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Developments in Computational Modeling of Turbulent Flows

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In this paper, we will discuss some recent model developments in two turbulence closure schemes. One is the Reynolds-stress algebraic equation model and the other is the Reynolds-stress transport equation model. Various model constraints required by the rapid distortion theory, the invariant theory and the realizability principle, etc. will be described in the model development.

1. INTRODUCTION

Ensemble averaged Navier-Stokes equations for incompressible turbulent flows can be written as

$$U_{i,i} = 0 \quad (1)$$

$$U_{i,t} + U_j U_{i,j} = -(\overline{u_i u_j})_{,j} - \frac{1}{\rho} P_{,i} + 2\nu\, S_{ij,j} \quad (2)$$

$$T_{,t} + U_i T_{,i} = -(\overline{\theta u_i})_{,i} + \gamma\, T_{,ii} \quad (3)$$

where "$_{,i}$" and "$_{,t}$" represent the derivatives with respect to coordinates x_i and time t, respectively. U_i and u_i are the mean and fluctuating velocities. T and θ are the mean and fluctuating temperatures. In Eqs. (2) and (3), the unknown terms, $\overline{u_i u_j}$ and $\overline{\theta u_i}$, are one-point (in space and time) turbulent correlations and are commonly referred to as Reynolds stresses and turbulent heat fluxes. In order to use Eqs. (1)-(3) for studying various turbulent flows, we need auxiliary relations (either algebraic or differential) which can express the Reynolds stresses and turbulent heat fluxes in terms of known quantities, say, the mean velocity U_i and the mean temperature T, etc. Unfortunately, auxiliary relations that are valid for all turbulent flows have not been (and probably never will be) found theoretically or experimentally. All that can be done, in this situation, is to propose approximate relations, called turbulence models, for certain turbulent flows using phenomenological analysis and supporting experimental or direct numerical simulation (DNS) data. We fully expect that the developed turbulence models may not be accurate for turbulent flows other than those used to develop the models. Nevertheless, much effort in phenomenological turbulence modeling has been performed to broaden the applicability of these models for generic turbulent flows. Turbulence modeling, from mixing length models [1], two-equation isotropic eddy viscosity models [2-9], high-order algebraic equation models [10-13] to second moment transport equation models [14-22] and many other models [23-24], has made a significant impact on computational fluid dynamics (CFD).

In this paper, some recent developments in turbulence modeling will be discussed. We will first discuss the Reynolds stress algebraic equation models in Section 2. This discussion will encompass a range of models from mixing length scale models, two-equation eddy viscosity models to high-order nonlinear Reynolds stress algebraic equation models. As will be seen, model development along this line can be unified to a general constitutive stress-strain relation. In Section 3, we will discuss recent developments of second moment transport equation models. A set of new models for the rapid-pressure strain and return-to-isotropy terms, based on realizability constraints and the results of a direct numerical simulation of homogeneous turbulence, will be presented. No discussion of near-wall turbulence will be presented. Interested readers may reference the recent papers by Perot and Moin [25, 26] which shed some physical insight on the behavior of near-wall turbulence from their DNS of shear-free turbulent boundary layer flows.

2. REYNOLDS STRESS ALGEBRAIC EQUATION MODELS

In this section, we will discuss Reynolds stress algebraic equation models, which algebraically relate the Reynolds stresses to mean velocity gradients. The simplest model of this type is the mixing length model [1]:

$$\overline{u_i u_j} = -\nu_T \, (U_{i,j} + U_{j,i})$$
$$\nu_T = \ell^2 \, |\frac{\partial U}{\partial y}| \quad \text{or} \quad \ell^2 \, |\omega| \tag{4}$$

where ℓ is the mixing length scale, which is specified from the flow geometry, and ω is the mean vorticity. Equation (4) is easy to use and, in fact, is still very commonly used in CFD calculations. However, this type of model contains little turbulence physics, hence, its applicability is quite limited.

A more general eddy viscosity formulation than Eq. (4) is

$$\overline{u_i u_j} = \frac{2}{3} \, k \, \delta_{ij} - \nu_T \, (U_{i,j} + U_{j,i})$$
$$\nu_T = C_\mu \frac{k^2}{\varepsilon} \tag{5}$$

where the eddy viscosity ν_T is not specified by the flow geometry, but determined by the turbulent kinetic energy k and its dissipation rate ε. The quantities k and ε will be determined by the equations governing the turbulent kinetic energy and its dissipation rate. In this way, a portion of the turbulence physics (contained in the k-ε transport equations) enters the eddy viscosity model. These types of models are commonly called two-(transport)equation eddy viscosity models.

The stress-strain relation in Eq. (5) is linear. However, there is no reason as to why the relation need be linear. In fact, a more general non-linear constitutive relation can be derived mathematically by using the generalized Cayley-Hamilton formulas and the invariant principles [27]. In what follows we will, based on this general constitutive relation, discuss the development of Reynolds stress algebraic equation models.

2.1 Constitutive relation for Reynolds stresses

Let us assume

$$\overline{u_i u_j} = F_{ij}(U_{i,j}, \; k, \; \varepsilon) \tag{6}$$

which indicates that the turbulent stresses depend only on the mean velocity gradient $U_{i,j}$ and the turbulent scales characterized by the turbulent kinetic energy k and its dissipation rate ε. Under this assumption, a general relation can be derived [27]:

$$\begin{aligned}
\overline{u_i u_j} &= \frac{2}{3}k\delta_{ij} + 2a_2\frac{K^2}{\varepsilon}(U_{i,j} + U_{j,i} - \frac{2}{3}U_{k,k}\delta_{ij}) \\
&+ 2a_4\frac{K^3}{\varepsilon^2}(U^2_{i,j} + U^2_{j,i} - \frac{2}{3}\Pi_1\delta_{ij}) \\
&+ 2a_6\frac{K^3}{\varepsilon^2}(U_{i,k}U_{j,k} - \frac{1}{3}\Pi_2\delta_{ij}) \\
&+ 2a_7\frac{K^3}{\varepsilon^2}(U_{k,i}U_{k,j} - \frac{1}{3}\Pi_2\delta_{ij}) \\
&+ 2a_8\frac{K^4}{\varepsilon^3}(U_{i,k}U^2_{j,k} + U^2_{i,k}U_{j,k} - \frac{2}{3}\Pi_3\delta_{ij}) \\
&+ 2a_{10}\frac{K^4}{\varepsilon^3}(U_{k,i}U^2_{k,j} + U_{k,j}U^2_{k,i} - \frac{2}{3}\Pi_3\delta_{ij}) \\
&+ 2a_{12}\frac{K^5}{\varepsilon^4}(U^2_{i,k}U^2_{j,k} - \frac{1}{3}\Pi_4\delta_{ij}) \\
&+ 2a_{13}\frac{K^5}{\varepsilon^4}(U^2_{k,i}U^2_{k,j} - \frac{1}{3}\Pi_4\delta_{ij}) \\
&+ 2a_{14}\frac{K^5}{\varepsilon^4}(U_{i,k}U_{l,k}U^2_{l,j} + U_{j,k}U_{l,k}U^2_{l,i} \\
&\qquad -\frac{2}{3}\Pi_5\delta_{ij}) \\
&+ 2a_{16}\frac{K^6}{\varepsilon^5}(U_{i,k}U^2_{l,k}U^2_{l,j} + U_{j,k}U^2_{l,k}U^2_{l,i} \\
&\qquad -\frac{2}{3}\Pi_6\delta_{ij}) \\
&+ 2a_{18}\frac{K^7}{\varepsilon^6}(U_{i,k}U_{l,k}U^2_{l,m}U^2_{j,m} \\
&\qquad +U_{j,k}U_{l,k}U^2_{l,m}U^2_{i,m} - \frac{2}{3}\Pi_7\delta_{ij})
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\Pi_1 &= U_{i,k}U_{k,i}\ ,\ \Pi_2 = U_{i,k}U_{i,k}\ , \\
\Pi_3 &= U_{i,k}U^2_{i,k}\ ,\ \Pi_4 = U^2_{i,k}U^2_{i,k}\ , \\
\Pi_5 &= U_{i,k}U_{l,k}U^2_{l,i}\ ,\ \Pi_6 = U_{i,k}U^2_{l,k}U^2_{l,i}\ , \\
\Pi_7 &= U_{i,k}U_{l,k}U^2_{l,m}U^2_{i,m} \qquad (8)
\end{aligned}$$

It should be emphasized that Eq. (7) is the most general relation that we can obtain under the assumption of Eq. (6). It is noticed that the first two terms on the right hand side of Eq. (7) represent the standard k-ε eddy viscosity model, and that the first five terms of Eq. (7) are similar to the models derived from both the two-scale DIA approach (Yoshizawa [28]) and the RNG method (Rubinstein and Barton [29]). A similar formulation to Eq. (7), in terms of the strain rate S_{ij} and the rotation rate Ω_{ij}, was also introduced by Pope [10].

Equation (7) contains eleven undetermined coefficients which are, in general, scalar functions of various invariants of the tensors in question, for example, $S_{ij}S_{ij}$ (strain rate) and $\Omega_{ij}\Omega_{ij}$ (rotation rate) which are $(\Pi_2+\Pi_1)/2$ and $(\Pi_2-\Pi_1)/2$ respectively. The detailed forms of these scalar functions need to be determined by other methods, for example, by using Rodi's relation as suggested by Pope [10], Taulbee [11] and Gatski and Speziale [12]. Rodi's relation [30] assumes a constant anisotropy of Reynolds stresses such that the transport equation of the Reynolds stress is reduced to an algebraic equation. The use of Rodi's relation permits some turbulence physics, contained in the Reynolds stress transport equations, to be introduced into the coefficients in Eq. (7). The models developed by these researchers are encouraging. The capability of modeling complex turbulent flows is currently being investigated. On the other hand, if we want to avoid the assumption of constant anisotropy, then we cannot use Rodi's relation. The only alternative is to directly determine these coefficients by using experimental data and other model constraints such as those required by rapid distortion theory and the realizability principle, as suggested by Shih, Zhu and Lumley [13].

In addition, Eq. (7) contains 12 terms; however, in practice, one may not need all of these terms. For example, the quadratic tensorial form is quite appropriate for turbulent shear flows [13]. Launder [31] found that the cubic terms in Eq. (7) are needed for turbulent flows with swirling. Recently, Hirsch and Khodak [32] found that a quadratic form with appropriate coefficients can also predict swirling flows quite well.

2.2 k-ε turbulence modeling

In order to use Eq. (7), we need to know the turbulent kinetic energy k and its dissipation rate ε, or the k and the ratio between k and ε. These two quantities are determined by two transport equations: k-ε equations, k-ω equations, k-τ equations or k-ℓ equations, etc., where $\omega = \varepsilon/k$, $\tau = k/\varepsilon$ and $\ell = k^{\frac{3}{2}}/\varepsilon$. All of these are commonly referred to as two-equation models. What is implied here is that the different two transport equations should not produce significant differences in view of the turbulence physics; however, in practice, the models require varying amounts of difficulty in specifying boundary conditions for different turbulent flows. In this paper, we only discuss the k-ε model equations.

2.2.1 Standard k-ε eddy viscosity model

Launder and Spalding [2] proposed the following k-ε equations:

$$k_{,t} + U_i k_{,i} = [(\nu + \frac{\nu_T}{\sigma_k})k_{,i}]_{,i} - \overline{u_i u_j}U_{i,j} - \varepsilon \quad (9)$$

$$\begin{aligned}
\varepsilon_{,t} + U_i\varepsilon_{,i} &= [(\nu + \frac{\nu_T}{\sigma_\varepsilon})\varepsilon_{,i}]_{,i} \\
&- C_{\varepsilon 1}\overline{u_i u_j}U_{i,j}\frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \quad (10)
\end{aligned}$$

where $\overline{u_i u_j}$ and ν_T are determined by Eq. (5) with the following model constants:

C_μ	σ_k	σ_ε	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$
0.09	1.0	1.3	1.44	1.92

This model is most commonly used in turbulent shear flow calculations with a great deal of success. Some deficiencies have been recognized, for example, in calculations of both plane and round jets. The predicted spreading rate of the round jet was unrealistically larger than that of the plane jet.

2.2.2 Realizable k-ε eddy viscosity model

Shih *et al* [8] proposed a new dissipation rate equation based on the dynamic equation of vorticity variance:

$$\begin{aligned}\varepsilon_{,t} + U_i \varepsilon_{,i} &= [(\nu + \frac{\nu_T}{\sigma_\varepsilon})\varepsilon_{,i}]_{,i} \\ &+ C_1\, S\, \varepsilon - C_2 \frac{\varepsilon^2}{k + \sqrt{\nu\, \varepsilon}} \end{aligned} \quad (11)$$

The equation for turbulent kinetic energy is Eq. (9). $\overline{u_i u_j}$ and ν_T are defined in Eq. (5). Combining these, we have

$$k_{,t} + U_i k_{,i} = [(\nu + \frac{\nu_T}{\sigma_k})k_{,i}]_{,i} - \overline{u_i u_j} U_{i,j} - \varepsilon \quad (12)$$

$$\overline{u_i u_j} = \frac{2}{3}\, k\, \delta_{ij} - \nu_T\, (U_{i,j} + U_{j,i}) \quad (13)$$

$$\nu_T = C_\mu \frac{k^2}{\varepsilon} \quad (14)$$

where the coefficients C_μ and C_1 in Eqs. (11) and (14) are not constant. Based on the realizability requirement, Shih *et al* [8] found that C_μ should depend on the mean flow and the turbulent field as follows,

$$C_\mu = \frac{1}{A_0 + A_s U^{(*)} \dfrac{k}{\varepsilon}} \quad (15)$$

where

$$\begin{aligned} U^{(*)} &= \sqrt{S_{ij}S_{ij} + \tilde{\Omega}_{ij}\tilde{\Omega}_{ij}} \\ \tilde{\Omega}_{ij} &= \Omega_{ij} - 2\varepsilon_{ijk}\omega_k \\ \Omega_{ij} &= \overline{\Omega_{ij}} - \varepsilon_{ijk}\omega_k \end{aligned} \quad (16)$$

where $\overline{\Omega_{ij}}$ is the mean rotation rate viewed in a rotating reference frame with the angular velocity ω_k. The relations in Eq. (16) enable the model to simulate rotating homogeneous shear flows [33] as shown in Figures 1 and 2.

