_1(w_j)(z + (1 - w_j^2)P(w_j)) + f_2(w_j)w_j)}{(z + (1 - w_j^2)P(w_j))^2 + w_j^2}.$$

We have not yet tested or analyzed these schemes. Finite time error estimates should be obtainable using the error formula for the quadrature rule, but the hope is that time independent estimates will hold.

Problem B: Find a function or class of functions, $P(w)$, such that the resulting scheme leads to well-posed problems and kernels which converge to $\mathcal{K}(t)$ in $L_1[(0, \infty)]$.

3. Generalizations.

3.1. Anisotropic Problems. The approximations discussed above are also applicable to problems with anisotropic wave propagation. As a first example, consider the convective wave equation with subsonic convection:

$$(3.1) \quad \left(\frac{\partial}{\partial t} + M \sum_l \omega_l \frac{\partial}{\partial x_l} \right)^2 u = \nabla^2 u, \quad x_1 \geq L.$$

We normalize so that $\sum_l \omega_l^2 = 1$, $0 \leq M < 1$. To formulate the exact boundary condition, we seek solutions of the form:

$$(3.2) \quad \hat{u} = A e^{\lambda x_1},$$

leading to the quadratic equation,

$$(3.3) \quad \left(s + iM \sum_{l>1} \omega_l k_l + M \omega_1 \lambda \right)^2 = \lambda^2 - |k|^2.$$

The relevant solution, that is the one with negative real part for $\Re(s)$ sufficiently large, is given by:

$$(3.4) \quad \lambda = -(1 - M^2 \omega_1^2)^{-1} \left((1 - M \omega_1) \tilde{s} + \left((\tilde{s}^2 + (1 - M^2 \omega_1^2) |k|^2)^{1/2} - \tilde{s} \right) \right),$$

where

$$(3.5) \quad \tilde{s} = s + iM \sum_{l>1} \omega_l k_l.$$

To conveniently express approximations to this condition, we define the *tangential material derivative*, D_{tan}/Dt , by

$$(3.6) \quad \frac{D_{\text{tan}}}{Dt} = \frac{\partial}{\partial t} + U_{\text{tan}} \cdot \nabla_{\text{tan}}, \quad U_{\text{tan}} = M \begin{pmatrix} \omega_2 & \omega_3 & \dots & \omega_n \end{pmatrix}^T,$$

and let D/Dt denote the standard material derivative with respect to the full velocity $U = M(\omega_1 \dots \omega_n)^T$. The exact condition is then given by:

$$(3.7) \quad (1 + M\omega_1) \frac{\partial \bar{u}}{\partial x_1} + \frac{\partial \bar{u}}{\partial t} + i(U_{\text{tan}} \cdot k) \bar{u} + (1 + M\omega_1) |k|^2 \mathcal{A} * \bar{u},$$

$$(3.8) \quad \mathcal{A}(k, t) = \frac{J_1(\sqrt{1 - M^2 \omega_1^2} |k| t)}{\sqrt{1 - M^2 \omega_1^2} |k| t} e^{-i(U_{\text{tan}} \cdot k)t}.$$

Using the rational approximation $\sqrt{1 - M^2 \omega_1^2} |k| R(z)$, with $z = \tilde{s}/(\sqrt{1 - M^2 \omega_1^2} |k|)$, we obtain, in analogy with (2.11-2.12),

$$(3.9) \quad (1 + M\omega_1) \frac{\partial \bar{v}}{\partial x_1} + \frac{\partial \bar{v}}{\partial t} + i(U_{\text{tan}} \cdot k) \bar{v} + \sum_{j=1}^{p+1} \bar{h}_j = 0,$$

$$(3.10) \quad \left(\frac{\partial}{\partial t} + i(U_{\text{tan}} \cdot k) - \sqrt{1 - M^2 \omega_1^2} |k| \rho_j \right) \bar{h}_j = \alpha_j |k|^2 (1 + M\omega_1) \bar{v}.$$

