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T. Hagstrom
Institute for Computational Mechanics in Propulsion
Cleveland, Ohio

and the University of New Mexico
Albuquerque, New Mexico

S.I. Hariharan
Institute for Computational Mechanics in Propulsion
Cleveland, Ohio

and The University of Akron
Akron, Ohio

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PROGRESSIVE WAVE EXPANSIONS AND OPEN BOUNDARY PROBLEMS

T HAGSTROM* AND S I HARIHARAN†

Abstract. In this paper we construct progressive wave expansions and asymptotic boundary conditions for wave-like equations in exterior domains, including applications to electromagnetics, compressible flows and aero-acoustics The development of the conditions will be discussed in two parts The first part will include derivations of asymptotic conditions based on the well-known progressive wave expansions for the two-dimensional wave equations A key feature in the derivations is that the resulting family of boundary conditions involve a single derivative in the direction normal to the open boundary These conditions are easy to implement and an application in electromagnetics will be presented The second part of the paper will discuss the theory for hyperbolic systems in two dimensions Here, the focus will be to obtain the expansions in a general way and to use them to derive a class of boundary conditions that involve only time derivatives or time and tangential derivatives Maxwell's equations and the compressible Euler equations are used as examples Simulations with the linearized Euler equations are presented to validate the theory

Key words. Progressive wave expansions, boundary conditions, Maxwell's Equations, Euler Equations, Numerical Simulations

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1. Introduction. Exterior problems are commonly posed for wave-like equations, and their numerical solution leads to the problem of open boundary conditions. We discuss both isotropic and nonisotropic cases as they arise in electromagnetics and fluid dynamics. These equations include first order hyperbolic systems such as Maxwell's equations, the Euler equations of compressible flows, or the linearized Euler equations, as well as second order reduced forms as appropriate Many work studies of this problem have appeared in the recent literature and we won't try to list them all There are fundamentally two different, though related, approaches that have usually been taken One is the use of high frequency asymptotics such as the geometrical optics approximation. The other is based on the far field structure of the solution. (For a third approach based on the direct approximation of the exact condition, see [5].) Progressive wave expansions were used as a tool to construct far field boundary conditions as early as the time of Sommerfeld In the modern computational point of view, they were put in use for the first time by Kriegsmann and Morawetz [8] Since

* DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF NEW MEXICO, ALBUQUERQUE NM 87131, SUPPORTED IN PART BY ICOMP, NASA LEWIS RESEARCH CENTER, CLEVELAND, OH AND NSF GRANT NO DMS-9304406

† DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF AKRON AKRON, OH 44325-4002, SUPPORTED IN PART BY A NASA COOPERATIVE AGREEMENT NCC - 283 AND BY ICOMP, NASA LEWIS RESEARCH CENTER, CLEVELAND, OH

then there have been many variations to this approach. For example, extensions to anisotropic propagation were first attempted by Bayliss and Turkel [2], and a generalization to the case of anisotropic wave equations in two and three dimensions was proved by the authors [7] Issues include the construction of the expansion for general systems, their use to construct stable boundary conditions of minimum order, and, finally, their practical implementation. Higher order conditions no matter which approach is used, are typically more complicated than the partial differential equation one starts with In particular, they tend to have higher order derivatives in the direction of the propagation. To avoid this problem, often the partial differential equation itself is used. Such a procedure is not known in general for problems governed by first order hyperbolic systems. Here we provide a systematic way of dealing with this issue using progressive wave expansions. Our attention focuses on first order systems, namely, Maxwell's equations and the linearized Euler equations To motivate the central ideas, we first consider the second order wave equation with the emphasis on progressive wave solutions.

2. Second order wave equation. As mentioned above, the goal here is to treat the problem of boundary conditions without having higher order normal derivatives. To illustrate the underlying procedure, let us consider the problem governed by the wave equation in two dimensions. We wish to construct the progressive wave solutions to this equation and exploit their structure to prescribe asymptotic boundary conditions. The equation written in cylindrical coordinates takes the form

$$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \tag{2.1}$$

We look for solutions that are periodic in the angular direction as follows

$$u(r,\theta,t) = \sum_{n=0}^{\infty} v_n(r,t)a_n(\theta) \tag{2.2}$$

where $a_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$. Substituting (2.2) in (2.1), we obtain

$$v_{n,tt} = v_{n,rr} + \frac{1}{r}v_{n,r} - \frac{n^2}{r^2}v_n \tag{2.3}$$

Following Friedlander [4], we construct solutions of (2.3) in the form

$$v_n(r,t) = \sum_{j=0}^{\infty} \frac{f_j^n(t-r)}{r^{j+\frac{1}{2}}} \tag{2.4}$$

Substitution of (2.4) in (2 3) results in the following recurrence relations

$$f_{j+1}^{n'}(t-r) = -\frac{(j+\frac{1}{2})^2 - n^2}{2(j+1)} f_j^n(t-r) \tag{2.5}$$

The goal is to examine the effect of the recurrence relations on constructions of asymptotic boundary conditions. First, we observe that substitution of (2.4) in (2.2) yields the following formal representation of the solution

$$u(r,\theta,t) = \sum_{n=0}^{\infty} a_n(\theta) \sum_{j=0}^{\infty} \frac{f_j^n(t-r)}{r^{j+\frac{1}{2}}} \tag{2.6}$$

Manipulations of this series, particularly increasing the order of the decay rate for boundary conditions, have been proposed by many authors (e.g. Bayliss and Turkel [1]). In fact, a different form of (2 6) has been used for these constructions, which will not be discussed here. We define a "basic boundary operator" from (2.6) as follows:

$$B = \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r} \tag{2.7}$$

It is immediately verified from (2 6) that

$$Bu = -\sum_{n=0}^{\infty} a_n(\theta) \sum_{j=1}^{\infty} \frac{j f_j^n(t-r)}{r^{j+\frac{3}{2}}} \tag{2.8}$$