Coefficient C_1 is calibrated against the experimental data of homogeneous shear flow [34] and channel flow [35]. It is found that C_1 should be a function of the time scale ratio of the turbulence to the mean strain, η:

$$C_1 = max\{0.43,\ \frac{\eta}{5+\eta}\} \quad (17)$$

where

$$\eta = \frac{S\, k}{\varepsilon}, \quad S = \sqrt{2 S_{ij} S_{ij}}. \quad (18)$$

Other model constants are listed as follows

C_μ	σ_k	σ_ε	C_1	C_2	A_0
Eq.(15)	1.0	1.2	Eq.(17)	1.9	4.0

This new k-ε eddy viscosity model has been tested in various flows: a planar mixing layer (in Figure 3), planar and round jets (in Figures 4, 5 and Table 1), a channel flow (in Figure 6), boundary layer flows with and without pressure gradients (in Figures 7, 8 and 9), and backward-facing step flows (in Figure 10, 11 and Table 2). As may be seen from reference to these figures and tables, the new k-ε eddy viscosity model produces consistently superior performance over the standard k-ε eddy viscosity model. It should be noted that the well-known spreading rate anomaly of planar and round jets is removed completely.

Case	Exp.	Standard	Present
Mixing	0.13-0.17	0.152	0.151
Planar jet	0.105-0.11	0.109	0.105
Round jet	0.085-0.095	0.116	0.094

Table 1. The spreading rates of turbulent free shear flows

Case	Exp.	Standard	Present
DS [36]	6.1	4.99	6.02
KKJ [37]	7 ± 0.5	6.35	7.5

Table 2. Comparison of the reattachment locations in back-step flows

2.2.3 A quadratic algebraic Reynolds stress equation model

In order to consider the effects of anisotropy on the mean flow, a nonlinear stress-strain relation must be employed. As we discussed in the previous section, the most general form is Eq. (7).

However, determination of all the coefficients is not an easy task. For practical purposes, Shih et al [13] suggested a quadratic stress-strain relation, which is also suggested by an RNG result [29]. The necessity of using a higher order stress-strain relation will be considered in future studies. Applying the constraint of rapid distortion theory for the rapid rotation [13, 38] and the constraint of realizability, a very simple nonlinear algebraic Reynolds stress model can be obtained:

$$\begin{aligned}\overline{u_i u_j} &= \frac{2}{3}k\delta_{ij} - C_\mu \frac{k^2}{\varepsilon} 2S^*_{ij} \\ &+ 2C_2 \frac{k^3}{\varepsilon^2}(-S^*_{ik}\Omega *_{kj} + \Omega^*_{ik}S^*_{kj}) \quad (19)\end{aligned}$$

where C_μ and C_2 are determined from the following relations:

$$C_\mu = \frac{1}{6.5 + A^*_s U^* \frac{k}{\varepsilon}} \quad (20)$$

$$C_2 = \frac{\sqrt{1 - 9C_\mu^2 (S^* \frac{k}{\varepsilon})^2}}{1 + 6S^*\Omega^* \frac{k^2}{\varepsilon^2}} \quad (21)$$

where

$$\begin{aligned} S^* &= \sqrt{S^*_{ij}S^*_{ij}}\,, \quad \Omega^* = \sqrt{\Omega^*_{ij}\Omega^*_{ij}}\,, \\ U^* &= \sqrt{S^*_{ij}S^*_{ij} + \tilde{\Omega}^*_{ij}\tilde{\Omega}^*_{ij}}\,, \\ W^* &= \frac{S^*_{ij}S^*_{jk}S^*_{ki}}{(S^*)^3} \end{aligned} \quad (22)$$

and

$$A^*_s = \sqrt{6}\ \cos\phi_1, \quad \phi_1 = \frac{1}{3} arc\cos(\sqrt{6}W^*) \quad (23)$$

In Eqs. (19)-(22), S^*_{ij}, Ω^*_{ij} and $\tilde{\Omega}^*_{ij}$ are defined as

$$S^*_{ij} = S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\,, \quad \tilde{\Omega}^*_{ij} = \Omega^*_{ij} - 2\varepsilon_{ijk}\omega_k \quad (24)$$

$$S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}),\ \Omega^*_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i}) - \varepsilon_{ijk}\omega_k \quad (25)$$

where ω_k is the angular velocity of the reference frame.

The k and ε in Eq. (19) are determined either by the standard k-ε equations listed in Section 2.2.1 or by the new k-ε equations described in Section 2.2.2.

The model represented in Eqs. (19)-(23) is quite simple but has several advantages compared to the standard k-ε eddy viscosity model. First, the present model is fully realizable. Accordingly, it will not produce negative energy components and will not violate the Schwarz' inequality between turbulent velocities. Second, the effective eddy viscosity, defined as $\overline{u_\alpha u_\beta}/2S^*_{\alpha\beta}$, is anisotropic, as it should be. Finally, the present model contains the effect of mean rotation on the Reynolds stresses with proper behavior that matches the RDT result: the pure mean rotation will not affect the isotropic turbulence.

The model of Eq. (19) gives $b_{12} = -0.156$, $b_{11} = -b_{22} = 0.123$ for Tavoularis and Corrsin's homogeneous shear flow [34] at $U_{1,2}k/\varepsilon = 6.08$ and gives $b_{12} = -0.122$, $b_{11} = -b_{22} = 0.14$ for the direct numerical simulation of channel flow (Kim *et al* [35]) in the inertial sublayer at $U_{1,2}k/\varepsilon = 3.3$. These results show that the present model gives reasonable anisotropy of the Reynolds stresses for both homogeneous shear flow and boundary layer flow compared to the standard k-ε eddy viscosity model. Detailed comparisons with the experimental and DNS data are shown in Tables 3 and 4.

	experiment	standard	present
b_{12}	-0.142	-0.273	-0.156
b_{11}	0.202	0.	0.123
b_{22}	-0.145	0.	-0.123

Table 3. Anisotropy in the homogeneous shear flow

	DNS	standard	present
b_{12}	-0.145	-0.149	-0.122
b_{11}	0.175	0.	0.14
b_{22}	-0.145	0.	-0.14

Table 4. Anisotropy in the channel flow

The model in Eq. (19) was tested together with the standard $k-\varepsilon$ equations for rotational homogeneous shear flows, backward-facing step flows, diffuser flows and confined jets, etc. The results are very encouraging compared to the results obtained from the standard isotropic eddy viscosity model. Figure 12 shows the Reynolds stresses for the KKJ [37] case. The anisotropy is well captured by the present model. Figures 13 and 14 show the pressure distribution and the velocity at the center of a diffuser. Clearly, a significant improvement over the standard k-ε model is achieved. Figure 15 shows a confined axisymmetric jet with back flows (Barchilon and Curtet [39]). The calculated velocity profiles at two downstream locations and the pressure distribution for two inflow conditions ($Ct = 0.976,\ 0.152$) are plotted and compared with experimental data in Figure 16.

2.3 Constitutive relation for heat fluxes

Here we assume

$$\overline{\theta u_i} = F_i(U_{i,j}\ ,\ T_{,i}\ ,\ k\ ,\ \varepsilon\ ,\ \overline{\theta^2}\ ,\ \varepsilon_\theta) \qquad (26)$$

where $\overline{\theta^2}$ is the variance of a fluctuating scalar (e.g., temperature) and ε_θ is its dissipation rate. Equation (26) indicates that the scalar flux depends on not only the mean scalar gradient $T_{,i}$, but also the mean velocity gradient $U_{i,j}$ and the scales of both velocity and scalar fluctuations characterized by k, ε, $\overline{\theta^2}$ and ε_θ. Using dimensional analysis, the arguments in Eq. (26) can be reduced to three groups: $\frac{k}{\varepsilon}U_{i,j}$, $\frac{k}{\sqrt{\varepsilon\varepsilon_\theta}}T_{,i}$ and $r = 2\frac{k/\varepsilon}{\overline{\theta^2}/\varepsilon_\theta}$. The latter group is a ratio of the mechanical time scale to the scalar time scale. Using methods of rational mechanics, Shih and Lumley [25] obtained a general relation:

$$\begin{aligned}
\overline{\theta u_i} &= a_1 k\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2} T_{,i} \\
&+ \frac{k^2}{\varepsilon}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_2 U_{i,j} + a_3 U_{j,i})T_{,j} \\
&+ \frac{k^3}{\varepsilon^2}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_4 U_{i,k}U_{k,j} + a_5 U_{j,k}U_{k,i} \\
&\qquad + a_6 U_{i,k}U_{j,k} + a_7 U_{k,i}U_{k,j}\ )\ T_{,j} \\
&+ \frac{k^4}{\varepsilon^3}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_8 U_{i,k}U^2_{j,k} + a_9 U^2_{i,k}U_{j,k} \\
&\qquad + a_{10} U_{k,i}U^2_{k,j} + a_{11} U^2_{k,i}U_{k,j}\ )\ T_{,j} \\
&+ \frac{k^5}{\varepsilon^4}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_{12} U^2_{i,k}U^2_{j,k} + a_{13} U^2_{k,i}U^2_{k,j} \\
&\qquad + a_{14} U_{i,k}U_{l,k}U^2_{l,j} + a_{15} U_{j,k}U_{l,k}U^2_{l,i}\ )\ T_{,j} \\
&+ \frac{k^6}{\varepsilon^5}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_{16} U_{i,k}U^2_{l,k}U^2_{l,j} \\
&\qquad + a_{17} U_{j,k}U^2_{l,k}U^2_{l,i})\ T_{,j} \\
&+ \frac{k^7}{\varepsilon^6}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2} a_{18}(U_{i,k}U_{l,k}U^2_{l,m}U^2_{j,m}\ )\ T_{,j}
\end{aligned} \qquad (27)$$

The coefficients $a_1 - a_{18}$ are, in general, functions of the time scale ratio r and the other invariants of tensors in question.

It is interesting to note that the conventional eddy diffusivity model for the scalar flux is just the first term on the right hand side of Eq. (27) and that the models derived from the two-scale DIA (Yoshizawa [40]) and the RNG method (Rubinstein and Barton [41]) are the first two terms of Eq. (27).

2.3.1 Heat flux algebraic equation model

Shabbir [42] used a linear form of Eq. (27), i.e.

$$\begin{aligned}
\overline{\theta u_i} &= a_1 k\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2} T_{,i} \\
&\quad + \frac{k^2}{\varepsilon}\left(\frac{k}{\varepsilon}\frac{\overline{\theta^2}}{\varepsilon_\theta}\right)^{1/2}(a_2 U_{i,j} + a_3 U_{j,i})T_{,j}
\end{aligned} \qquad (28)$$

The proposed coefficients a_1 and a_2 are based on the experiment of Tavoularis and Corrsin [34]

$$\begin{aligned}
a_1 &= -\frac{2 + 2r + 0.5r^2}{26 + 3.2\eta^2 + 2\xi^2} \\
a_2 &= 0.024
\end{aligned} \qquad (29)$$

where

$$r = 2\frac{k/\varepsilon}{\overline{\theta^2}/\varepsilon_\theta},\ \eta = \frac{S\,k}{\varepsilon},\ \xi = \frac{k}{\varepsilon}\sqrt{\frac{k}{\overline{\theta^2}}}\,S_T \qquad (30)$$

$S = \sqrt{2S_{ij}S_{ij}}$ and $S_T = \sqrt{T_{,i}T_{,i}}$. the coefficient a_3 cannot be calibrated with Tavoularis and Corrsin's experiment, because the term which contains a_3 disappears in this case. Other experiments are needed to determine a_3. In the test cases shown below, a_3 does not appear.

The $\overline{\theta^2}$ and its dissipation rate ε_θ will be determined by the following transport equations:

$$\overline{\theta^2}_{,t} + U_i\overline{\theta^2}_{,i} = \left(\frac{\alpha_T}{\sigma_t}\overline{\theta^2}_{,i}\right)_{,i} - 2\,\overline{\theta u_i}T_{,i} - 2\,\varepsilon_\theta \quad (31)$$

$$\varepsilon_{\theta,t} + U_{,i}\varepsilon_{\theta,i} = \left(\frac{\alpha_T}{\sigma_\phi}\varepsilon_{\theta,i}\right)_{,i} + C_{\theta 1}\,\varepsilon_\theta\,S + C_{\theta 2}\sqrt{\frac{\varepsilon_\theta\,\varepsilon}{P_T}}\,S_T - C_{\theta 3}\frac{\varepsilon_\theta\,\varepsilon}{k} \quad (32)$$

where Pr is the Prandtl number of the fluid. The coefficients in Eqs. (31) and (32) are listed below:

$C_{\theta 1}$	$C_{\theta 2}$	$C_{\theta 3}$	σ_t	σ_ϕ
$C_1 - 0.33$	0.63	$C_2 - 1 + r$	1.0	1.8

The coefficients C_1 and C_2 appear in the dissipation rate equation, Eq. (11). The scalar dissipation rate equation of ε_θ, Eq. (32), is derived based on the dynamic equation of the variance of the temperature gradient $\overline{\theta_{,i}\theta_{,i}}$ [43].

This model has been tested against the heat transfer in homogeneous flows [44, 34] and boundary layer flows [45]. As Figures 17, 18 and 19 show, the model of Eqs. (28)-(32) can simulate these flows quite reasonably. Further tests for complex flows are currently in progress.