The approximate condition is local in space under the same conditions for locality in the isotropic case, that is (2.13). Then we have:

$$(3.11) \quad \frac{\partial v}{\partial x_1} + \frac{Dv}{Dt} + \sum_{j=1}^q \phi_j = 0,$$

$$(3.12) \quad \left(\frac{D_{\text{tan}}^2}{Dt^2} - (1 - M^2 \omega_1^2) \beta_j^2 \nabla_{\text{tan}}^2 \right) \phi_j = -(1 + M\omega_1) \gamma_j \nabla_{\text{tan}}^2 \frac{D_{\text{tan}} v}{Dt}.$$

We are confident that issues of consistency and convergence for these approximations could be handled as in the isotropic case, but we have not yet carried out the details. Below we see that the same operators appear in boundary conditions for the linearized compressible Euler and Navier-Stokes systems.

3.2. Applications to Fluid Dynamics. We now consider the compressible Euler equations linearized about a uniform flow in two space dimensions:

$$(3.13) \quad \frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0,$$

$$(3.14) \quad \frac{Dv}{Dt} + \frac{\partial p}{\partial y} = 0,$$

$$(3.15) \quad \frac{Dp}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

where $D/Dt = \partial/\partial t + M\omega_x \partial/\partial x + M\omega_y \partial/\partial y$, $\omega_x^2 + \omega_y^2 = 1$, $0 < M < 1$. We suppose $\omega_x > 0$ and put artificial boundaries at $x = \pm L$; that is inflow at $-L$ and outflow at L .

To compute exact conditions we Laplace transform in t , Fourier transform in y , and rewrite the system in the form:

$$(3.16) \quad \frac{\partial}{\partial x} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \frac{M\omega_x \bar{s}}{1-M^2\omega_x^2} & -\frac{ik}{1-M^2\omega_x^2} & -\frac{\bar{s}}{1-M^2\omega_x^2} \\ 0 & -\frac{\bar{s}}{M\omega_x} & -\frac{ik}{M\omega_x} \\ -\frac{\bar{s}}{1-M^2\omega_x^2} & \frac{ikM\omega_x}{1-M^2\omega_x^2} & \frac{M\omega_x \bar{s}}{1-M^2\omega_x^2} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{pmatrix}.$$

Here, $\bar{s} = s + ikM\omega_y$. Eigenvalues and left eigenvectors of the coefficient matrix are given by:

$$(3.17) \quad \lambda_1 = \frac{\hat{A} + M\omega_x \bar{s}}{1 - M^2\omega_x^2}, \quad l_1^T = (\bar{s} \quad -ikM\omega_x \quad -\hat{A}),$$

$$(3.18) \quad \lambda_2 = -\frac{\hat{A} - M\omega_x \bar{s}}{1 - M^2\omega_x^2}, \quad l_2^T = (\bar{s} \quad -ikM\omega_x \quad \hat{A}),$$

$$(3.19) \quad \lambda_3 = -\frac{\bar{s}}{M\omega_x}, \quad l_3^T = (ikM\omega_x \quad \bar{s} \quad ik),$$

where

$$(3.20) \quad \hat{A} = (\bar{s}^2 + (1 - M^2\omega_x^2)k^2)^{1/2}.$$

Noting that $\Re(\lambda_1) > 0$ and $\Re(\lambda_{2,3}) < 0$ for $\Re(s) > 0$ we have, setting $w = (u \ v \ p)^T$:

$$(3.21) \quad l_1^T w = 0, \quad x = L; \quad l_2^T w = l_3^T w = 0, \quad x = -L.$$

The exact outflow boundary condition is then given by:

$$(3.22) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y \right) (\bar{u} - \bar{p}) - ikM\omega_x \bar{v} - (1 - M^2\omega_x^2)k^2 \mathcal{A} * \bar{p} = 0.$$

The exact inflow conditions are given by:

$$(3.23) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y \right) (\bar{u} + \bar{p}) - ikM\omega_x \bar{v} + (1 - M^2\omega_x^2)k^2 \mathcal{A} * \bar{p} = 0,$$

$$(3.24) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y \right) \bar{v} + ikM\omega_x \bar{u} + ik\bar{p} = 0.$$