Direct approximation of (2 8) is the radiation condition $Bu = 0$, a popular condition in the literature noted by many researchers (e g. Bayliss and Turkel [1], Engquist and Majda [3]). Asymptotic accuracy of such a condition is $0(r^{-5/2})$, which is evident from (2.8). Higher order conditions in general require higher order normal derivatives or derivatives in the dominant direction of propagation This may not be a desirable feature numerically, particularly for nonlinear generalizations. Here we obtain higher order conditions that involve Bu, u, $u_{\theta\theta}$, and their time derivatives on the artificial boundary. We begin with the construction of higher order conditions by differentiating (2 8). This yields:

$$\frac{\partial}{\partial t} Bu = -\sum_{n=0}^{\infty} a_n(\theta) \sum_{j=0}^{\infty} \frac{j f_j^{n'}(t-r)}{r^{j+\frac{3}{2}}} \tag{2.9}$$

Noting that the inner summation may be written in the form

$$\sum_{j=0}^{\infty} \frac{(j+1) f_{j+1}^{n'}(t-r)}{r^{j+\frac{5}{2}}}$$

and using the recurrence relation (2 5) yields:

$$\frac{\partial}{\partial t} Bu = \sum_{n=0}^{\infty} a_n(\theta) \sum_{j=0}^{\infty} \frac{(j+\frac{1}{2})^2 - n^2}{2r^{j+\frac{5}{2}}} f_j^n$$

A simple manipulation of the right hand side yields:

$$\frac{\partial}{\partial t}Bu - \frac{1}{8r^2}u - \frac{1}{2r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{2}\sum_{n=0}^{\infty} a_n(\theta)\sum_{j=1}^{\infty}\frac{j(j+1)}{r^{j+\frac{5}{2}}}f_j^n \tag{2.10}$$

The highlight here is the observation

$$u_{\theta\theta} = -\sum_{n=0}^{\infty} n^2 a_n(\theta)\sum_{j=0}^{\infty}\frac{f_j^n(t-r)}{r^{j+\frac{1}{2}}} \tag{2.11}$$

Note that the asymptotic accuracy of the candidate boundary condition (2 10) is increased further to $0(r^{-\frac{7}{2}})$ Let

$$B_1 u = \frac{\partial}{\partial t}Bu - \frac{u}{8r^2} - \frac{u_{\theta\theta}}{r^2} \tag{2.12}$$

Then (2 10) becomes

$$B_1 u = \frac{1}{2}\sum_{n=0}^{\infty} a_n(\theta)\sum_{j=0}^{\infty}\frac{(j+1)(j+2)}{r^{j+\frac{7}{2}}}f_{j+1}^n \tag{2.13}$$

This form again suggests the use of the recurrence relations (2 5) by differentiating the equation with respect to time Doing so, we obtain

$$\frac{\partial}{\partial t}B_1 u = -\frac{1}{4}\sum_{n=0}^{\infty} a_n(\theta)\sum_{j=0}^{\infty}\frac{(j+2)[(j+\frac{1}{2})^2-n^2]}{r^{j+\frac{7}{2}}}f_j^n \tag{2 14}$$

which is equivalent to

$$\frac{\partial}{\partial t}B_1 u + \frac{1}{8r^3}u + \frac{1}{2r^3}u_{\theta\theta} = -\frac{1}{4}\sum_{n=0}^{\infty} a_n(\theta)\sum_{j=1}^{\infty}\frac{j[(j+\frac{3}{2})^2-n^2]}{r^{j+\frac{7}{2}}}f_j^n \tag{2.15}$$

We note that $-n^2$ translates into the second tangential derivative Defining

$$B_2 u = \frac{\partial}{\partial t}B_1 u + \frac{1}{8r^3}u + \frac{1}{2r^3}u_{\theta\theta} \tag{2.16}$$

it is clear that equation (2.15) yields a one asymptotic order higher boundary condition (to $O(r^{-\frac{9}{2}})$). Moreover, noting

$$B_2 u = -\frac{1}{4}\sum_{n=0}^{\infty} a_n(\theta)\sum_{j=0}^{\infty}\frac{(j+1)[(j+\frac{5}{2})^2-n^2]}{r^{j+\frac{9}{2}}}f_{j+1}^n \tag{2.17}$$

and applying the time derivative again, the process becomes clear and it yields

$$B_3 u := \frac{\partial}{\partial t}B_2 u - \frac{25}{128r^4}u - \frac{13}{16r^4}u_{\theta\theta} - \frac{1}{8r^4}u_{\theta\theta\theta\theta} = O(\frac{1}{r^{\frac{11}{2}}}) \tag{2 18}$$

Remark 1: As far as numerical implementation of these conditions are concerned, one may consider a sequence of operations to update u at the current time:

$$u_t + u_r + \frac{1}{2r}u = z \tag{2.19}$$

$$z_t - \frac{1}{8r^2}u + \frac{1}{r^2}u_{\theta\theta} = v \tag{2.20}$$

$$v_t + \frac{1}{8r^3}u + \frac{1}{2r^3}u_{\theta\theta} = w \tag{2.21}$$

$$w_t - \frac{25}{128r^4}u - \frac{13}{16r^4}u_{\theta\theta} - \frac{1}{8r^4}u_{\theta\theta\theta\theta} = 0 \tag{2.22}$$

The above sequence of equations (which provides a boundary condition asymptotically accurate to $O(r^{-\frac{11}{2}})$) as a system of first order equations to march in time. (At the time this article was written one of the students of the second author has implemented such a procedure and obtained the indicated asymptotic improvement. The details will appear elsewhere).
Remark 2: The procedure above coincides with the high frequency approximations of the exact condition in the radially symmetric case In the Laplace transform domain, the exact operator has the form (see [6])

$$B_e u = -\frac{1}{r}\frac{sr K_0'(sr)}{K_0(sr)}. \tag{2.23}$$

Where $K_0(z)$ is the modified Bessel function of order 0. Moreover we find as $sr \to \infty$

$$B_e u = \frac{1}{r}(sr + \frac{1}{2} - \frac{1}{8sr} + \frac{1}{8(sr)^2} + O((sr)^{-3})) \tag{2.24}$$

The Laplace transform of the derived operators coincides with the large sr approximations of the exact boundary operator B_e We can, then, interpret the expansions both as a long-range and as a high-frequency approximation.