3. REYNOLDS STRESS TRANSPORT EQUATION MODELS

Transport equation models for second moments such as the Reynolds stress and the heat flux terms are considered as advanced tools for studying complex turbulent flows where the anisotropy of the second moments, induced by strong shear, swirl, the rotation and/or curvature, etc. are dominating phenomena. The effect of anisotropy due to the above physical phenomena is very difficult for k-ε based models, but can be easily modeled by second moment transport equation models. This is due to the fact that most of the important physics in complex turbulent flows can be represented by the terms in the second moment equations, such as the turbulent diffusion tensor, the production tensor, the pressure-strain correlation tensor and the dissipation rate tensor.

The second moment equations of $\overline{u_iu_j}$, $\overline{\theta u_i}$ and $\overline{\theta^2}$ for incompressible flows without buoyancy can be written as

$$\begin{aligned}\frac{D}{Dt}\overline{u_iu_j} = &-[\,\overline{u_iu_ju_k} + \frac{1}{\rho}(\overline{pu_i}\delta_{jk} + \overline{pu_j}\delta_{ik})\,]_{,k}\\ &-(\overline{u_iu_k}\,U_{j,k} + \overline{u_ju_k}\,U_{i,k})\\ &+\frac{1}{\rho}\,\overline{p(u_{i,j}+u_{j,i})} - 2\nu\,\overline{u_{i,k}u_{j,k}}\\ &-2\Omega_l\,(\varepsilon_{ilk}\overline{u_ku_j} + \varepsilon_{jlk}\overline{u_ku_i})\end{aligned} \quad (33)$$

$$\begin{aligned}\frac{D}{Dt}\overline{\theta u_i} = &-[\overline{\theta u_iu_k} + \frac{1}{\rho}\overline{p\theta}\delta_{ik}]_{,k}\\ &-(\overline{\theta u_k}U_{i,k} + \overline{u_iu_k}T_{,k})\\ &+\frac{1}{\rho}\,\overline{p\theta_{,i}} - (\nu+\gamma)\,\overline{\theta_{,k}u_{i,k}}\\ &-2\varepsilon_{ilk}\Omega_l\overline{\theta u_k}\end{aligned} \quad (34)$$

$$\frac{D}{Dt}\overline{\theta^2} = -(\overline{\theta^2u_k})_{,k} - 2\overline{\theta u_k}\,T_{,k} - 2\varepsilon_\theta \quad (35)$$

where the molecular diffusion terms in the above equations have been neglected. Ω_i is the angular velocity of the reference frame, ε_{ijk} is an alternating operator and $D(\;)/Dt$ is the substantial derivative: $(\;)_{,t} + U_k(\;)_{,k}$. Unfortunately, most of the terms on the right hand side of the above equations (except the production terms) are themselves unknowns:

$$\frac{1}{\rho}\,\overline{p(u_{i,j}+u_{j,i})},\ \frac{1}{\rho}\,\overline{p\theta_{,i}}$$
(pressure-"strain" correlation terms) (36)

$$\nu\,\overline{u_{i,k}u_{j,k}},\ (\nu+\gamma)\,\overline{\theta_{,k}u_{i,k}},\ \varepsilon_\theta$$
(dissipation terms) (37)

$$\frac{1}{\rho}\,\overline{pu_i},\ \frac{1}{\rho}\,\overline{p\theta}$$
(pressure transport terms) (38)

$$\overline{u_iu_ju_k},\ \overline{\theta u_iu_k},\ \overline{\theta^2u_k}$$
(triple correlations terms) (39)

In turn, these must be modeled in terms of known quantities such as the mean flow quantities, U_i, T and the second moments, $\overline{u_iu_j}$, $\overline{\theta\,u_i}$. The correct modeling of these terms will enable us to capture important turbulent flow physics which will enable us to improve the prediction of complex turbulent flows.

Over the past several decades, numerous studies have been directed towards the development of models for the pressure-"strain" correlation terms which are considered as the most important model terms in the second moment equations. Many attractive models have been proposed by using a variety of theories and principles, such as invariant theory, rapid distortion theory and realizability, etc. In this section, we will concentrate on recent developments in modeling of the pressure-strain correlation and the dissipation rate tensors for the velocity field.

3.1 Rapid pressure-strain and return-to-isotropy terms

In modeling the pressure-strain correlation term, the following equation for the instantaneous pressure is often used for guiding the development:

$$-\frac{1}{\rho}p_{,ii} = 2U_{i,j}u_{j,i} + u_{i,j}u_{j,i} + 2\varepsilon_{ilj}\Omega_l u_{j,i} \quad (40)$$

which indicates that the pressure fluctuation can be immediately affected by the mean shear and the rotation of the reference frame. Since this equation is linear in p, the pressure can be decomposed into two components:

$$p = p^r \quad \text{(rapid)} \quad + p^s \quad \text{(slow)} \quad (41)$$

The components are defined as follows

$$-\frac{1}{\rho}\, p^r_{,ii} = 2(U_{i,j} + \varepsilon_{ilj}\Omega_l)u_{j,i} \quad (42)$$

$$-\frac{1}{\rho}\, p^s_{,ii} = u_{i,j}u_{j,i} \quad (43)$$

With Eqs. (42) and (43), the pressure-strain correlation can be decomposed into two parts: the rapid part and the slow part. For example, in terms of the solution of Eq. (42) the rapid pressure-strain correlation for homogeneous turbulence can be written as

$$\frac{1}{\rho}\,\overline{p^r(u_{i,j} + u_{j,i})} = 2(U_{p,q} + \varepsilon_{plq}\Omega_l)(X_{pjqi} + X_{piqj}) \quad (44)$$

where X_{pjqi} is defined as

$$X_{pjqi} = -\frac{1}{4\pi}\int\int\int_v [\overline{u_q(r)u_i(r')}]_{,pj}\frac{1}{|r - r'|}dv \quad (45)$$

In addition, the modeling of the slow pressure-strain correlation is traditionally combined with the dissipation rate tensor and the combination is called the return-to-isotropy term:

$$-\Phi_{ij}\,\varepsilon = \frac{1}{\rho}\,\overline{p^s(u_{i,j} + u_{j,i})} - 2\nu\,\overline{u_{i,k}u_{j,k}} + \frac{2}{3}\varepsilon\,\delta_{ij} \quad (46)$$

where $\varepsilon = \nu\,\overline{u_{i,k}u_{i,k}}$ is the dissipation rate of the turbulent kinetic energy $k = \overline{u_i u_i}/2$.

3.2 Realizability constraints

Before discussing the modeling of the rapid pressure-strain and the return-to-isotropy terms, let us briefly describe the realizability concept in second moment closures. Later, we will see how the realizability constraints can help in constructing a turbulence model.

Schumann [46], Lumley [15, 47], Shih and Lumley [16] and Shih *et al* [48] have discussed the concept of realizability for second moments and its application in the second moment closures. For Reynolds stresses, realizability means that the normal stresses must be positive and that the Schwarz' inequality must be satisfied. It can be shown that realizability will be satisfied if the eigenvalues of the Reynolds stress tensor, $\overline{u_\alpha u_\alpha}$, remain positive. To ensure realizability, Lumley [47], Shih and Lumley [16] suggested the following constraint:

$$(\overline{u_\alpha u_\alpha})_{,t} = C\,(\overline{u_\alpha u_\alpha})^\alpha \quad \text{if} \quad \overline{u_\alpha u_\alpha} \to 0 \quad (47)$$

where $C > 0$ and $0 < \alpha \leq 1/2$. In application of realizability, Lumley's parameter F has been found to be very useful and is defined as

$$F = 1 + 9\,II_b + 27\,III_b \quad (48)$$

It can be shown that $0 \leq F \leq 1$ for all realistic turbulent flows and that $F \to 0$ if $\overline{u_\alpha u_\alpha} \to 0$. Using the latter property, $\overline{u_\alpha u_\alpha}$ on the right hand side of Eq. (47) can be replaced by F. Applying Eq. (47) to the Reynolds stress equation Eq. (33), we obtain the following constraints for the rapid pressure-strain and return-to-isotropy terms [16]:

$$U_{p,q}X_{p\alpha q\alpha} = 0 \quad \text{if} \quad \overline{u^2_\alpha} \to 0 \quad (49)$$

$$-(\Phi_{\alpha\alpha}+2/3) = C\,F^{\alpha} \quad \text{if} \quad \overline{u_\alpha^2} \to 0 \qquad (50)$$

3.3 Modeling of the rapid pressure-strain

From a modeling point of view, we need to establish a relation, similar to a constitutive relation in continuum mechanics, which expresses the fourth rank tensor X_{pjqi} in Eq. (44) in terms of the "known" quantities in question, such as the Reynolds stress tensor, at the local point and the present time. It is well-known that this kind of constitutive relation (an "local equilibrium" relation) does not exist for a general turbulent flow, because turbulence is in general a non-local phenomenon. A general turbulent relation is at least a functional. That is, it contains the history of the arguments in question. However, the functional form of the turbulent relation will not help to solve the turbulence closure problem. In turbulence modeling, therefore, we formally derive a constitutive relation under certain assumptions that are approximately valid for particular turbulent flows. As a result, the constitutive relation or the model can be quite accurate for the selected flows, but cannot be guaranteed for other flows.

Let us assume X_{pjqi} is a function of $\overline{u_iu_j}$ which has six components involving one dimension. From the π theorem it is possible to form five independent dimensionless groups as Lumley [15] suggested:

$$b_{ij} = \frac{\overline{u_iu_j}}{2k} - \frac{1}{3}\delta_{ij} \qquad (51)$$

where δ_{ij} is Kronecker's delta and b_{ij} represents the anisotropy of the Reynolds stress tensor. The trace of this tensor is zero, i.e., $b_{ii}=0$. Therefore, we may write

$$\frac{X_{pjqi}}{2k} = F_{pjqi}(\, b_{ij}\,) \qquad (52)$$

The most general form of Eq. (52) can be formally obtained by using invariant theory [16, 49]:

$$\begin{aligned}
\frac{X_{pjqi}}{2k} &= \alpha_1\delta_{qi}\delta_{pj} + \alpha_2(\delta_{pq}\delta_{ij}+\delta_{qj}\delta_{pi}) \\
&+ \alpha_3\delta_{qi}b_{pj} + \alpha_4\delta_{pj}b_{qi} \\
&+ \alpha_5(\delta_{pq}b_{ij}+\delta_{ij}b_{pq}+\delta_{jq}b_{pi}+\delta_{pi}b_{qj}) \\
&+ \alpha_6\delta_{qi}b^2_{pj} + \alpha_7\delta_{pj}b^2_{qi} \\
&+ \alpha_8(\delta_{pq}b^2_{ij}+\delta_{ij}b^2_{pq}+\delta_{jq}b^2_{pi}+\delta_{pi}b^2_{qj}) \\
&+ \alpha_9 b_{qi}b_{pj} + \alpha_{10}(b_{pq}b_{ij}+b_{qj}b_{pi}) \\
&+ \alpha_{11}b_{qi}b^2_{pj} + \alpha_{12}b_{pj}b^2_{qi} \\
&+ \alpha_{13}(b_{pq}b^2_{ij}+b_{ij}b^2_{pq}+b_{qj}b^2_{pi}+b_{pi}b^2_{qj}) \\
&+ \alpha_{14}b^2_{qi}b^2_{pj} + \alpha_{15}(b^2_{pq}b^2_{ij}+b^2_{qj}b^2_{pi})
\end{aligned} \qquad (53)$$

where $b^2_{ij} = b_{ik}b_{kj}$ and the symmetry properties of X_{pjqi}, i.e., $X_{pjqi}=X_{pjiq}$, $X_{pjqi}=X_{jpqi}$ have been used. It should be pointed out that all the coefficients in Eq. (53) are in general not constant, but functions of the independent invariants of b_{ij}:

$$II_b = -\frac{1}{2}b_{ij}b_{ji}, \; III_b = \frac{1}{3}b_{ij}b_{jk}b_{ki} \qquad (54)$$

and that the fifteen coefficients α_i are not all independent to each other, because the X_{pjqi} must satisfy certain mathematical relations according to its definition, Eq. (45), i.e.,

$$X_{ppqi} = \overline{u_qu_i}, \quad X_{pjpi} = 0 \qquad (55)$$

Using these constraints, the number of coefficients will be reduced to nine. In addition, when the term $(X_{pjqi}+X_{piqj})U_{p,q}$ is formed, two more coefficients may be dropped because of the following tensorial identity relations (Reynolds [18], Johansson and Hallbäck [21]):

$$\begin{aligned}
&(\, b^2_{ip}\delta_{jq} + b^2_{jp}\delta_{iq} - b_{ij}b_{pq} + b_{iq}b_{jp} \\
&\quad -b^2_{pq}\delta_{ij} - \frac{1}{2}II_b\delta_{ip}\delta_{jq}\,)S_{pq} = 0
\end{aligned} \qquad (56)$$

$$\begin{aligned}
&(\, b^2_{ip}b_{jq} + b^2_{jp}b_{iq} - b_{ij}b^2_{pq} - b^2_{ij}b_{pq} \\
&\quad +\frac{1}{3}III_b\delta_{ip}\delta_{jq}\,)S_{pq} = 0
\end{aligned} \qquad (57)$$

$$\begin{aligned}
&[\, b^2_{iq}b^2_{jp} + b^2_{jq}b^2_{ip} - 2b^2_{ij}b^2_{pq} - II_b(b_{ij}b_{pq} - b_{iq}b_{jp}) \\
&\quad +\frac{2}{3}III_b(b_{ip}\delta_{jq} + b_{jp}\delta_{iq} - b_{pq}\delta_{ij})\,]S_{pq} = 0
\end{aligned} \qquad (58)$$

where $S_{pq} = (U_{p,q}+U_{q,p})/2$ is the mean strain rate.

The rest of the task is to determine the detailed forms of the remaining seven coefficients.