Here,

$$(3.25) \quad \mathcal{A}(k, t) = \frac{J_1(\sqrt{1 - M^2\omega_x^2}kt)}{\sqrt{1 - M^2\omega_x^2}kt} e^{-ikM\omega_y t}.$$

Using the rational approximation $\sqrt{1 - M^2\omega_x^2}|k|R(z)$, for $z = \tilde{s}/(\sqrt{1 - M^2\omega_x^2}|k|)$, equations (3.22- 3.23) become, in analogy with (3.9-3.10),

$$(3.26) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y \right) (\bar{u} - \bar{p}) - ikM\omega_x \bar{v} - \sum_{j=1}^{p+1} \bar{h}_j = 0,$$

$$(3.27) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y - \sqrt{1 - M^2\omega_x^2}|k|\rho_j \right) \bar{h}_j = \alpha_j k^2 (1 - M^2\omega_x^2) \bar{p},$$

at outflow and at inflow,

$$(3.28) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y \right) (\bar{u} + \bar{p}) - ikM\omega_x \bar{v} + \sum_{j=1}^{p+1} \bar{g}_j = 0,$$

$$(3.29) \quad \left(\frac{\partial}{\partial t} + ikM\omega_y - \sqrt{1 - M^2\omega_x^2}|k|\rho_j \right) \bar{g}_j = \alpha_j k^2 (1 - M^2\omega_x^2) \bar{p}.$$

The approximate conditions are local in space under the same conditions for locality in the scalar case, that is (2.13). In particular, if we use the Engquist-Majda-Padé-Gaussian approximation, (2.51), we have, at outflow:

$$(3.30) \quad \frac{D_{\tan}}{Dt}(u - p) - M\omega_x \frac{\partial v}{\partial y} - \sum_{j=1}^q \phi_j = 0,$$

$$(3.31) \quad \left(\frac{D_{\tan}}{Dt} + \sqrt{1 - M^2\omega_x^2} \cos \frac{j\pi}{q+1} \frac{\partial}{\partial y} \right) \phi_j = -\frac{1}{q+1} \sin^2 \frac{j\pi}{q+1} (1 - M^2\omega_x^2) \frac{\partial^2 p}{\partial y^2}.$$

The inflow conditions become:

$$(3.32) \quad \frac{D_{\tan}}{Dt}(u + p) - M\omega_x \frac{\partial v}{\partial y} + \sum_{j=1}^q \psi_j = 0,$$

$$(3.33) \quad \left(\frac{D_{\tan}}{Dt} + \sqrt{1 - M^2 \omega_x^2} \cos \frac{j\pi}{q+1} \frac{\partial}{\partial y} \right) \psi_j = -\frac{1}{q+1} \sin^2 \frac{j\pi}{q+1} (1 - M^2 \omega_x^2) \frac{\partial^2 p}{\partial y^2}.$$

$$(3.34) \quad \frac{D_{\tan} v}{Dt} + M \omega_x \frac{\partial u}{\partial y} + \frac{\partial p}{\partial y} = 0.$$

These conditions have been implemented by Goodrich [9] for channel flows, and shown to be very accurate even for moderate q . Note that (3.34) implies zero vorticity at inflow and, for the linearized problem, it is exact. Also, we have used an alternative representation of the approximate conditions with the property that the auxiliary functions are computed by solving first order equations.

A similar construction has been carried out for the linearized, isentropic, compressible Navier-Stokes equations in the low Mach number limit by the author and Lorenz [18]. Here, we require two boundary conditions at outflow and three at inflow. The conditions (3.22), (3.23) and (3.34) remain the same to leading order. The additional condition at outflow is

$$(3.35) \quad \frac{Dv}{Dt} + \frac{\partial p}{\partial y} = 0,$$

and at inflow is given by:

$$(3.36) \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0.$$

We note that here we generally expect that long time computations, as measured on the time scale of the sound waves, will be of interest. Therefore, nonlocal approximations may be efficient. As part of his doctoral dissertation, L. Xu is looking into such approximations and their applications in acoustics. Some of his results will appear in [19].