We also note that the Fourier transform of the operator $B_1 u$ coincides with the second order operator proposed by Kriegsmann et al. [9] in conjunction with on surface radiation conditions. As an example we consider the computation of the surface current calculation in electromagnetic scattering. Let Γ be the boundary of a perfect conductor Then the magnitude of the total current is given by the formula (see [9])

$$J = |\frac{i}{k}\frac{\partial}{\partial n}(u_s + u_{inc})|_\Gamma \tag{2.25}$$

where u_s is the scattered field, and u_{inc} is the known incident field For perfect conductors $u_s = -u_{inc}$ on the boundary of the scatterer Γ The principle of the on surface boundary procedure consists of bringing the far field boundary exactly on the interface of the scatterer The advantage is rather clear. Since the total current is a functional of the normal derivative of the scattered field and the radiation boundary operators on the surface directly express the normal derivatives in terms of the incident field We note that in the formula for the surface current k is the wave number which arose from the Fourier transform of the wave equation We list the Fourier transform of the operators derived in our theory They are:
Condition 1

$$-ik u + u_r + \frac{1}{2r}u = 0, \tag{2 26}$$

Condition 2.

$$-ik(-iku + u_r + \frac{1}{2r}u) = \frac{1}{8r^2} + \frac{1}{2r^2}u_{\theta\theta}, \tag{2 27}$$

Condition 3.

$$(-ik)^2(-iku + u_r + \frac{1}{2r}u) = -ik(\frac{1}{8r^2}u + \frac{1}{2r^2}u_{\theta\theta}) - \frac{1}{8r^3}u - \frac{1}{2r^3}u_{\theta\theta} \tag{2.28}$$

The first two operators are used in [9], and the third one is, so far as we know, new A plane wave incident upon a unit cylinder is considered for the calculation of J and results are shown in Figures 2.1 ($k = 5$) and 2 2 ($k = 2$) respectively. The incident field has the specific form $u_{inc} = e^{ikr\cos\theta}$

Remark 3.: For anisotropic equations, such as convective wave equation, an analogous procedure may be derived. The use of the resulting conditions are more pertinent to systems of equations such as the linearized Euler equation This is discussed in section 4.

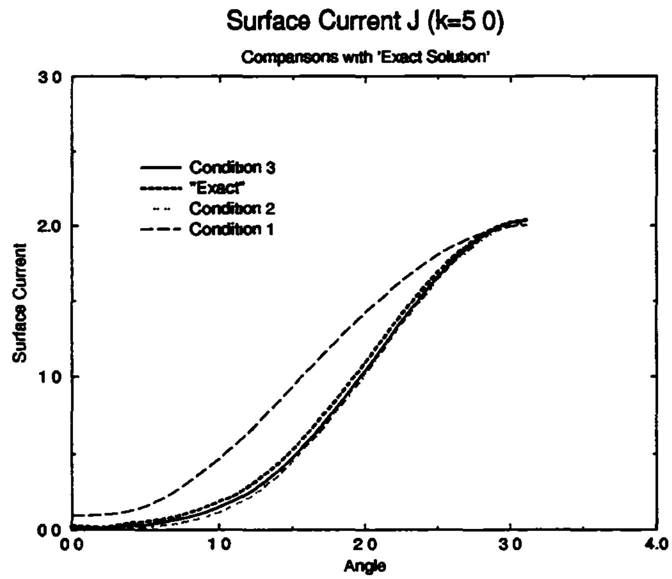

FIG 2 1 *Comparison of results with exact solution, $k = 5$*

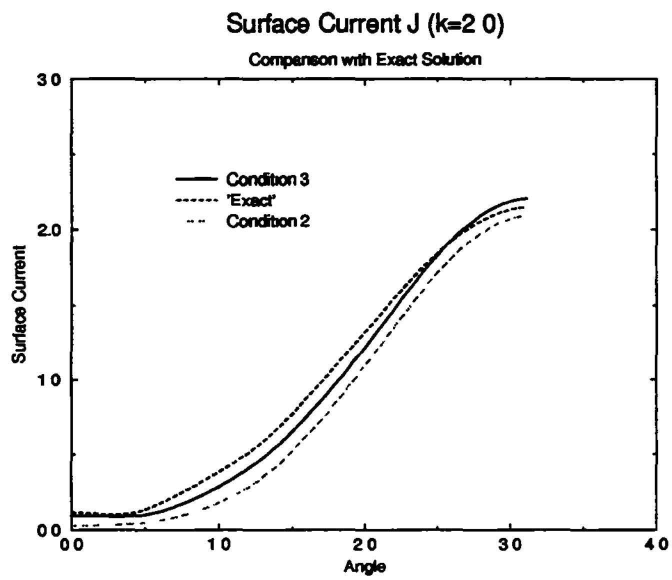

FIG 2 2 *Comparison of results with exact solution, $k = 2$*

3. First order hyperbolic systems - Isotropic case. Here our focus is to extend the ideas to systems of first order equations The progressive wave expansions may be carried out directly in the time domain as we did for the second order wave equations or in the Laplace transform domain. In this section we present the construction using the Laplace transform. The direct approach is illustrated in Section 4.