3.3.1 Expansion in terms of II_b and III_b

As mentioned previously, the seven undetermined coefficients in Eq. (53) are, in general, functions of II_b and III_b. According to the definitions of Eq. (54), the absolute values of II_b and III_b are always less than unity. Therefore, one may naturally expand the coefficients as a power series of II_b and III_b as suggested by Reynolds [18] and Johansson and Hallbäck [21]:

$$\alpha_i = C_i^{(1)} + C_i^{(2)}\ II_b + C_i^{(3)}\ III_b + C_i^{(4)}\ II_b^2 + C_i^{(5)}\ II_b\ III_b + C_i^{(6)}\ III_b^2 + \cdots \quad (i = 1, 15) \tag{59}$$

The coefficients in Eq. (59) are independent of II_b and III_b, hence, are considered to be constant. Therefore, if their values are determined by using a particular flow, then the values should be valid for other flows as well. That is, Eq. (59) (with an infinite number of terms) will provide us with a universal rapid pressure-strain model. However, in practice, we must truncate the series to a certain power of II_b and III_b and use certain flows and constraints (such as homogeneous flows and realizability) to determine the values of the coefficients. Having done that, we then must verify whether such determined coefficients are constant and are valid for other flows as well.

3.3.2 Johansson and Hallbäck's model

Johansson and Hallbäck [21] suggested an expansion in which the terms in Eq. (53) are retained up to the order of $O(b^4)$. This will require the following truncated power series for Eq. (59):

$$\begin{aligned}
\alpha_i &= C_i^{(1)} + C_i^{(2)}\ II_b + C_i^{(3)}\ III_b + C_i^{(4)}\ II_b^2 && (i = 1, 2) \\
\alpha_i &= C_i^{(1)} + C_i^{(2)}\ II_b + C_i^{(3)}\ III_b && (i = 3, 4, 5) \\
\alpha_i &= C_i^{(1)} + C_i^{(2)}\ II_b && (i = 6, 7, 8, 9, 10) \\
\alpha_i &= C_i^{(1)} && (i = 11, 12, 13, 14, 15)
\end{aligned} \tag{60}$$

Formally, there are thirty two constants $C_i^{(1)}$, $C_i^{(2)}, \cdots, C_i^{(4)}$ for fifteen coefficients α_i; however, there are only seven independent coefficients α_i, hence, the number of independent constants are much less than thirty two. Using the realizability constraint Eq. (49), the number of the constants can be substantially reduced. Johansson and Hallbäck found that only four undetermined constants remain unresolved. The rapid pressure-strain correlation model becomes

$$\begin{aligned}
\frac{\Pi_{ij}^{(1)}}{4\ k} &= \frac{Q_1}{4} S_{ij} + \frac{Q_2}{2}(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}\delta_{ij}b_{kl}S_{kl}) \\
&+ Q_3 b_{kl}S_{kl}b_{ij} + Q_4(b_{kj}b_{li}S_{kl} - \frac{1}{3}b_{kl}^2 S_{kl}\delta_{ij}) \\
&+ 2\ Q_5 b_{kl}^2 S_{kl} b_{ij} \\
&+ (2Q_5 b_{kl}S_{kl} + 4Q_6 b_{kl}^2 S_{kl})(b_{ij}^2 + \frac{2}{3}II_b\delta_{ij}) \\
&+ \frac{Q_7}{2}(b_{ik}\Omega_{kj} + b_{jk}\Omega_{ki}) + Q_8(b_{jk}^2\Omega_{ki} + b_{ik}^2\Omega_{kj}) \\
&+ 2Q_9(b_{jk}^2 b_{il}\Omega_{kl} + b_{ik}^2 b_{jl}\Omega_{kl})
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
Q_1 &= \frac{4}{5} + \frac{16}{5}(4B_2 + 15B_3)\ II_b - \frac{48}{5}B_5\ III_b - \frac{304}{55}B_6\ II_b^2 \\
Q_2 &= -12B_1 + 4B_5\ II_b - 12B_6\ III_b \\
Q_3 &= -8B_2 + 36B_3 - \frac{28}{11}B_6\ II_b \\
Q_4 &= 96B_2 - 36B_3 + \frac{28}{11}B_6\ II_b \\
Q_7 &= -\frac{4}{3} - \frac{28}{3}B_1 - \frac{4}{3}(2B_4 - B_5)\ II_b - 4B_6\ III_b \\
Q_8 &= -16B_2 + 28B_3 - \frac{12}{11}B_6\ II_b \\
Q_5 &= B_5, \quad Q_6 = B_6, \quad Q_9 = B_4
\end{aligned}$$

and

$$\begin{aligned}
&B_1 = \beta_1 - 8\beta_4 II_b + 24\beta_7 III_b, \\
&B_2 = \beta_2 - 8\beta_8 II_b, \quad B_3 = \beta_3 - 8\beta_9 II_b, \\
&B_4 = \beta_5, \quad B_5 = \beta_6, \quad B_6 = \beta_{10}
\end{aligned}$$

and

$$\begin{aligned}
\beta_4 &= -\frac{3}{160}(3 + 60\beta_1 + 48\beta_2 - 40\beta_3), \\
\beta_5 &= -\frac{3}{2} - 132\beta_2 + \frac{2}{3}\beta_{10}, \\
\beta_6 &= \frac{3}{2} + 60(\beta_2 + \beta_3),
\end{aligned}$$

$$\beta_7 = \frac{9}{8}\beta_1 - \frac{9}{4}(\beta_2 + \beta_3),$$
$$\beta_8 = -\frac{3}{88}(\frac{3}{8} + 21\beta_2 + 10\beta_3 + \frac{1}{18}\beta_{10}),$$
$$\beta_9 = -\frac{9}{220}(\frac{3}{8} + 21\beta_2 + 10\beta_3)$$

The four undetermined constants are $\beta_1, \beta_2, \beta_3$ and β_{10}. To determine these constants, Johansson and Hallbäck used rapid distortion theory (RDT) to analyze an irrotational strain imposed on initially isotropic turbulence and a pure rotation imposed on an anisotropic turbulence. Comparing the model with the RDT results, they suggested that

$$\beta_1 = -\frac{1}{7}, \quad \beta_2 = 0.0295,$$
$$\beta_3 = -0.0484, \quad \beta_{10} = 14 \qquad (62)$$

This fourth order power truncation model, Eq. (61), was shown to be in excellent agreement with irrotational RDT results. The results indicate that for the RDT irrotational flows, the fourth order power truncation is appropriate and the truncation error is quite small. However, the applicability for other flows needs to be substantiated.

3.3.3 Expansion in terms of F and III_b

Recently, Shih and Lumley [22] proposed another truncation method for X_{pjqi}. It is a perturbation about two-component (2-C) turbulence, an extremely anisotropic turbulence state where $F = 0$. The idea is to emphasize the behavior of the rapid pressure-strain model in the anisotropic state. They expand the coefficients in Eq. (53) as a power series of F:

$$\alpha_i = c_i^{(0)}(III_b) + c_i^{(1)}(III_b)\, F + c_i^{(2)}(III_b)\, F^2 + \cdots \quad (i = 1, 15) \qquad (63)$$

where the coefficients $c_i^{(0)}, c_i^{(1)}, c_i^{(2)}, \cdots$ are in general not constant but functions of the third invariant III_b and they can be expanded as a power series of III_b:

$$c_i^{(j)}(III_b) = c_{i;0}^{(j)} + c_{i;1}^{(j)}\, III_b + c_{i;2}^{(j)}\, III_b^2 + \cdots \qquad (64)$$

However, for small III_b, such as in most turbulent shear flows, $c_i^{(j)}$ can be considered a constant as a first order approximation. Let us now start with the most general form of X_{pjqi}. Using the conditions $X_{ppqi} = \overline{u_q u_i}$ and $X_{pjpi} = 0$, we may obtain six relations between the fifteen coefficients which will reduce the number of coefficients to nine. As a result, we obtain

$$\alpha_1 = \frac{2}{15} + \frac{II_b}{5}(4\alpha_6 - 2\alpha_8) + \frac{III_b}{5}(\alpha_{11} + \alpha_{12} - 6\alpha_{13})$$
$$\alpha_2 = -\frac{1}{30} - \frac{II_b}{5}(\alpha_6 - 3\alpha_8) - \frac{III_b}{10}(3\alpha_{11} + 3\alpha_{12} + 2\alpha_{13})$$
$$\alpha_3 = -\frac{1}{3} - \frac{11}{3}\alpha_5 + \frac{II_b}{3}(\alpha_{11} + 3\alpha_{12} + 8\alpha_{13}) - \frac{III_b}{3}(3\alpha_{14} + \alpha_{15})$$
$$\alpha_4 = \frac{1}{3} - \frac{4}{3}\alpha_5 + \frac{2II_b}{3}(\alpha_{11} + 2\alpha_{13}) - \frac{2}{3}III_b\alpha_{15}$$
$$\alpha_7 = -\frac{2}{3}\alpha_{10} - \frac{4}{3}\alpha_8 + \frac{2II_b}{3}(\alpha_{14} + \alpha_{15})$$
$$\alpha_9 = -\frac{11}{3}\alpha_8 - \frac{1}{3}\alpha_{10} - \alpha_6 + \frac{II_b}{3}(\alpha_{14} + 7\alpha_{15}) \qquad (65)$$

Equation (65) will ensure the rapid pressure-strain correlation model satisfies basic mathematical conditions while the values of the nine coefficients: α_5, α_6, α_8, α_{10}, α_{11}, α_{12}, α_{13}, α_{14} and α_{15} can be arbitrary. Now let us apply the realizability condition of Eq. (49) to X_{pjqi}. This provides three additional relations for the $\alpha's$ at the 2-C state, i.e., $F = 0$:

$$\alpha_5 = -\frac{1}{10} + \frac{1}{30}(\alpha_{11} + \alpha_{12} + 4\alpha_{13} - 10\alpha_{15}/9) + \frac{II_b}{10}(3\alpha_{11} + 3\alpha_{12} + 8\alpha_{13} - 2\alpha_{15} - 6\alpha_6)$$
$$\alpha_8 = \frac{1}{3}\alpha_{13} - \frac{1}{9}\alpha_{15}$$
$$\alpha_{10} = -\frac{3}{10} + \frac{1}{90}(9\alpha_{11} + 9\alpha_{12} + 66\alpha_{13} - 10\alpha_{15}) + \frac{II_b}{10}(9\alpha_{11} + 9\alpha_{12} + 24\alpha_{13} - 6\alpha_{15} - 18\alpha_6) \qquad (66)$$

Equation (66) is valid only at the 2-C turbulence state and will ensure that the rapid pressure-strain correlation model satisfies the realizability constraint while the five coefficients α_6, α_{11}, α_{12}, α_{13} and α_{15} can be arbitrary. Therefore, we conclude that Eq. (65) together with Eq. (66) will ensure the model satisfies both the basic mathematical conditions of Eq. (55) and the realizability condition, Eq. (49), with arbitrary values of the six coefficients: $\alpha_6, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}$ and α_{15}. This will allow us to set any of these six coefficients to zero in order to simplify the model while maintaining its basic properties described by Eqs. (55) and (49).

3.3.4 Shih and Lumley's model

Shih and Lumley [22] set all the six undetermined coefficients $\alpha_6, \cdots, \alpha_{15}$ to zero to obtain the most simplified model. The necessity of retaining these coefficients is left for a future study. Following this approach, Eq. (65) becomes

$$\begin{aligned}
&\alpha_1 = \frac{2}{15} - \frac{2\,II_b}{5}\alpha_8, \quad \alpha_2 = -\frac{1}{30} + \frac{3\,II_b}{5}\alpha_8, \\
&\alpha_3 = -\frac{1}{3} - \frac{11}{3}\alpha_5, \quad \alpha_4 = \frac{1}{3} - \frac{4}{3}\alpha_5, \\
&\alpha_7 = -\frac{2}{3}\alpha_{10} - \frac{4}{3}\alpha_8, \quad \alpha_9 = -\frac{11}{3}\alpha_8 - \frac{1}{3}\alpha_{10}.
\end{aligned} \tag{67}$$

and Eq. (66) becomes

$$\alpha_5 = -\frac{1}{10}, \ \alpha_8 = 0, \ \alpha_{10} = -\frac{3}{10} \quad \text{if} \quad F = 0 \tag{68}$$

For $F \neq 0$, the coefficients α_5, α_8 and α_{10} could in general be a function of F and III_b. We propose to expand these coefficients in a power series of F and determine them by using DNS data of turbulent shear flows (Rogers et al. [50]). The coefficients α_5, α_8 and α_{10} can be approximately expressed by a truncated fourth order power series of F for small III_b:

$$\begin{aligned}
\alpha_5 &= -\,0.1 - 0.5\ F + 1.57\ F^2 - 2.22\ F^3 \\
&\quad + 1.07\ F^4 \\
\alpha_8 &= -\,1.6\ F + 4.69\ F^2 - 5.92\ F^3 \\
&\quad + 2.62\ F^4 \\
\alpha_{10} &= -\,\frac{3}{10}\ (1.0 + 12.95\ F - 37.0\ F^2 \\
&\quad + 44.3\ F^3 - 18.65\ F^4)
\end{aligned} \tag{69}$$

Using Eq. (67), the rapid pressure-strain model becomes

$$\begin{aligned}
\frac{\Pi_{ij}^{(1)}}{4\,k} = \frac{1}{5}\ S_{ij} &- 3\alpha_5\ (b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}\delta_{ij}b_{kl}S_{kl}) \\
&+ \alpha_8\ (2b^2_{kl}S_{kl}\delta_{ij} + \frac{2}{3}b^2_{ki}S_{kj} + \frac{2}{3}b^2_{kj}S_{ki} \\
&\qquad - \frac{22}{3}b_{ki}b_{lj}S_{kl} + \frac{2}{5}II_b\ S_{ij}) \\
&- \frac{2}{3}\alpha_{10}\ (b^2_{il}S_{jl} + b^2_{jl}S_{il} \\
&\qquad -2b_{kj}b_{li}S_{kl} - 3b_{ij}b_{kl}S_{kl}) \\
&+ \frac{1}{3}(2 + 7\alpha_5)(b_{ik}\Omega_{jk} + b_{jk}\Omega_{ik}) \\
&- (\frac{4}{3}\alpha_8 + \frac{2}{3}\alpha_{10})(b^2_{il}\Omega_{jl} + b^2_{jl}\Omega_{il})
\end{aligned} \tag{70}$$

As shown in Figures 20, 21, 22 and 23, this model is very encouraging compared against the DNS data of turbulent shear flows and other models. However, its applicability for other flows needs to be examined.