3.3. Corner Conditions. The boundary conditions, as discussed so far, only apply to half-space or periodic problems. To generalize their applicability, one must understand how to treat the case of an artificial boundary intersecting another part of the boundary at a corner. Collino [4] has solved this problem for the important special case of the isotropic wave equation, two artificial boundaries intersecting at a right angle, and spatially local boundary conditions. (We note that our formulation of the exact conditions and spatially nonlocal approximations is not valid in this case.)

As a first attempt to generalize Collino's results, we have considered the convective wave equation in a rectangular domain with all boundaries artificial and the spatially local boundary conditions (3.11-3.12). Collino's

construction does not directly apply, as it is based on special exact solutions of the wave equation in a corner. We have, instead, looked at power series expansions in the corner. These do lead to compatibility conditions which can be used to relate the auxiliary functions associated with distinct boundaries. However, it seems that the expansion must be carried out to order greater than $4q$ to produce the q required conditions. We have tried to do this symbolically, but so far have succeeded only for $q \leq 2$.

Problem C: Derive corner compatibility relations for local approximate boundary conditions for anisotropic systems.

We also note that the theory of exact conditions and nonlocal approximations is still undeveloped.

Problem D: Characterize the exact boundary condition for the wave equation and convective wave equation with a rectangle or rectangular parallelepiped as artificial boundary. Construct convergent temporally local approximate conditions.

Another case of practical importance should be analyzed is the intersection of an artificial boundary with a physical boundary, such as a solid wall. As a first example consider the convective wave equation in two space dimensions with convection in the x direction and walls at $y = 0$ and $y = H$:

$$(3.37) \quad \left(\frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right)^2 u = \nabla^2 u, \quad (x, y) \in (0, \infty) \times (0, H),$$

$$(3.38) \quad \alpha_0 \frac{\partial u}{\partial y} - \beta_0 u = 0, \quad y = 0; \quad \alpha_1 \frac{\partial u}{\partial y} + \beta_1 u = 0, \quad y = H.$$

We can expand the solution:

$$(3.39) \quad u = \sum_l \bar{u}_l Y_l(y), \quad Y_l(y) = A_l \cos k_l y + B_l \sin k_l y,$$

where the eigenvalues, k_l , are determined by the boundary conditions. Then, \bar{u}_l satisfies (3.7), and for the approximate local boundary condition (3.11)-(3.12) we have:

$$(3.40) \quad \phi_j = \sum_l \bar{\phi}_{jl} Y_l(y).$$

Since the eigenfunctions, Y_l , satisfy (3.38) so would the ϕ_j , under the assumption that the expansion is sufficiently regular. That is:

$$(3.41) \quad \alpha_0 \frac{\partial \phi_j}{\partial y} - \beta_0 \phi_j = 0, \quad y = 0; \quad \alpha_1 \frac{\partial \phi_j}{\partial y} + \beta_1 \phi_j = 0, \quad y = H.$$

We should emphasize that this is generally not a true solution of the compatibility problem, but simply an approximation which we have used.

This approximation can be extended to the linearized, compressible Euler system, again with the physical assumption that the base flow is parallel to the walls, $y = 0, H$. The physical boundary condition at this characteristic boundary is:

$$(3.42) \quad v = 0,$$

while a second relation, implied by the y -momentum equation is:

$$(3.43) \quad \frac{\partial p}{\partial y} = 0.$$

Using these we see that the solution can be expanded in the form:

$$(3.44) \quad \begin{aligned} u &= \sum_l \bar{u}_l \cos l\pi y/H, \quad v = \sum_l \bar{v}_l \sin l\pi y/H, \\ p &= \sum_l \bar{p}_l \cos l\pi y/H. \end{aligned}$$

To relate these expansions to expansions of the auxiliary variables, we note that:

$$(3.45) \quad \cos \frac{j\pi}{q+1} + \cos \frac{(q+1-j)\pi}{q+1} = 0.$$

Hence, equations (3.31) and (3.33) imply:

$$(3.46) \quad \phi_j + \phi_{q+1-j} = \sum_l \bar{\Phi}_l \cos l\pi y/H,$$

$$(3.47) \quad \psi_j + \psi_{q+1-j} = \sum_l \bar{\Psi}_l \cos l\pi y/H,$$

with boundary conditions:

$$(3.48) \quad \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_{q+1-j}}{\partial y} = 0, \quad \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_{q+1-j}}{\partial y} = 0, \quad y = 0, H.$$

These are the corner conditions used by Goodrich in [9].