Maxwell's equations offer an interesting example of an isotropic system Here we confine our attention to Transverse Magnetic (TM) fields for simplicity. The full field equations are:

$$div\ \epsilon\mathbf{E} = div\ \mu\mathbf{H} = 0, \tag{3 1}$$

$$curl\ \mathbf{H} = \epsilon\frac{\partial\mathbf{E}}{\partial t}, \tag{3.2}$$

$$curl\ \mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \tag{3.3}$$

We shall consider TM fields as follows:

$$\mathbf{E} = E(x,y,t)\ \mathbf{k} \tag{3 4}$$

$$\mathbf{H} = H_1(x,y,t)\ \mathbf{i} + H_2(x,y,t)\ \mathbf{j}. \tag{3 5}$$

Equation (3.4) indicates that the electric field propagates in the direction perpendicular to the $x - y$ plane and is transverse to the magnetic field Under these assumptions, equations (3.2) and (3.3) become

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = \epsilon\frac{\partial E}{\partial t} \tag{3 6}$$

and

$$\frac{\partial E}{\partial y}\mathbf{i} - \frac{\partial E}{\partial x}\mathbf{j} = -\mu\left(\frac{\partial H_1}{\partial t}\mathbf{i} + \frac{\partial H_2}{\partial t}\mathbf{j}\right) \tag{3.7}$$

respectively Rearranging equations (3 6) and (3 7), we obtain the following system

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right]$$

$$\frac{\partial H_1}{\partial t} = -\frac{1}{\mu}\frac{\partial E}{\partial y}$$

$$\frac{\partial H_2}{\partial t} = \frac{1}{\mu}\frac{\partial E}{\partial x}$$

This can be put in the conventional form:

$$\mathbf{u}_t = \mathbf{A}\ \mathbf{u}_x + \mathbf{B}\ \mathbf{u}_y \tag{3.8}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \frac{1}{\epsilon} \\ 0 & 0 & 0 \\ \frac{1}{\mu} & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & -\frac{1}{\epsilon} & 0 \\ -\frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where $\mathbf{u} = (E, H_1, H_2)^T$ Converting to polar coordinates we obtain.

$$\frac{\partial}{\partial t}\begin{pmatrix} E \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\sin\theta & \frac{1}{\epsilon}\cos\theta \\ -\frac{1}{\mu}\sin\theta & 0 & 0 \\ \frac{1}{\mu}\cos\theta & 0 & 0 \end{pmatrix}\frac{\partial}{\partial r}\begin{pmatrix} E \\ H_1 \\ H_2 \end{pmatrix}$$

$$+\frac{1}{r}\begin{pmatrix} 0 & -\frac{1}{\epsilon}\cos\theta & -\frac{1}{\epsilon}\sin\theta \\ -\frac{1}{\mu}\cos\theta & 0 & 0 \\ -\frac{1}{\mu}\sin\theta & 0 & 0 \end{pmatrix}\frac{\partial}{\partial\theta}\begin{pmatrix} E \\ H_1 \\ H_2 \end{pmatrix} \tag{3.9}$$

We take the Laplace transform of (3.9). With the change of variable $\tilde{r} = rs$ we have

$$\begin{pmatrix} \hat{E} \\ \hat{H}_1 \\ \hat{H}_2 \end{pmatrix} = \left(R\frac{\partial}{\partial\tilde{r}} + \frac{1}{\tilde{r}}\Theta\frac{\partial}{\partial\theta} \right)\begin{pmatrix} \hat{E} \\ \hat{H}_1 \\ \hat{H}_2 \end{pmatrix} \tag{3.10}$$

where

$$R = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\sin\theta & \frac{1}{\epsilon}\cos\theta \\ -\frac{1}{\mu}\sin\theta & 0 & 0 \\ \frac{1}{\mu}\cos\theta & 0 & 0 \end{pmatrix}$$

and

$$\Theta = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\cos\theta & -\frac{1}{\epsilon}\sin\theta \\ -\frac{1}{\mu}\cos\theta & 0 & 0 \\ -\frac{1}{\mu}\sin\theta & 0 & 0 \end{pmatrix}$$

We seek an expansion of solutions of (3 10) in the form

$$\begin{pmatrix} \hat{E} \\ \hat{H}_1 \\ \hat{H}_2 \end{pmatrix} = \frac{e^{-\tilde{r}g(\theta)}}{\tilde{r}^{\alpha}}\left(\mathbf{a} + \frac{1}{\tilde{r}}\mathbf{b} + \cdots \right) \tag{3.11}$$

We note that this form is similar to Friedlander's form that applies to the second order wave equation in the Laplace transform domain Also, we have introduced a decay rate constant α which turns out in two dimensions to equal $\frac{1}{2}$, as expected Substituting (3.11) into (3.10) yields to leading order:

$$A = I + gR + g'\Theta \tag{3 12}$$

and

$$A\mathbf{a} = \mathbf{0} \tag{3 13}$$

For this requirement to be true, clearly it must follow that·

$$0 = det(A) = det \begin{pmatrix} 1 & -\frac{1}{\epsilon}(g\sin\theta + g'\cos\theta) & \frac{1}{\epsilon}(g\cos\theta - g'\sin\theta) \\ -\frac{1}{\mu}(g\sin\theta + g'\cos\theta) & 1 & 0 \\ \frac{1}{\mu}(g\cos\theta - g'\sin\theta) & 0 & 1 \end{pmatrix}$$

$$= (1 - \frac{1}{\epsilon\mu}((g\sin\theta + g'\cos\theta)^2 + (g\cos\theta - g'\sin\theta)^2))$$

$$= 1 - \frac{1}{\epsilon\mu}(g^2 + (g')^2) \tag{3.14}$$

The roots of equation (3.14) are:

$$g = \pm\sqrt{\epsilon\mu}, \sqrt{\epsilon\mu}\cos(\theta + \phi),$$

ϕ arbitrary. For waves propagating to infinity in all directions we choose $g = \sqrt{\epsilon\mu}$ as the allowoable root With this value of g, the matrix A becomes

$$A = \begin{pmatrix} 1 & -\sqrt{\frac{\mu}{\epsilon}}\sin\theta & \sqrt{\frac{\mu}{\epsilon}}\cos\theta \\ -\sqrt{\frac{\epsilon}{\mu}}\sin\theta & 1 & 0 \\ \sqrt{\frac{\epsilon}{\mu}}\cos\theta & 0 & 1 \end{pmatrix}$$

whose right nullvector is

$$\mathbf{a} = \begin{pmatrix} 1 \\ \sqrt{\frac{\epsilon}{\mu}}\sin\theta \\ -\sqrt{\frac{\epsilon}{\mu}}\cos\theta \end{pmatrix} a_1(\theta)$$

with left nullvector

$$\mathbf{l}^T = (1, \sqrt{\frac{\mu}{\epsilon}}\sin\theta, -\sqrt{\frac{\mu}{\epsilon}}\cos\theta)$$

The next order terms in the asymptotic expansion yield the following relation:

$$\text{(3.15)} \qquad A\mathbf{b} = -\alpha R\mathbf{a} + \Theta\frac{\partial \mathbf{a}}{\partial\theta},$$

and α is determined by:

$$\text{(3.16)} \qquad \alpha = \frac{\mathbf{1}^T\Theta\frac{\partial \mathbf{a}}{\partial\theta}}{\mathbf{1}^T R\mathbf{a}},$$

i e., by requiring $(\mathbf{1}, A\mathbf{b}) = 0$

Noting the following calculations:

$$R\mathbf{a} = a_1(\theta)\begin{pmatrix} -\frac{1}{\sqrt{\epsilon\mu}} \\ -\frac{1}{\mu}\sin\theta \\ \frac{1}{\mu}\cos\theta \end{pmatrix},$$

$$\frac{\partial \mathbf{a}}{\partial\theta} = \frac{\partial a_1}{\partial\theta}\begin{pmatrix} 1 \\ \sqrt{\frac{\epsilon}{\mu}}\sin\theta \\ -\sqrt{\frac{\epsilon}{\mu}}\cos\theta \end{pmatrix} + a_1\begin{pmatrix} 0 \\ \sqrt{\frac{\epsilon}{\mu}}\cos\theta \\ \sqrt{\frac{\epsilon}{\mu}}\sin\theta \end{pmatrix},$$

$$\Theta\frac{\partial \mathbf{a}}{\partial\theta} = \frac{\partial a_1}{\partial\theta}\begin{pmatrix} 0 \\ -\frac{1}{\mu}\cos\theta \\ -\frac{1}{\mu}\sin\theta \end{pmatrix} + a_1\begin{pmatrix} -\frac{1}{\sqrt{\epsilon\mu}} \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{1}^T R\mathbf{a} = a_1(\theta)(\frac{-2}{\sqrt{\epsilon\mu}}), \quad \mathbf{1}^T\Theta\frac{\partial \mathbf{a}}{\partial\theta} = a_1(\theta)(\frac{-1}{\sqrt{\epsilon\mu}}),$$

it follows that

$$\alpha = \frac{1}{2}$$

Choose

$$\mathbf{b} = \begin{pmatrix} 0 \\ b_2 \\ b_3 \end{pmatrix},$$

and use the last two equations to obtain

$$b_2 = (\frac{1}{2\mu}\sin\theta)a_1 - \frac{1}{\mu}\cos\theta\frac{\partial a_1}{\partial\theta},$$

$$b_3 = (-\frac{1}{2\mu}\cos\theta)a_1 - \frac{1}{\mu}\sin\theta\frac{\partial a_1}{\partial\theta}.$$

Substituting **a** and **b** in (3.11) we obtain

$$\hat{E} = \frac{e^{-\tilde{r}\sqrt{\mu\epsilon}}}{\tilde{r}^{\frac{1}{2}}}a_1, \quad \frac{\partial\hat{E}}{\partial\theta} = \frac{e^{-\tilde{r}\sqrt{\mu\epsilon}}}{\tilde{r}^{\frac{1}{2}}}\frac{\partial a_1}{\partial\theta},$$

$$\hat{H}_1 = \sqrt{\frac{\epsilon}{\mu}}\sin\theta\hat{E} + \frac{1}{\mu r s}(\frac{\sin\theta}{2}\hat{E} - \cos\theta\frac{\partial\hat{E}}{\partial\theta}),$$

$$\hat{H}_2 = -\sqrt{\frac{\epsilon}{\mu}}\cos\theta\hat{E} + \frac{1}{\mu r s}(\frac{-\cos\theta}{2}\hat{E} - \sin\theta\frac{\partial\hat{E}}{\partial\theta}).$$

Multiplying through by s and taking the inverse transform finally yields

$$\frac{\partial H_1}{\partial t} = (\sqrt{\frac{\epsilon}{\mu}}\sin\theta\frac{\partial}{\partial t} + \frac{\sin\theta}{2\mu r} - \frac{\cos\theta}{\mu r}\frac{\partial}{\partial\theta})E, \tag{3.17}$$

$$\frac{\partial H_2}{\partial t} = -(\sqrt{\frac{\epsilon}{\mu}}\cos\theta\frac{\partial}{\partial t} + \frac{\cos\theta}{2\mu r} + \frac{\sin\theta}{\mu r}\frac{\partial}{\partial\theta})E \tag{3.18}$$

As only one boundary condition is required, we convert these into a single condition Multiplying (3.17) $\sin\theta$ and subtracting (3.18) multiplied by $\cos\theta$, we obtain our final form·

$$\frac{\partial H_1}{\partial t}\sin\theta - \frac{\partial H_2}{\partial t}\cos\theta - \sqrt{\frac{\epsilon}{\mu}}\frac{\partial E}{\partial t} = \frac{1}{2\mu r}E \tag{3.19}$$

This construction is easily extended to higher order, though we have not devised a unified approach to the implementation of the higher order conditons.