3.4 Modeling of the return to isotropy

Now let us discuss the modeling of the return-to-isotropy term Φ_{ij} in Eq. (46). To establish a constitutive relation, we assume $\varepsilon\ \Phi_{ij}$ be a second rank tensor function of the Reynolds stress $\overline{u_i u_j}$, dissipation rate ε, and kinematic viscosity ν. Dimensional analysis tells us that the eight arguments (with two dimensions) can be regrouped into six: five independent b_{ij} and one turbulent Reynolds number R_l,

$$R_l = \frac{(2k)^2}{9\varepsilon\nu} \tag{71}$$

Therefore, we may write

$$\Phi_{ij} = F_{ij}(\ b_{ij},\ R_l\ ) \tag{72}$$

Using invariant analysis, a general expression for Φ_{ij} can be obtained

$$\Phi_{ij} = a_2\ b_{ij} + a_3\ (\ \frac{2}{3}\ II_b\ \delta_{ij} + b^2_{ij}\ ) \tag{73}$$

where the property of $\Phi_{ii} = 0$ has been used. The coefficients a_2 and a_3 are, in general, functions of II_b, III_b and the Reynolds number R_l. Equation

(73) is the most general form of Φ_{ij} which was first derived by Lumley [15].

To determine the coefficients in Eq. (73), we first apply the realizability constraint, Eq. (50), to the return-to-isotropy model, Eq. (73), and obtain

$$a_2 = 2 + a_3(\frac{1}{3} + 2\ II_b) + C_\beta\ F^\alpha \quad \text{if } F \to 0 \quad (74)$$

Then, Φ_{ij} becomes

$$\Phi_{ij} = (2 + C_\beta\ F^\alpha) b_{ij} + a_3[(\frac{1}{3} + 2II_b) b_{ij} + b_{ij}^2 + \frac{2}{3} II_b \delta_{ij}] \quad (75)$$

where $C_\beta > 0$ and $0 \leq \alpha \leq 1/2$. The coefficient a_3 is still unknown. Lumley [15] let $a_3 = 0$, $\alpha = 1$ and

$$C_\beta = \frac{1}{9} exp[-7.77/R_l^{1/2}] \Big\{ 72/R_l^{1/2} + 80.1\ ln[1 + 62.4(-II_b + 2.3\ III_b)] \Big\} \quad (76)$$

based on the experimental data available at that time. This model is known to be a quite good model for cases where $III_b < 0$, but not very good for cases where $III_b > 0$ at large Reynolds numbers.

Here we may adopt a truncation method similar to those proposed by Reynolds [18] and Johansson and Hallbäck [21]. We expand a_3 as a power series of II_b and III_b:

$$a_3 = C_3^{(1)} + C_3^{(2)}\ II_b + C_3^{(3)}\ III_b + \cdots \quad (77)$$

where the coefficients, $C_3^{(1)}, C_3^{(2)}, \cdots$, are independent of II_b, III_b and are functions of R_l only. Now if we insert Eq. (77) into Eq. (75) and retain terms up to the order of $O(b^6)$, then there are only four coefficients $C_3^{(1)}$, $C_3^{(2)}$, $C_3^{(3)}$ and C_β which need to be determined. DNS of homogeneous turbulence and the final period decaying turbulence and other experimental data could be used for calibrating these coefficients.

An alternative approach is to expand a_2, a_3 in Eq. (73) in terms of F as suggested by Shih and Lumley [22] in constructing the model for the rapid pressure-strain term:

$$a_2 = C_2^{(1)} + C_2^{(2)}\ F + C_2^{(3)}\ F^2 + \cdots \quad (78)$$

$$a_3 = C_3^{(1)} + C_3^{(2)}\ F + C_3^{(3)}\ F^2 + \cdots \quad (79)$$

where the coefficients C's are in general functions of III_b and R_l. Using the constraint of realizability, Eq. (74), we obtain

$$C_2^{(1)} = 2 + C_3^{(1)}(\frac{1}{3} + 2II_b) + C_\beta\ F^\alpha \quad (80)$$

then Φ_{ij} can be written as

$$\Phi_{ij} = (2 + C_\beta\ F^\alpha + \beta)\ b_{ij} + \gamma\ [b_{ij}^2 + \frac{2}{3} II_b \delta_{ij} + (\frac{1}{3} + 2II_b) b_{ij}] \quad (81)$$

where

$$\beta = C_2^{(2)}\ F + C_2^{(3)}\ F^2 + \cdots - (C_3^{(2)}\ F + C_3^{(3)}\ F^2 + \cdots)(\frac{1}{3} + 2II_b)$$

$$\gamma = C_3^{(1)} + C_3^{(2)}\ F + C_3^{(3)}\ F^2 + \cdots \quad (82)$$

Using DNS data of homogeneous shear flows [50] and the Reynolds number dependence suggested by Lumley [15], β, γ and C_β can be approximated for small III_b as

$$\beta = \frac{8\ \exp(-\frac{7.77}{\sqrt{R_l}})}{\sqrt{R_l}} (-2.6\ F + 11.8\ F^2 - 23.3\ F^3 + 21.7 F^4 - 7.6\ F^5) + \frac{\exp(-\frac{7.77}{\sqrt{R_l}})}{11} (423\ F - 11112\ F^2 + 1528\ F^3 - 854\ F^4)$$

$$\gamma = -45\ (0.2 - 0.9\ F + 2.1\ F^2 - 2.4\ F^3 + 1.0\ F^4)$$

$$C_\beta = \frac{8}{\sqrt{R_l}} \exp(-\frac{7.77}{\sqrt{R_l}}) \quad (83)$$

The model expressed in Eq. (81) labeled SL with the coefficients of Eq. (83) and $\alpha = 1/2$ are compared with the DNS data [50] and other return-to-isotropy models [51-55] in Figures 24-27. As these figures show the present return-to-isotropy model is very encouraging. However, the applicability for other flows must await further examination.

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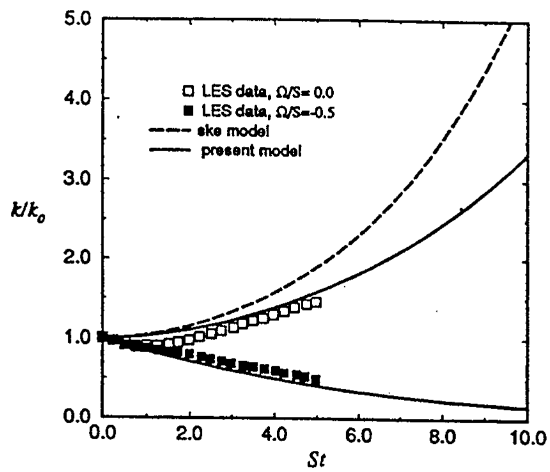

Fig. 1. Time evolution of turbulent kinetic energy in rotating homogeneous shear flows. $\Omega/S = 0.0$ and -0.5 [33].

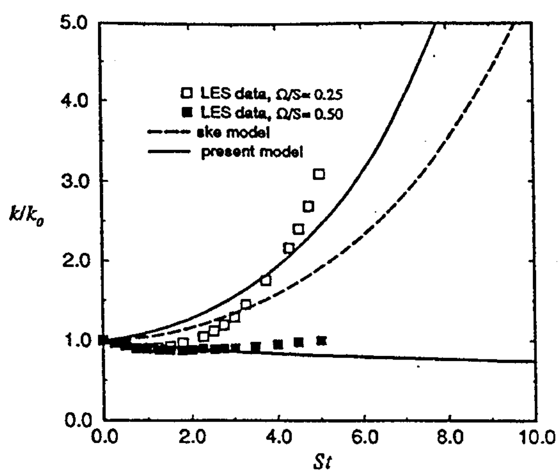

Fig. 2. Time evolution of turbulent kinetic energy in rotating homogeneous shear flows. $\Omega/S = 0.25$ and 0.5 [33].

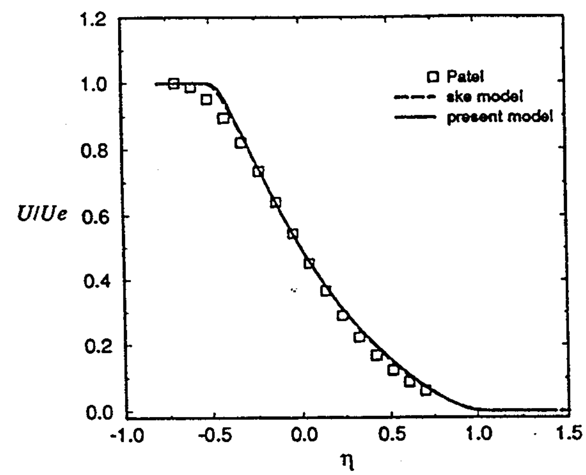

Fig. 3. Self-similar mean velocity profile for a planar mixing layer [56].

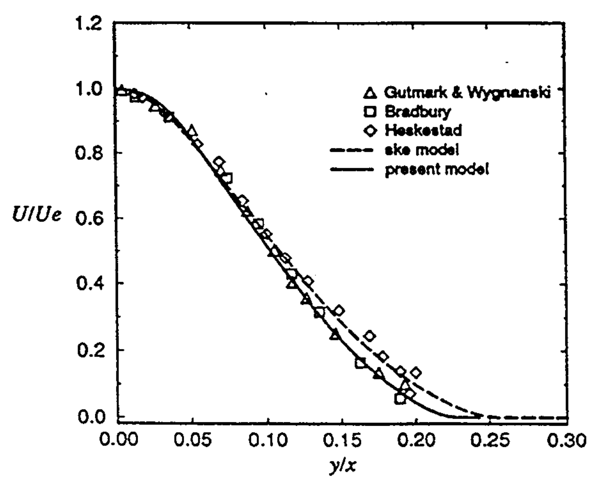

Fig. 4. Self-similar mean velocity profile for a planar jet [57].

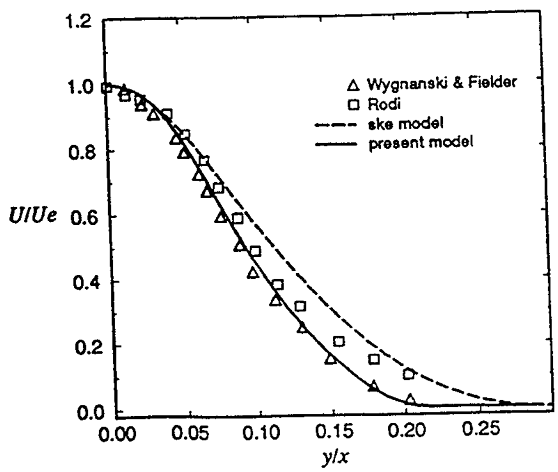

Fig. 5. Self-similar mean velocity profile for a round jet [58].

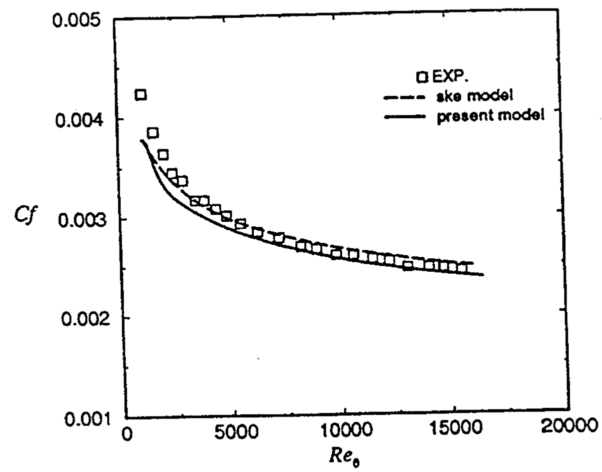

Fig. 7. Skin friction coefficient for the zero pressure gradient turbulent boundary layer [59].

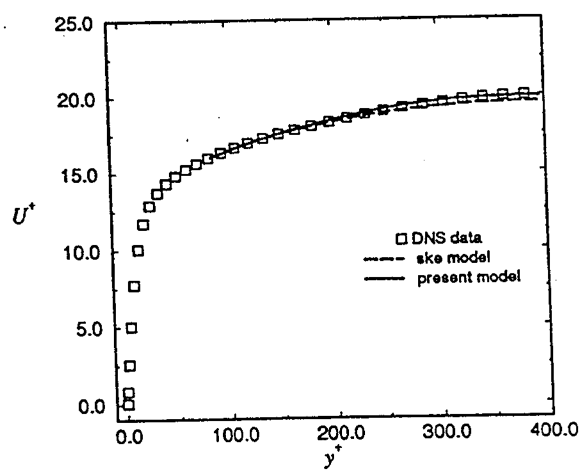

Fig. 6. Mean velocity profile for the turbulent channel flow [35].

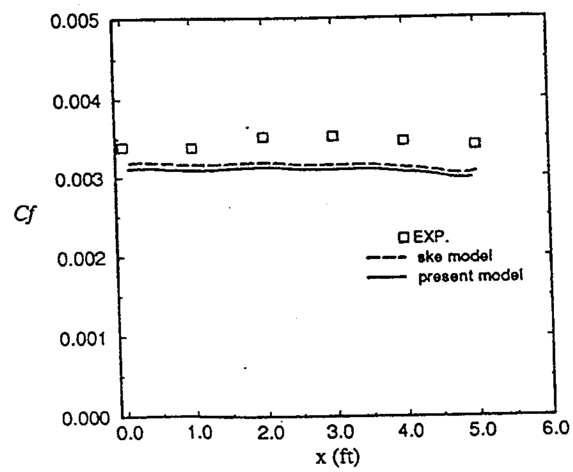

Fig. 8. Skin friction coefficient for the Herring and Norbury flow [60] (with a favorable pressure gradient).

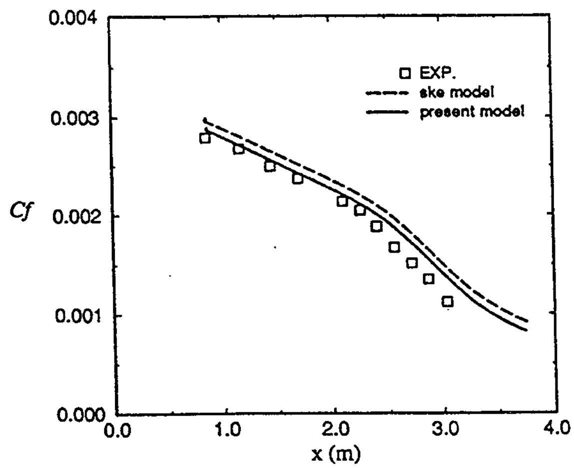

Fig. 9. Skin friction coefficient for the Samuel and Joubert flow [61] (with an adverse pressure gradient).

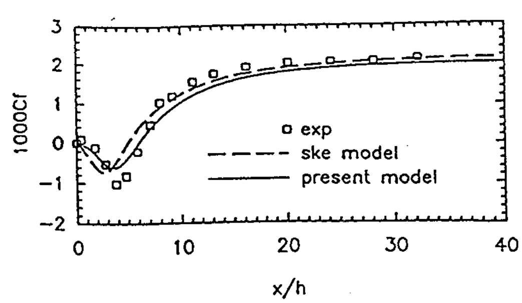

Fig. 10 Skin friction coefficient along the bottom wall (DS-case) [36].

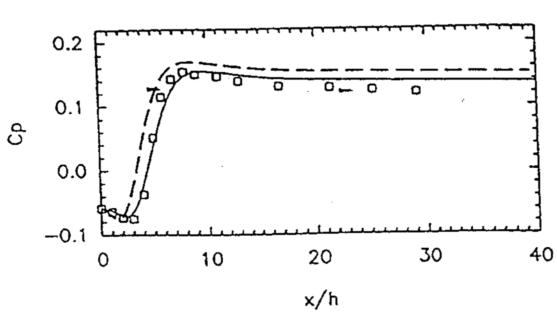

Fig. 11 Static pressure coefficient along the bottom wall (DS-case) [36].

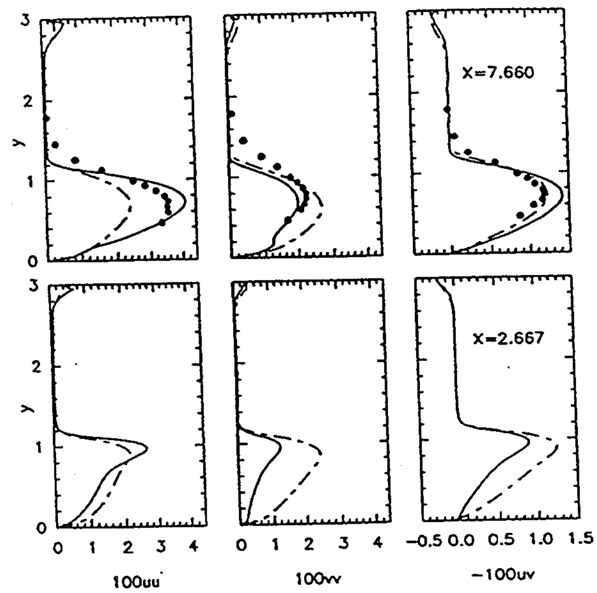

Fig. 12. Turbulent stress profiles in KKJ-case [37]. ——: present model; - - -: SKE; ●: experiment.

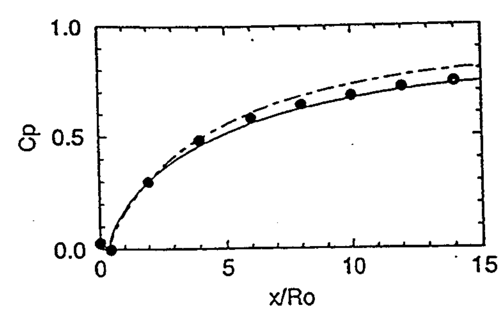

Fig. 13. Pressure distribution in a diffuser [62]. See figure 14 for legend.

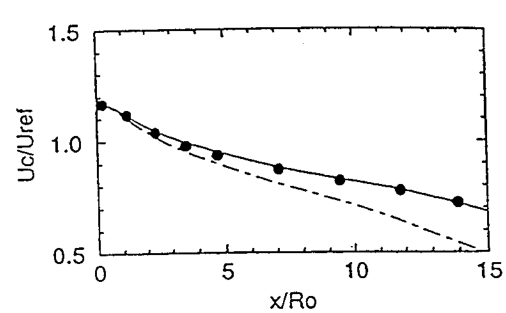

Fig. 14. Center velocity distribution in a diffuser [62]. ——: present model; - - -: SKE; ●: experiment.

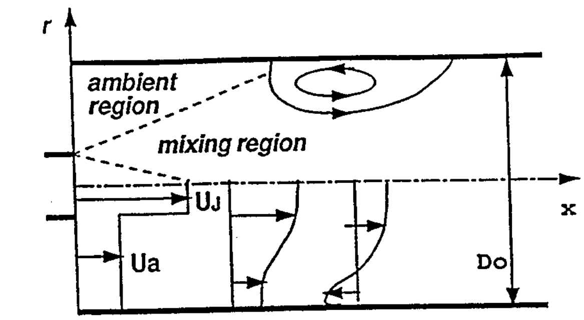

Fig. 15. A configuration of confined axsymmetric jet with back flows.

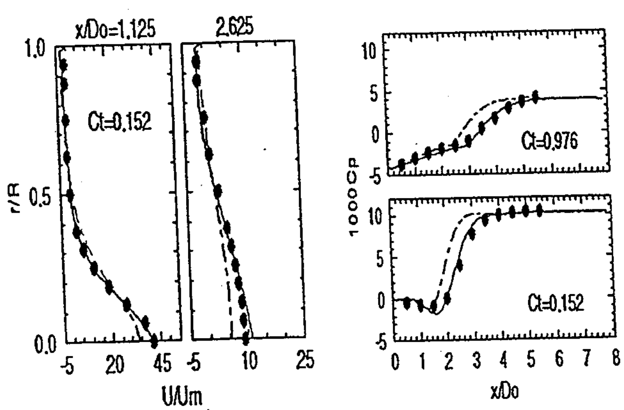

Fig. 16. Velocity profiles at two downstream locations (x/Do=1.125, 2.625) and pressure distributions for two inflow conditions (Ct=0.976, 0.152).

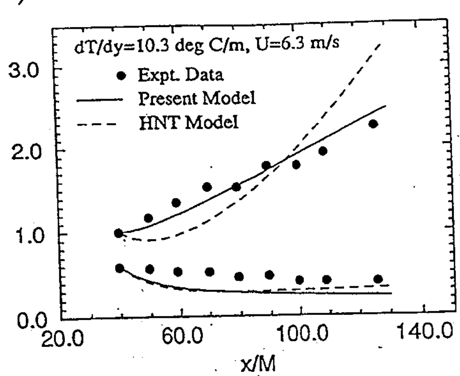

Fig. 17. Evolution of normalized temperature variance and its dissipation rate (Sirivat and Warhaft [44]).

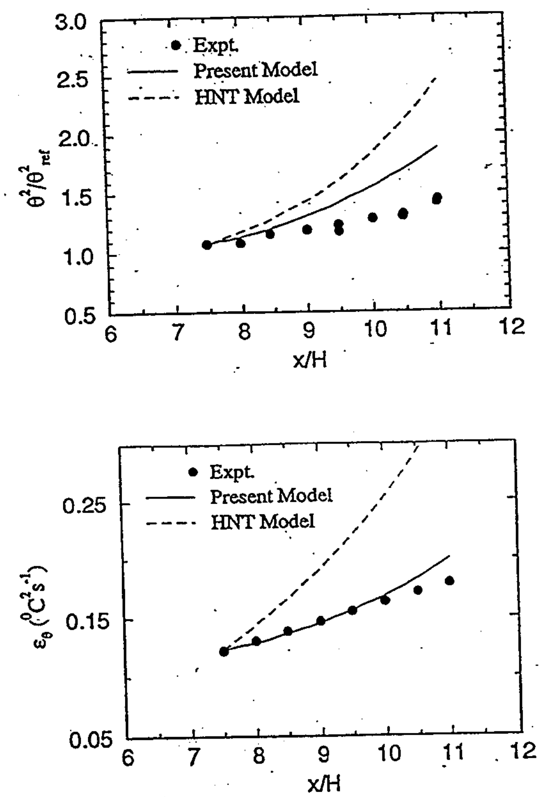

Fig. 18. Evolution of normalized temperature variance and its dissipation rate (Tavoularis and Corrsin [34]).

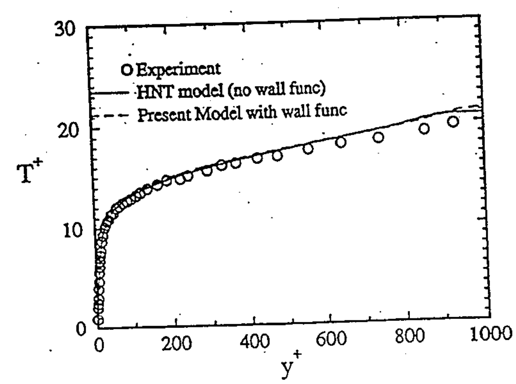

Fig. 19. Mean temperature profile for the constant temperature flat plate boundary layer (Gibson [45]).

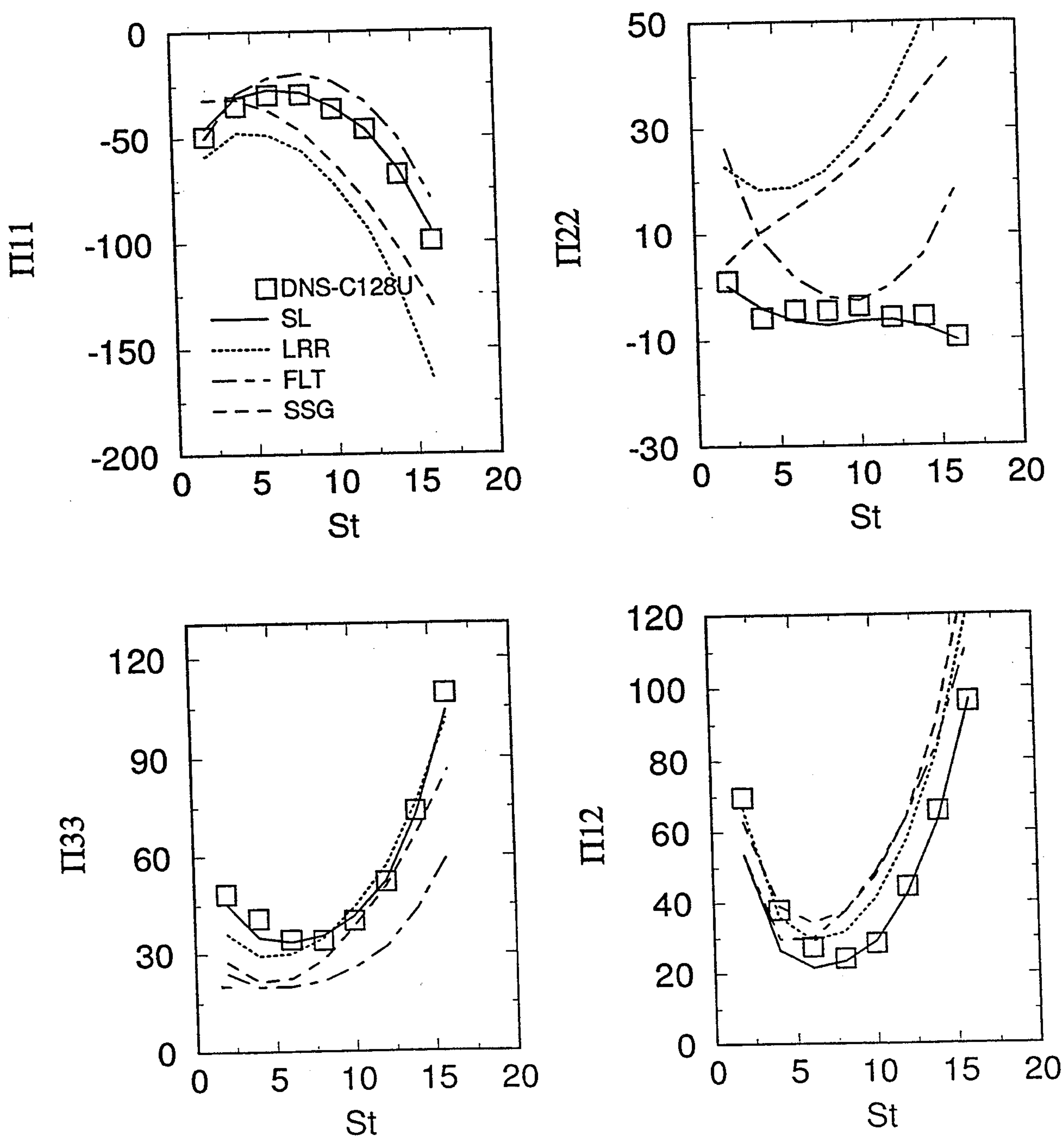

Fig. 20. Direct comparison of the rapid models with DNS data of homogeneous shear flow C128U (Rogers *et al.* [50]).

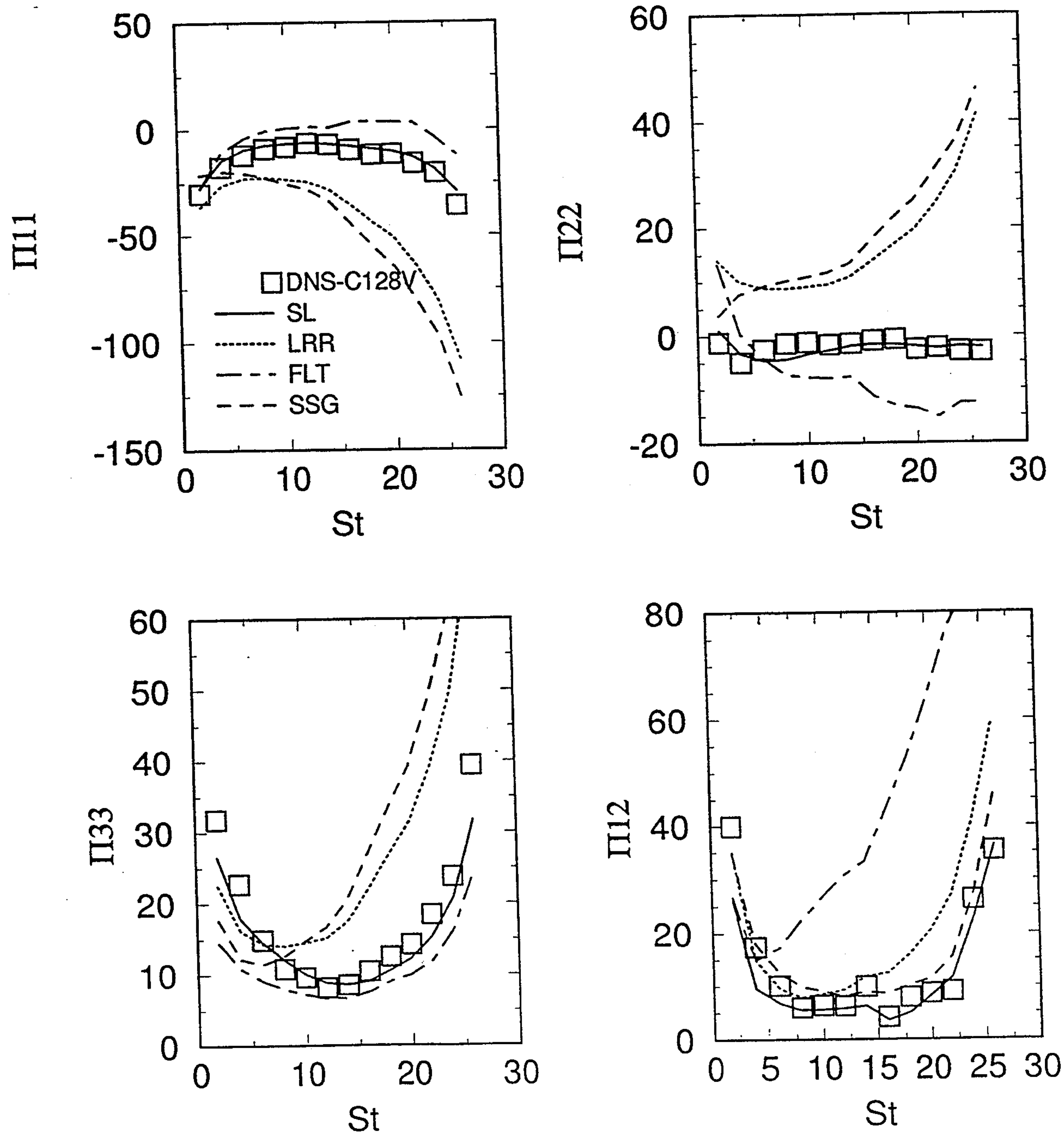

Fig. 21. Direct comparison of the rapid models with DNS data of homogeneous shear flow C128V (Rogers *et al.* [50]).

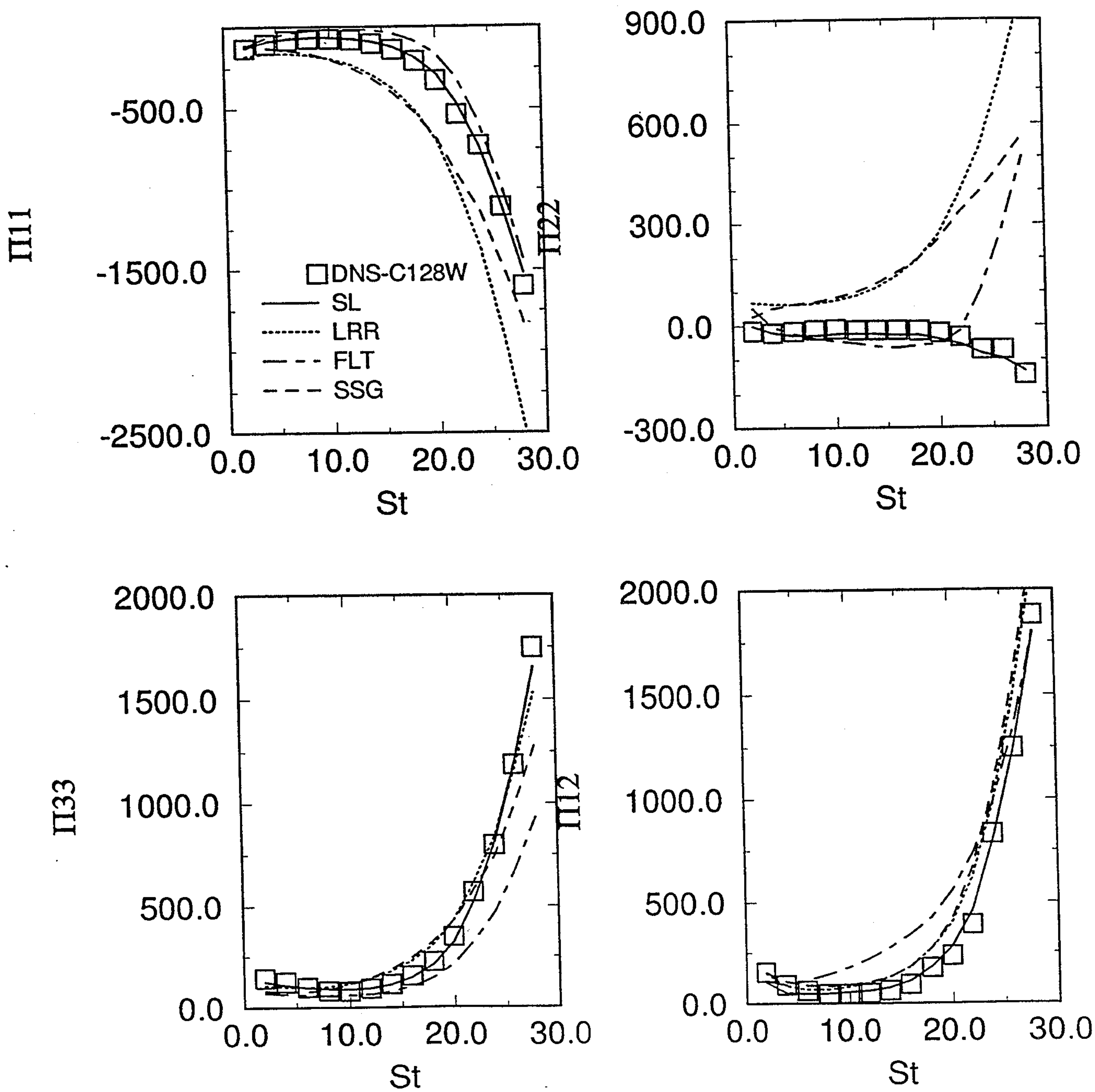

Fig. 22. Direct comparison of the rapid models with DNS data of homogeneous shear flow C128W (Rogers *et al.* [50]).

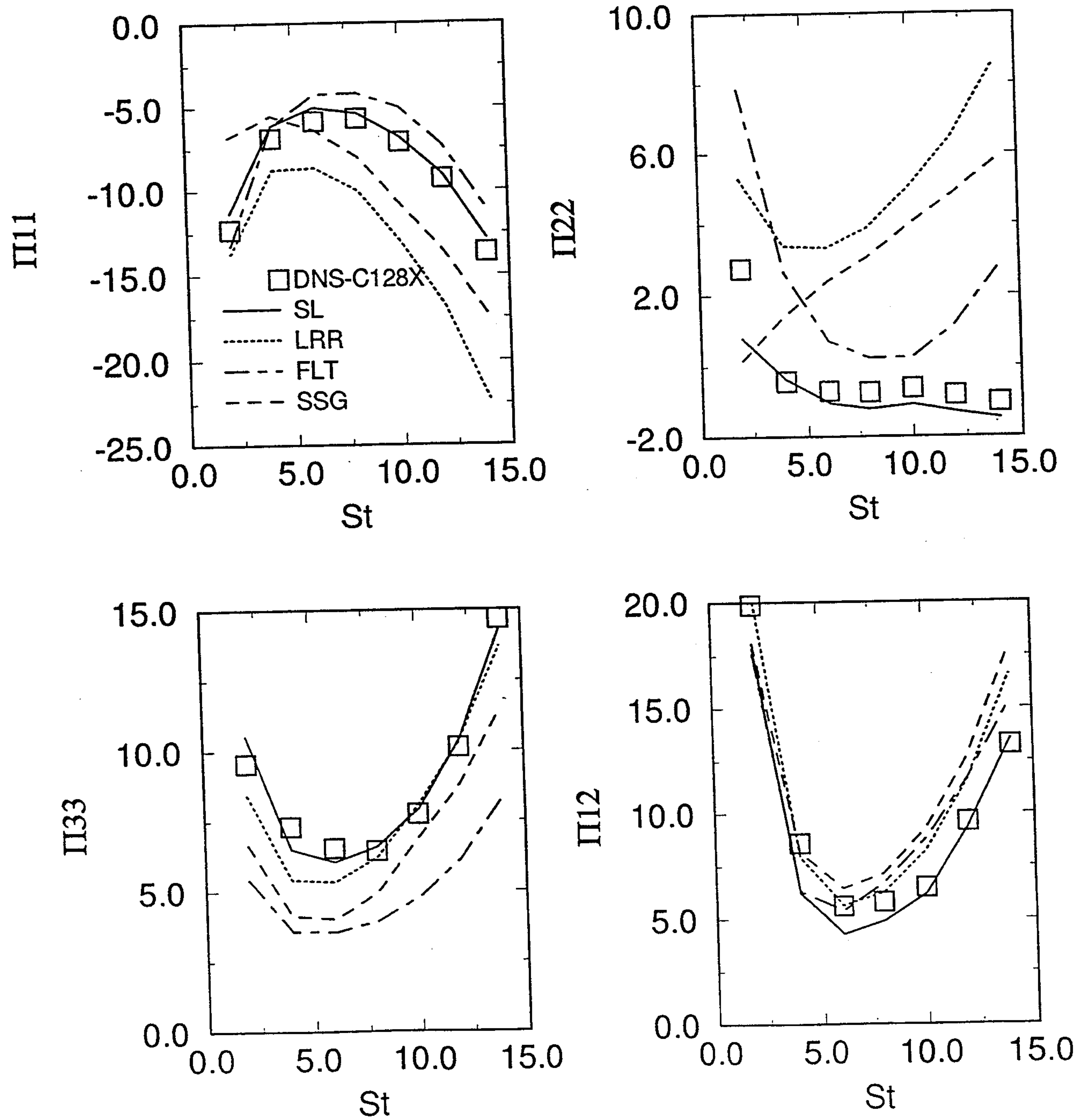

Fig. 23. Direct comparison of the rapid models with DNS data of homogeneous shear flow C128X (Rogers *et al.* [50]).

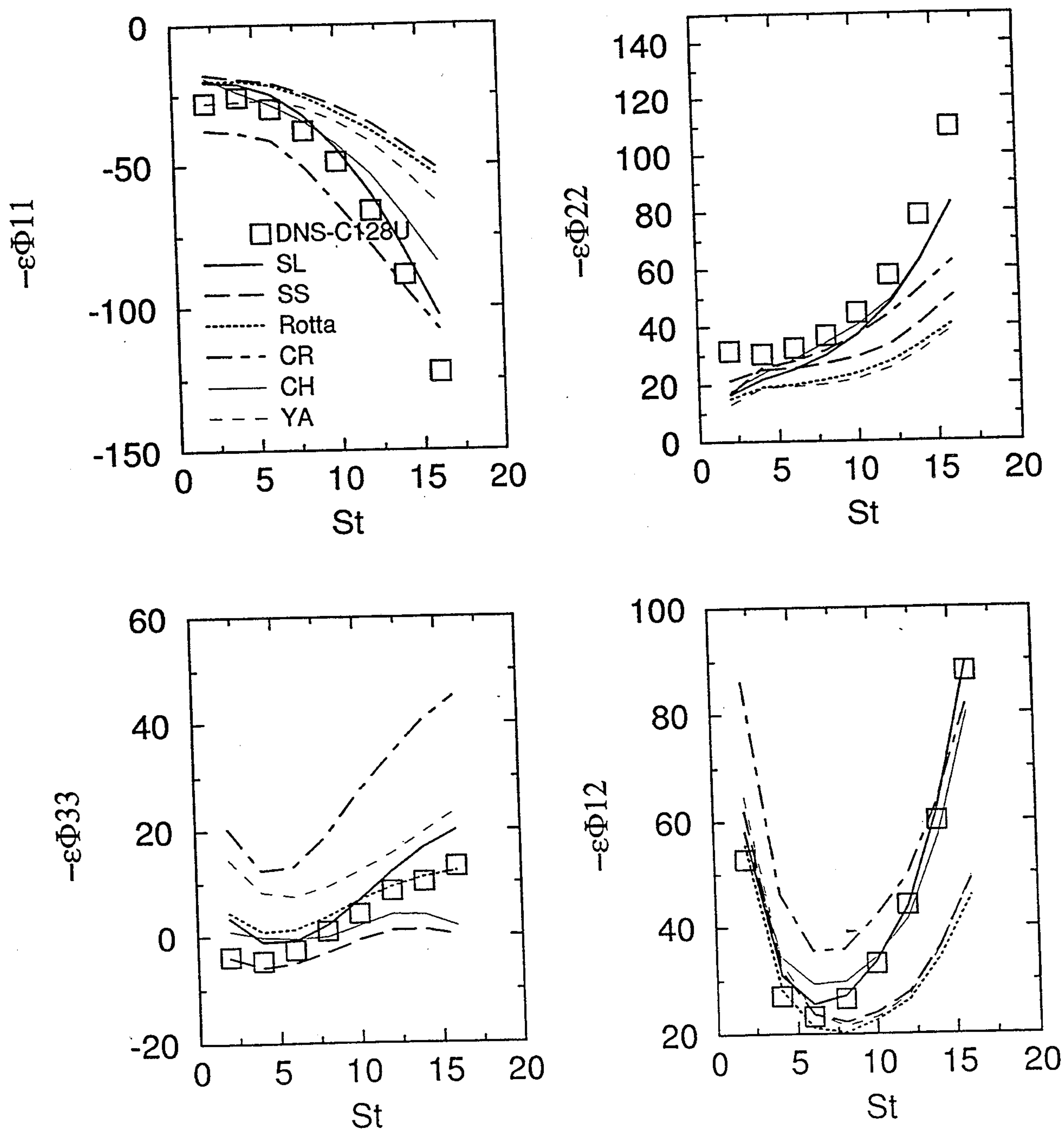

Fig. 24. Direct comparison of the return-to-isotropy models with DNS data of homogeneous shear flow C128U (Rogers *et al.* [50]).

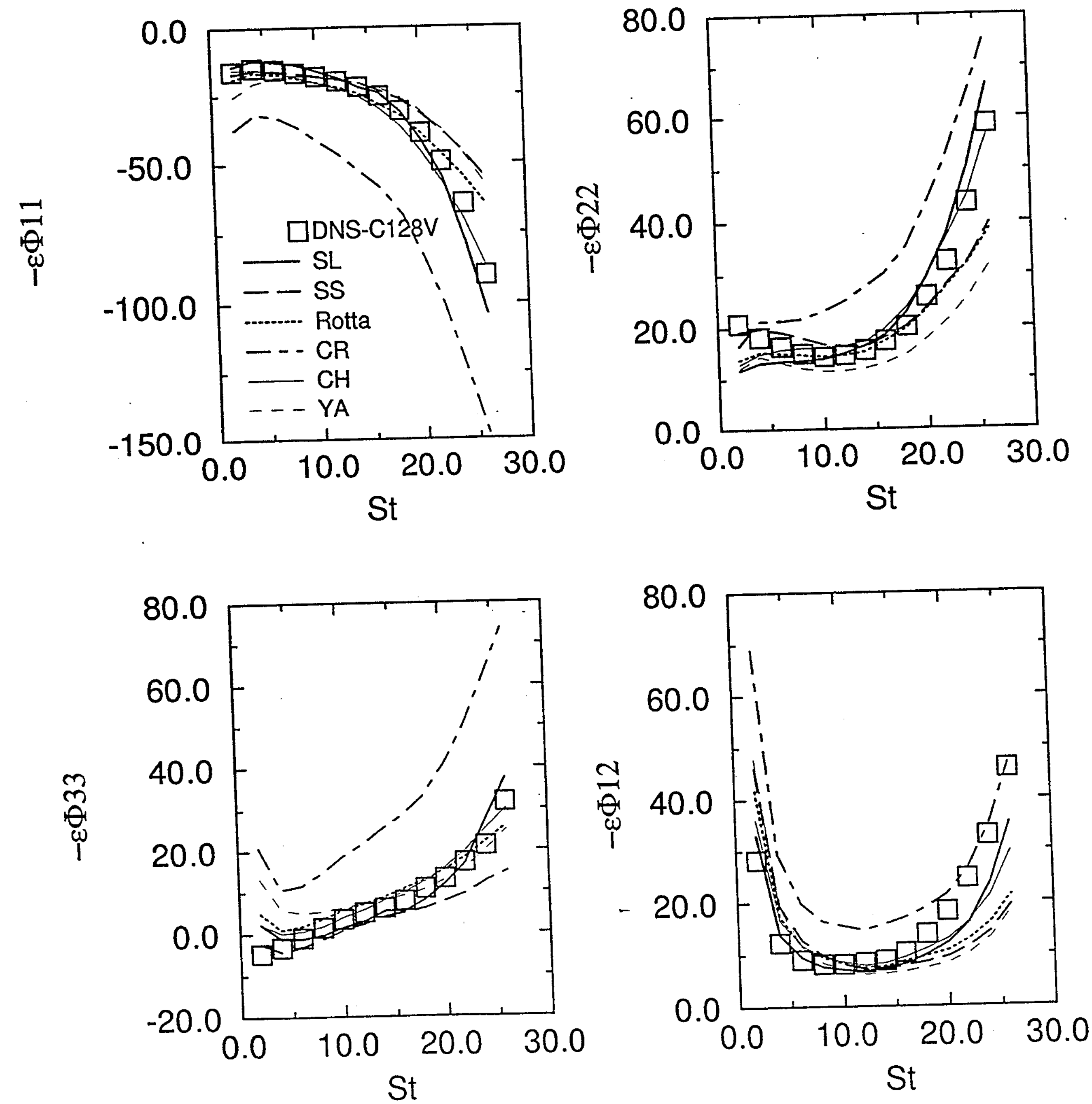

Fig. 25. Direct comparison of the return-to-isotropy models with DNS data of homogeneous shear flow C128V (Rogers *et al.* [50]).

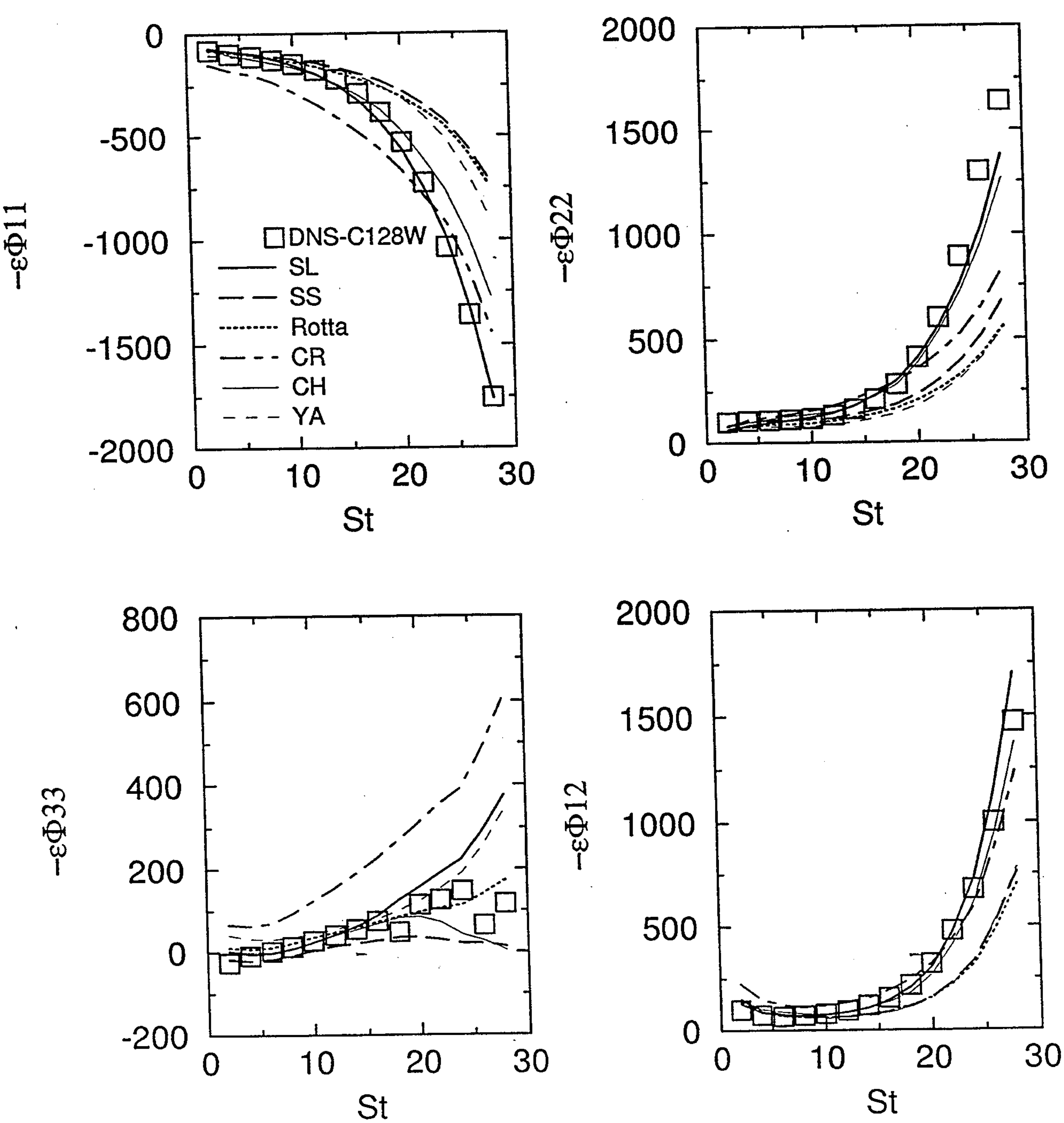

Fig. 26. Direct comparison of the return-to-isotropy models with DNS data of homogeneous shear flow C128W (Rogers *et al.* [50]).

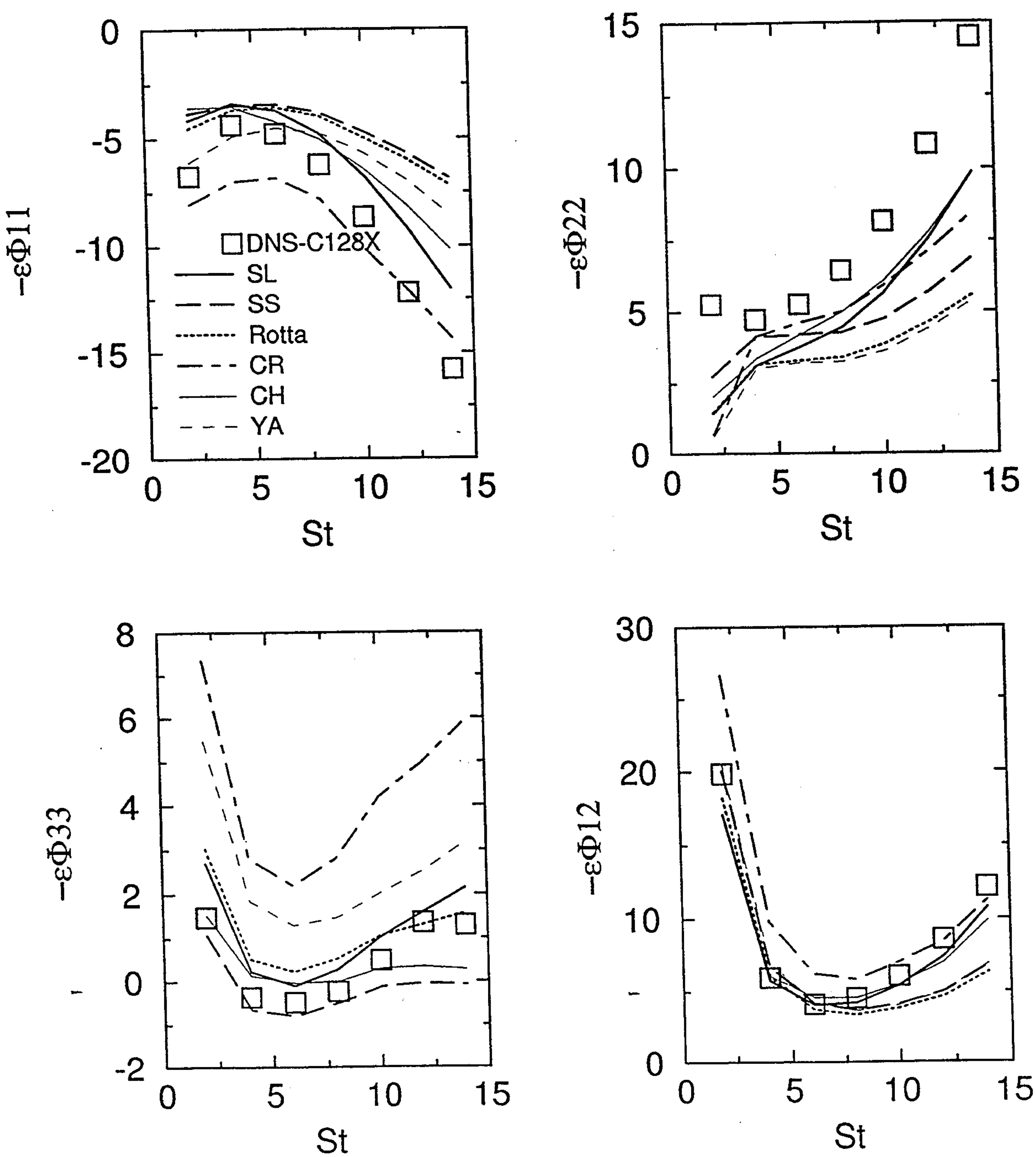

Fig. 27. Direct comparison of the return-to-isotropy models with DNS data of homogeneous shear flow C128X (Rogers *et al.* [50]).

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In this paper, we will discuss some recent model developments in two turbulence closure schemes. One is the Reynolds-stress algebraic equation model and the other is the Reynolds-stress transport equation model. Various model constraints required by the rapid distortion theory, the invariant theory and the realizability principle, etc. will be described in the model development.

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