3.4. Smooth Boundaries. Given the theoretical and practical difficulties of high-order boundary conditions at artificial boundaries with corners, smooth artificial boundaries are an attractive alternative. Representations of the exact boundary condition for the wave equation at circular and spherical artificial boundaries are easily obtained. (See [17].)

Using polar coordinates in two dimensions, we place the artificial boundary at $r = R$ and expand the solution, u , in a Fourier series in θ :

$$(3.49) \quad u(r, \theta, t) = \sum_l \bar{u}_l e^{il\theta}.$$

The exact boundary condition is then given by:

$$(3.50) \quad \frac{\partial \bar{u}_l}{\partial r} + \frac{\partial \bar{u}_l}{\partial t} + \frac{1}{2R} \bar{u}_l + \frac{1}{R^2} A_l(t/R) * \bar{u}_l = 0,$$

$$(3.51) \quad \hat{A}_l(z) = -z \left(\frac{K'_l(z)}{K_l(z)} + 1 + \frac{1}{2z} \right).$$

(Here, $K_l(z)$ is the modified Bessel function [1].)

We have not, as yet, found a closed form expression for A_l . We note that conditions based on far-field expansions (e.g. [2,16]) correspond to large z expansions of \hat{A}_l . By the results of [17], long time accuracy is difficult to achieve using time-local conditions, as A_0 decays slowly as $t \rightarrow \infty$.

In three dimensions, on the other hand, the exact condition takes a much simpler form. Here we use spherical coordinates and take $r = R$ as our artificial boundary. The solution, u , is now expanded in spherical harmonics:

$$(3.52) \quad u(r, \theta, \phi, t) = \sum_l \bar{u}_l Y_l(\theta, \phi),$$

where

$$(3.53) \quad \nabla_{\text{sphere}}^2 Y_l = -l(l+1)Y_l.$$

In analogy with the two-dimensional case, the exact condition is related to the inverse Laplace transform of the logarithmic derivative of spherical Bessel functions. In particular we have:

$$(3.54) \quad \frac{\partial \bar{u}_l}{\partial r} + \frac{\partial \bar{u}_l}{\partial t} + \frac{1}{R} \bar{u}_l + \frac{1}{R^2} S_l(t/R) * \bar{u}_l = 0,$$

$$(3.55) \quad \hat{S}_l(z) = -z \left(\frac{(z^{-1/2} K_{l+1/2}(z))'}{z^{-1/2} K_{l+1/2}(z)} + 1 + \frac{1}{z} \right) = \frac{P_l(z)}{Q_l(z)},$$

where, for $l \neq 0$,

$$(3.56) \quad P_l(z) = \sum_{k=0}^{l-1} \frac{(2l-k)!}{k!(l-k-1)!} (2z)^k, \quad Q_l(z) = \sum_{k=0}^l \frac{(2l-k)!}{k!(l-k)!} (2z)^k,$$

and $S_0 = 0$.

We immediately observe that $S_l(z)$ is rational and, therefore, (3.54) may be localized in time. (The fact that the exact boundary condition can be localized in time for a finite number of spherical harmonics is also noted by Grote and Keller [10].) What has not been accomplished so

far is to derive a convenient factorization or expansion of $P_t(z)/Q_t(z)$, to facilitate numerical implementation. However, even if this must be done numerically, the availability of simple, exact conditions makes the use of a spherical boundary very attractive.

We note that an important advantage of rectangular boundaries is the possibility of adjusting the aspect ratio of the domain. To do this with a smooth boundary, while maintaining a natural, separable coordinate system, one might need elliptical and spheroidal coordinates. This suggests our final problem.

Problem E: Characterize and approximate the exact boundary condition for the wave equation at elliptical and spheroidal boundaries.