4. The linearized Euler equations - An anisotropic example. The construction of asymptotic boundary conditions for the linearized and nonlinear compressible Euler equations is also of interest, particularly for applications in aeroacoustics. In this section, we construct the expansions in the time domain directly. Again, the system takes the form:

$$\mathbf{u}_t = \mathbf{A}\ \mathbf{u}_x + \mathbf{B}\ \mathbf{u}_y \tag{4.1}$$

where **A** and **B** are constant matrices In cylindrical coordinates we have

$$\mathbf{u}_t = \mathbf{R}\ \mathbf{u}_r + \frac{1}{r}\mathbf{T}\ \mathbf{u}_\theta, \tag{4.2}$$

where

$$\mathbf{R} = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta, \tag{4.3}$$

$$\mathbf{T} = -\mathbf{A}\sin\theta + \mathbf{B}\cos\theta \tag{4.4}$$

A far field asymptotic expansion may be constructed in the following form for the solution vector $\mathbf{u}$:

$$\mathbf{u} = \sum_{n=0}^{\infty} \frac{f_n(t - rg(\theta))}{r^{n+\frac{1}{2}}}\mathbf{a}_n, \tag{4.5}$$

where the scalar function $g(\theta)$ and the vectors $\mathbf{a}_n(\theta)$ are to be determined The function f_0 is analogous to the radiation function discussed in [4] The other functions are recursively determined by substitution of the expansion into equation (4.2). The $O(\frac{1}{r})$ terms yield:

$$\mathbf{C}\mathbf{a}_0 = \mathbf{0}, \tag{4.6}$$

where $\mathbf{C} = \mathbf{I} + g(\theta)\mathbf{R} + g'(\theta)\mathbf{T}$ For $\mathbf{a}_0$ to be nontrivial one must have $det(\mathbf{C}) = 0$, yielding an 'eikonal function' $g(\theta)$. The next order correction yields

$$f_0\mathbf{C}\mathbf{a}_1 = -f_1'\frac{1}{2}\mathbf{R}\mathbf{a}_0 + \mathbf{T}\frac{\partial \mathbf{a}_0}{\partial\theta} \tag{4.7}$$

This imposes a necessary restriction that $f_0 = f_1'$ In general, it follows $f_{n-1} = f_n', n \geq 1$. At this point, we turn to the isentropic, linearized, compressible, Euler equations to illustrate the actual calculations involved in solving these algebraic problems For a uniform base flow in the x direction they are:

$$(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x})p + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.8}$$

$$(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x})u + \frac{\partial p}{\partial x} = 0, \tag{4 9}$$

$$(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x})v + \frac{\partial p}{\partial y} = 0 \tag{4 10}$$

Conversion of this system to cylindrical coordinates (4 8)-(4.10) takes the form (4 2) where

$$\mathbf{u} = \begin{pmatrix} p \\ u \\ v \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} M\cos\theta & \cos\theta & \sin\theta \\ \cos\theta & M\cos\theta & 0 \\ \sin\theta & 0 & M\cos\theta \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} -M\sin\theta & -\sin\theta & \cos\theta \\ -\sin\theta & -M\sin\theta & 0 \\ \cos\theta & 0 & -M\sin\theta \end{pmatrix}$$

Calculation of $g(\theta)$ for these equations yield.

$$g(\theta) = \frac{1}{\sqrt{1 - M^2\sin^2\theta} + M\cos\theta}, \tag{4.11}$$

and the matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} 1 - MQ & -Q & -R \\ -Q & 1 - MQ & 0 \\ -R & 0 & 1 - MQ \end{bmatrix},$$

where

$$Q = g\cos\theta - g'\sin\theta,$$

$$R = g\sin\theta + g'\cos\theta$$

Solutions of (4.6) are given by

$$\mathbf{a}_0 = h_0(\theta) \begin{pmatrix} 1 \\ \frac{Q}{1-MQ} \\ \frac{R}{1-MQ} \end{pmatrix} = h_0(\theta) \begin{pmatrix} 1 \\ r_2 \\ r_3 \end{pmatrix},$$

and the solutions of (4 7) are given by:

$$\mathbf{a}_1 = h_1(\theta) \begin{pmatrix} 1 \\ r_2 \\ r_3 \end{pmatrix} + h_0(\theta) \begin{pmatrix} 0 \\ b_2 \\ b_3 \end{pmatrix} + h_0'(\theta) \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix} \tag{4.12}$$

Here $h_0(\theta)$ and $h_1(\theta)$ are arbitrary functions of θ and the coefficients b_i and c_i are given by

$$b_2 = \frac{(\cos\theta + Mr_2\cos\theta)/2 + Mr_2'\sin\theta}{1 - MQ},$$

$$b_3 = \frac{(\sin\theta + Mr_3\cos\theta)/2 + Mr_3'\sin\theta}{1 - MQ},$$

$$c_2 = \frac{\sin\theta + M r_2 \sin\theta}{1 - MQ},$$

$$c_3 = \frac{-\cos\theta + M r_3 \sin\theta}{1 - MQ}.$$

Collection of these results in the asymptotic expansions yields (to $O(r^{-5/2})$):

$$p = \frac{h_0 f_0}{r^{1/2}} + \frac{h_1 f_1}{r^{3/2}}, \tag{4.13}$$

$$u = r_2 p + \frac{h_0 f_1}{r^{3/2}} b_2 + \frac{h_0' f_1}{r^{3/2}} c_2, \tag{4.14}$$

$$v = r_3 p + \frac{h_0 f_1}{r^{3/2}} b_3 + \frac{h_0' f_1}{r^{3/2}} c_3 \tag{4.15}$$

Differentiating u and v with respect to t and using the result $f_0 = f_1'$ to $O(r^{-5/2})$, we have

$$u_t = r_2 p_t + \frac{p}{r} b_2 + \frac{h_0' f_0}{r^{3/2}} c_2, \tag{4 16}$$

$$v_t = r_3 p_t + \frac{p}{r} r_3 + \frac{h_0' f_0}{r^{3/2}} c_3 \tag{4 17}$$

Finally, noting the term involving h_0' can be eliminated from the last two equations, we have

$$\alpha\, p_t + \beta\, u_t + \gamma\, v_t = \frac{p}{r}\delta, \tag{4.18}$$

where

$$\alpha = r_2 c_3 - r_3 c_2,$$

$$\beta = -c_2,$$

$$\gamma = c_3,$$

$$\delta = b_3 c_2 - b_2 c_3$$

The higher order conditions are obtained in a similar manner In fact, one can show that the next order condition is of the form

$$(\alpha\, p + \beta\, u + \gamma\, v)_{tt} = \frac{p_t}{r}\delta + \frac{p}{r^2}\epsilon, \tag{4.19}$$

which is accurate to $O(r^{-7/2})$. Note that these conditions do not involve any spatial derivatives As such they are ideal for rectangular domains where typically one has to pay special attention to the corners, particularly when high order numerical schemes are used These conditions correspond to the primary acoustic boundary condition. In addition, one must impose at inflow boundaries, a vorticity condition For nonisentropic flows, in addition to the vorticity, entropy must also be specified at inflow At inflow the y momentum equation and zero vorticity condition yield an exact relation

$$v_t + (p + Mu)_y = 0. \qquad (4\ 20)$$

5. A Model Problem. In this section we begin with the linearized Euler equations with mean velocity convection. The scaled form of these equations are identical to the one that we used to derive the conditions, except they contain forcing terms that characterize a driving source They are.

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)p + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (5\ 1)$$

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)u + \frac{\partial p}{\partial x} = g_1(x, y, t), \qquad (5.2)$$

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)v + \frac{\partial p}{\partial y} = g_2(x, y, t), \qquad (5.3)$$

where $g1$ and $g2$ model a Gaussian momentum source, which both oscillates sinusoidally and decays algebraically in time. Typical examples include, a quadrupole sound distribution. Here g_1 and g_2 are the gradient of a potential ϕ Such a function is given by

$$\phi = A(t)e^{-\alpha R^2} cos(2\theta)$$

where $\tan\theta = \frac{y-y_0}{x-x_0}$, $R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, (x_0, y_0) is the location of the source, α is a positive constant, and $A(t)$ is the amplitude and a function of t. (In the numerical experiments $A = \sin 2\pi t/(1+t^2)$)

In a sample computation which was computed for a time length of 100 periods (22415 time steps), the solution obtained with the second order conditions was compared with the exact solution, a solution obtained by setting incoming characteristic variables to zero, and one obtained using the first order condition. The exact solution was computed in a large domain in which, within the time of computations, the waves could not reflect off the artificial boundaries and return to the small domain The maximum error in pressure calculations observed for the characteristic conditions was 10.3%, for the first order asymptotic condition it was 3.3%, and for the second order condition the error was 1.3%; indicating the expected improvement. In Figures 5.1 and 5.2, the exact solution for the pressure is given after 5 periods and 10 periods of time respectively. Subsequent pairs of figures (5 3-5.4, 5 5-5 6 and 5 7-5.8) indicate the solution at these times for the characteristic boundary condition, the first order asymptotic condition, and the second order asymptotic conditions respectively At 10 periods the errors are visible in the first two cases and their orders of the magnitude indeed are as indicated above. Clearly the higher order condition improved the results

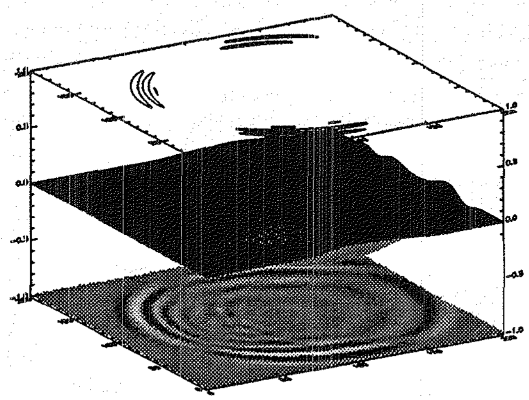

FIG. 5.1. *Exact Solution at 5 periods of time*

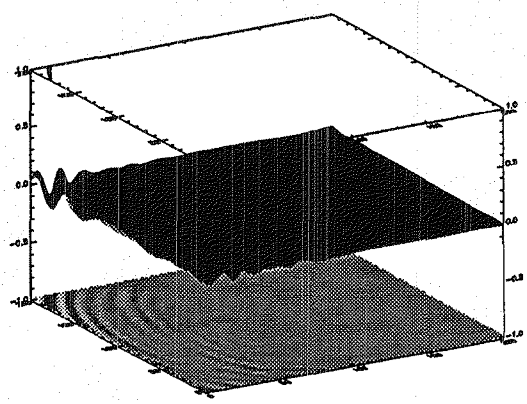

FIG. 5.2. *Exact Solution at 10 periods of time*

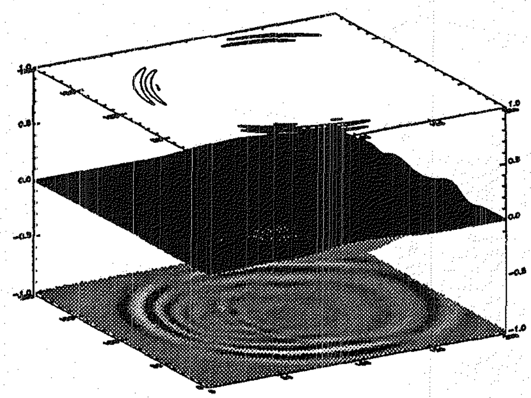

FIG. 5.3. *Solution with characteristics based boundary condition* $t = 5$ *periods*

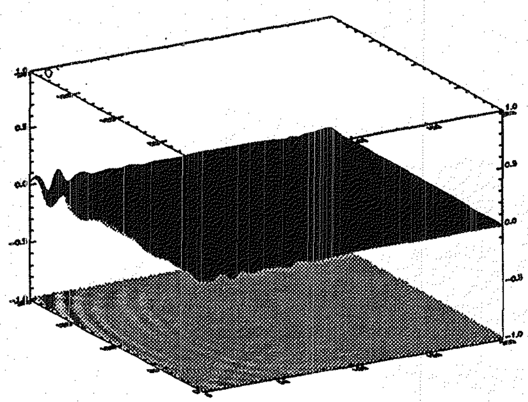

FIG. 5.4. *Solution with characteristics based boundary condition* $t = 10$ *periods*

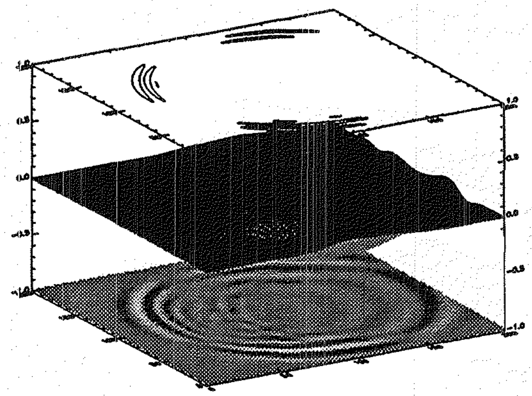

FIG. 5.5. *Solution with first asymptotic boundary condition* $t = 5$ *periods*

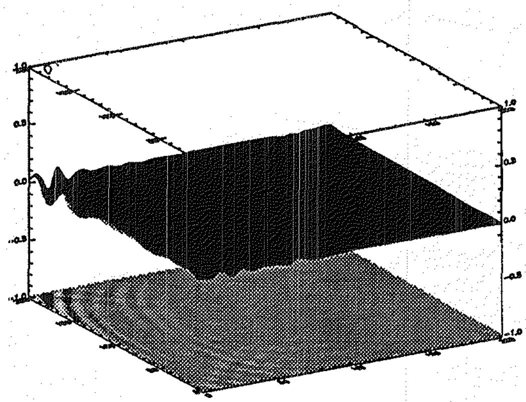

FIG. 5.6. *Solution with first asymptotic boundary condition* $t = 10$ *periods*

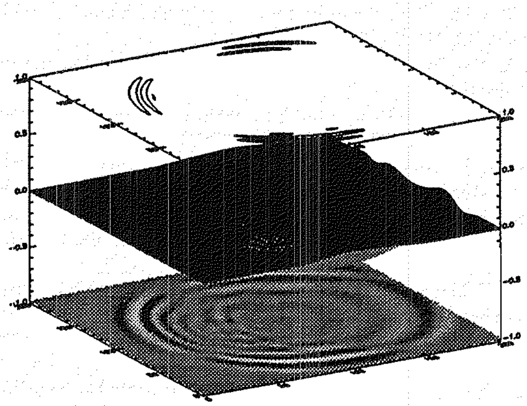

FIG. 5.7. *Solution with second asymptotic boundary condition t = 5 periods*

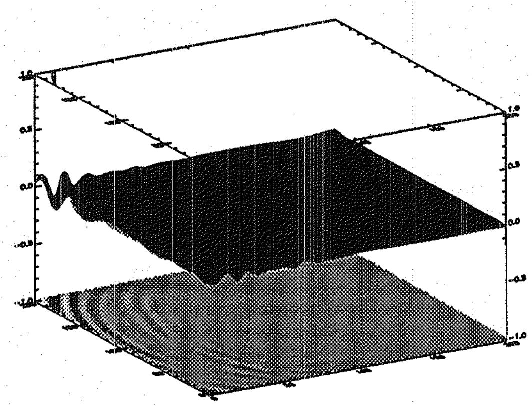

FIG. 5.8. *Solution with second asymptotic boundary condition t = 10 periods*

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T. Hagstrom and S.I. Hariharan

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Institute for Computational Mechanics in Propulsion
22800 Cedar Point Road
Cleveland, Ohio 44142

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13. ABSTRACT *(Maximum 200 words)*

In this paper we construct progressive wave expansions and asymptotic boundary conditions for wave-like equations in exterior domains, including applications to electromagnetics, compressible flows and aero-acoustics. The development of the conditions will be discussed in two parts The first part will include derivations of asymptotic conditions based on the well-known progressive wave expansions for the two-dimensional wave equations A key feature in the derivations is that the resulting family of boundary conditions involve a single derivative in the direction normal to the open boundary These conditions are easy to implement and an application in electromagnetics will be presented. The second part of the paper will discuss the theory for hyperbolic systems in two dimensions Here, the focus will be to obtain the expansions in a general way and to use them to derive a class of boundary conditions that involve only time derivatives or time and tangential derivatives. Maxwell's equations and the compressible Euler equations are used as examples Simulations with the linearized Euler equations are presented to validate the theory

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