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A. Derivation of Equation (2.4). Our goal is to compute a useful representation of the inverse Laplace transform of:

$$(A.1) \quad \hat{\mathcal{K}}(s) = (s^2 + 1)^{1/2} - s.$$

Although the inverse is available in standard tables, its importance to our work suggests the inclusion of a direct verification.

Following the ideas given in [5, Ch. 38], we will derive a simple differential equation satisfied by $\mathcal{K}(t)$. We first note that $\hat{\mathcal{K}}$ satisfies a first order differential equation with quadratic coefficients:

$$(A.2) \quad (s^2 + 1) \frac{d\hat{\mathcal{K}}}{ds} = s\hat{\mathcal{K}} - 1.$$

Recall that:

$$(A.3) \quad \frac{d\hat{f}}{ds} = -\widehat{tf(t)}, \quad s^p \hat{f} - \sum_{k=0}^{p-1} f^{(k)}(0^+) s^{p-k-1} = \widehat{f^{(p)}(t)}.$$

Therefore, (A.2) is equivalent to:

$$(A.4) \quad \widehat{\frac{d^2}{dt^2}(t\mathcal{K})} + \widehat{\frac{d\mathcal{K}}{dt}} + t\widehat{\mathcal{K}} = 1 - 2\mathcal{K}(0^+).$$

That is, $\mathcal{K}(t)$ satisfies the differential equation:

$$(A.5) \quad t \frac{d^2 \mathcal{K}}{dt^2} + 3 \frac{d\mathcal{K}}{dt} + t\mathcal{K} \equiv M\mathcal{K} = 0,$$

subject to the initial condition,

$$(A.6) \quad \mathcal{K}(0) = \frac{1}{2},$$

in addition to a growth condition at infinity.

We now verify that:

$$(A.7) \quad \mathcal{K}(t) = \frac{1}{\pi} \int_{-1}^1 \sqrt{1-w^2} \cos wtdw,$$

solves this problem. That the initial condition is satisfied may be checked directly. Applying the differential operator we obtain:

$$\begin{aligned} \pi M \mathcal{K} &= \int_{-1}^1 (1-w^2)^{3/2} t \cos wtdw - 3 \int_{-1}^1 \sqrt{1-w^2} w \sin wtdw \\ (A.8) \quad &= \int_{-1}^1 (1-w^2)^{3/2} t \cos wtdw + \int_{-1}^1 \frac{d}{dw} (1-w^2)^{3/2} \sin wtdw \\ &= 0. \end{aligned}$$

Finally, we recall the identity,

$$(A.9) \quad J_1(t) = \frac{t}{\pi} \int_{-1}^1 \sqrt{1-w^2} \cos wtdw,$$

which may be found, e.g., in [1].

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE September 1995	3. REPORT TYPE AND DATES COVERED Contractor Report		
4. TITLE AND SUBTITLE On High-Order Radiation Boundary Conditions		5. FUNDING NUMBERS WU-505-90-5K NCC3-370		
6. AUTHOR(S) Thomas Hagstrom				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Institute for Computational Mechanics in Propulsion 22800 Cedar Point Road Cleveland, Ohio 44142		8. PERFORMING ORGANIZATION REPORT NUMBER E-9918		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135-3191		10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA CR-198404 ICOMP-95-15		
11. SUPPLEMENTARY NOTES ICOMP Program Director, Louis A. Povinelli, organization code 2600, (216) 433-5818.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Categories 34 and 64 This publication is available from the NASA Center for Aerospace Information, (301) 621-0390.			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) In this paper we develop the theory of high-order radiation boundary conditions for wave propagation problems. In particular, we study the convergence of sequences of time-local approximate conditions to the exact boundary condition, and subsequently estimate the error in the solutions obtained using these approximations. We show that for finite times the Padé approximants proposed by Engquist and Majda lead to exponential convergence if the solution is smooth, but that good long-time error estimates cannot hold for spatially local conditions. Applications in fluid dynamics are also discussed.				
14. SUBJECT TERMS Radiation boundary conditions; Integral equations; Hyperbolic systems			15. NUMBER OF PAGES 23	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT	