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THE EVOLUTION OF FINITE AMPLITUDE WAVETRAINS IN PLANE CHANNEL FLOW

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Abstract

We consider a viscous incompressible fluid flow driven between two parallel plates by a constant pressure gradient. The flow is at a finite Reynolds number, with an $O(1)$ disturbance in the form of a travelling wave. A phase equation approach is used to discuss the evolution of slowly varying fully nonlinear two dimensional wavetrains. We consider uniform wavetrains in detail, showing that the development of a wavenumber perturbation is governed by Burgers equation in most cases. The wavenumber perturbation theory, constructed using the phase equation approach for a uniform wavetrain, is shown to be distinct from an amplitude perturbation expansion about the periodic flow. In fact we show that the amplitude equation contains only linear terms and is simply the heat equation. We review, briefly, the well known dynamics of Burgers equation, which imply that both shock structures and finite time singularities of the wavenumber perturbation can occur with respect to the slow scales. Numerical computations have been performed to identify areas of the {wavenumber, Reynolds number, energy} neutral surface for which each of these possibilities can occur. We note that the evolution equations will breakdown under certain circumstances, in particular for a weakly nonlinear secondary flow. Finally we extend the theory to three dimensions and discuss the limit of a weak spanwise dependence for uniform wavetrains, showing that two functions are required to describe the evolution. These unknowns are a phase and a pressure function which satisfy a pair of linearly coupled partial differential equations. The results obtained from applying the same analysis to the fully three dimensional problem are included as an appendix.

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1 Introduction

Fully nonlinear travelling wave solutions in plane Poiseuille flow (PPF) have been discussed by many authors in recent years (Zahn *et al* [17], Herbert [9], Pugh & Saffman [14]) with emphasis placed upon the secondary instability problem. This instability of the finite amplitude two-dimensional periodic flow to three-dimensional infinitesimal disturbances has been put forward traditionally as an explanation of transition. Orzsag & Patera [13] have also suggested that, even at Reynolds numbers below the subcritical minimum for these nonlinear two dimensional solutions, the time scale for ultimate decay of the disturbance is sufficiently long for a small three dimensional perturbation to grow strongly.

Subsequent work on superharmonic instability by Pugh & Saffman [14] has shown that a more complex structure is to be expected than a simple stability transition at the Reynolds number limit point of the neutral surface. They show that parameterization of the problem is important and that bifurcations to quasi–periodic flows exist at points on the upper branch of the neutral surface, leaving open the possibility of such finite amplitude solutions existing at a Reynolds number lower than the subcritical minimum for the periodic flow.

It should be noted that other approaches have been discussed that do not rely on the existence of finite amplitude two–dimensional travelling waves. Although the possibility of resonant growth caused by linear mechanisms has been recognized for some time, it is only through recent work (Gustavsson [6] [5], Gustavsson & Hultgren [7], Butler & Farrell [2], Trefethen *et al* [15]) concerning the initial value approach that the large amplification involved in three–dimensional problems has been revealed. The term "bypass" has been attached to these methods since they do not follow the more traditional idea of transition arising from two–dimensional Tollmien Schlichting waves with three dimensional effects appearing only at a secondary instability stage.

In this discussion we return to the finite amplitude travelling wave solutions described at the beginning of this section, and subsequently develop an evolution equation for a phase instability of the $O(1)$ flow. Since these nonlinear solutions are used frequently in some areas of both theory and computation, it seems sensible to try and discover something of their stability and evolution. We show, in fact, that uniform wavetrains will not be observed under certain classes of initial condition as they are susceptible to slow scale effects, with the wavenumber developing both singularities and shock structures after a finite time in the slow scale.

The method we use to determine the evolution equation is based upon the phase equation technique applied by numerous authors to Bénard convection problems with $O(1)$ amplitudes (e.g. Newell , Passot & Lega [12]). These same methods have been applied to wave problems by Howard & Kopell [10], Whitham [16], subsequently applied explicitly to the Ginzburg-Landau equation by Bernoff [1] and utilized in an investigation of boundary layer instability theory by Hall [8].

A detailed discussion of the phase equation method is given by Hall [8], Bernoff [1] and is not repeated here. The essential idea is that we have a finite amplitude wavetrain solution, which is locally periodic in space and time, allowing wavenumber and frequency to be functions varying on appropriate slow scales. The resulting equations of motion can then be rewritten in terms of the new scales and a phase function that is related to the wavenumber and frequency,

through which it also satisfies a conservation equation. An expansion in terms of the slow scale parameter will now yield a leading order system that is a nonlinear eigenvalue problem. This relationship determines the local frequency of the wavetrain as a function of local wavenumber, Reynolds number (and indirectly amplitude) yielding both supercritical and subcritical equilibria for PPF. The next order problem will then provide a linear inhomogeneous system that determines the frequency correction term through a solvability condition; this technique also allows for continuation to higher orders. Now since the wavetrain evolves according to the phase conservation equation, we can, by expanding appropriately, give a slow scale asymptotic approximation to the evolution equation.

In §2 of this discussion the above technique is applied explicitly to the finite Reynolds number *two*-dimensional PPF problem. In §2.1 we discuss the implications of the phase equation theory when applied to the stability of a *uniform* two-dimensional wavetrain. We show that the stability of a small wavenumber perturbation is governed by Burgers equation,

$$\Delta_\tau + \Delta\Delta_\zeta = \pm\Delta_{\zeta\zeta}\,, \tag{1}$$

for $O(1)$ problems that correspond to distinct points on either the upper or lower branch of the neutral surface and away from the linear neutral curve. In §3 we discuss the stability of the $O(1)$ flow to an amplitude perturbation, showing that, for these length scales, nonlinear terms are not introduced and that the amplitude equation is simply the heat transfer equation. Section 4 provides a short description of the numerical methods involved in the uniform stability calculations, presenting the $O(1)$ results which are consistent with those of other authors Herbert [9], Pugh & Saffman [14]. Computational results, which show the behaviour of the viscous diffusion term from Burgers equation, are presented for differing leading order problems. Section 5 returns to the analysis of a uniform wavetrain and briefly discusses how the theory breaks down for weakly nonlinear secondary flow solutions. In §6 we redevelop the phase equation theory for a three–dimensional problem and consider the stability of wavenumber perturbations in the limit of a weak spanwise dependence, finally in §7 we discuss the implications and future extensions of the work.

2 Formulation of the phase equation approach

We wish to consider a finite amplitude solution to the plane Poiseuille flow (PPF) problem, in the form of a travelling wave, then allow for a slow modulation on the new scales,

$$X = \delta x \quad \text{and} \quad T = \delta t\,. \tag{2}$$

We now use the methods presented in Howard & Kopell [10], Bernoff [1] and applied to asymptotic suction boundary layer flow by Hall [8]. This analysis follows closely the finite Reynolds number case of Hall [8], except for a few technical differences associated with bounded domains, namely the need for a pressure eigenfunction term (discussed later). We first introduce a phase function, $\Theta(X,T) = \delta\theta(x,t)$, which allows a definition of the local frequency and wavenumber as

$$\alpha = \frac{\partial\Theta}{\partial X}\,, \quad \Omega = -\frac{\partial\Theta}{\partial T}\,, \tag{3}$$

where $\alpha = \alpha(X,T)$ and $\Omega = \Omega(X,T)$ are allowed to vary on the slow scales. Thus the partial derivatives transform as

$$\frac{\partial}{\partial x} \to \alpha\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial X}\,, \quad \frac{\partial}{\partial t} \to -\Omega\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial T}\,, \tag{4}$$

and a conservation of phase condition must be satisfied,

$$\frac{\partial\alpha}{\partial T} + \frac{\partial\Omega}{\partial X} = 0\,. \tag{5}$$

We can now develop a perturbation scheme about the fully nonlinear leading order solution by introducing a slow scale expansion of the stream function,

$$\psi = \hat{\psi}_0 + \delta\psi_1 + \ldots\,, \tag{6}$$

which forces a similar expansion for the frequency

$$\Omega = \Omega_0 + \delta\Omega_1 + \ldots\,; \tag{7}$$

where $\hat{\psi}_0 = \bar{\Psi} + \psi_0$ the basic flow potential plus a leading order spatially periodic flow. We shall use the vorticity equation formulation to describe the flow,

$$\frac{\partial(\nabla^2\psi)}{\partial t} + \frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)} - \frac{1}{Re}\nabla^4\psi = 0\,, \tag{8}$$

with a Reynolds number, Re, defined as $\frac{\bar{U}^* h}{\nu}$; where $\bar{U}^*$ is the centerline velocity, $2h$ the plate separation and ν the kinematic viscosity. Thus at leading order, using the above expansions, we obtain

$$-\Omega_0\hat{\nabla}^2\hat{\psi}_{0\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\hat{\psi}_{0\theta} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\hat{\psi}_{0y} - \frac{1}{Re}\hat{\nabla}^4\hat{\psi}_0 = 0\,, \tag{9}$$

with subscripts $\{\theta, y, \ldots\}$ denoting the respective derivatives, where it is unambiguous to do so, and

$$\hat{\nabla}^2 \equiv \alpha^2\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial y^2}\,. \tag{10}$$

The boundary conditions are simply impermeability and no–slip at the parallel boundaries. In terms of the streamfunction (at this order) this gives

$$\hat{\psi}_{0\theta} = \hat{\psi}_{0y} = 0 \text{ at } y = \pm 1\,. \tag{11}$$

Similar situations have been discussed previously by Herbert [9], Pugh & Saffman [14], and this nonlinear eigenvalue problem can be solved using the same numerical techniques, here we have an eigenrelation

$$\Omega_0 = \Omega_0(\alpha, Re)\,, \tag{12}$$

which defines a "neutral surface" in $\{\alpha, Re, \text{Amplitude}\}$ parameter space.

Now at next order, after some rearrangement, we obtain

$$L\{\psi_1\} = \Big[\Omega_0(2\alpha\hat{\psi}_{0\theta\theta\alpha} + \hat{\psi}_{0\theta\theta}) + \frac{\partial\Omega_0}{\partial\alpha}\frac{\partial\hat{\nabla}^2\hat{\psi}_0}{\partial\alpha} + \alpha\frac{\partial(\hat{\psi}_0, 2\alpha\hat{\psi}_{0\theta\alpha} + \hat{\psi}_{0\theta})}{\partial(\theta, y)} + \frac{\partial(\hat{\psi}_0, \hat{\nabla}^2\hat{\psi}_0)}{\partial(\alpha, y)} + \frac{1}{Re}(4\alpha\hat{\nabla}^2\hat{\psi}_{0\theta\alpha} + 4\alpha^2\hat{\psi}_{0\theta\theta\theta} + 2\hat{\nabla}^2\hat{\psi}_{0\theta})\Big]\frac{\partial\alpha}{\partial X} + \Omega_1\hat{\nabla}^2\hat{\psi}_{0\theta}\,, \tag{13}$$

with boundary conditions $\psi_{1y} = \psi_{1\theta} = 0$ at $y = \pm 1$. To obtain the above form, (13), we have introduced the operator L defined as

$$L \equiv -\Omega_0\hat{\nabla}^2\frac{\partial}{\partial\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\frac{\partial}{\partial\theta} + \alpha\hat{\nabla}^2\hat{\psi}_{0\theta}\frac{\partial}{\partial y} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\frac{\partial}{\partial y} - \alpha\hat{\nabla}^2\hat{\psi}_{0y}\frac{\partial}{\partial\theta} - \frac{1}{Re}\hat{\nabla}^4\,, \tag{14}$$

and the $\partial/\partial T$ term has been replaced using the conservation of phase as

$$\frac{\partial}{\partial T} \to -\frac{\partial}{\partial\alpha}\left(\Omega_0 + \delta\Omega_1 + \dots\right)\frac{\partial\alpha}{\partial X}\frac{\partial}{\partial\alpha}\,. \tag{15}$$

We now consider the mean flow, which from the Navier-Stokes is governed by

$$\frac{1}{Re}\frac{\partial^2 u_{00}}{\partial y^2} = \sum_{j=1}^{\infty}\left(v_{0j}\frac{\partial u_{0j}^{(c)}}{\partial y} + v_{0j}^{(c)}\frac{\partial u_{0j}}{\partial y}\right) + \frac{\partial q_{-1}}{\partial X}\,. \tag{16}$$

Here we have expanded the velocity field and pressure as

$$\underline{\mathbf{u}} = \hat{\underline{\mathbf{u}}}_0(X, T, \theta, y) + \delta\underline{\mathbf{u}}_1(X, T, \theta, y) + \dots\,, \tag{17}$$

$$p = [GX + q_{-1}(X, T)]\delta^{-1} + [p_0(X, T, \theta, y) + q_0(X, T)] + \dots\,, \tag{18}$$

with

$$\hat{\underline{\mathbf{u}}}_0 = (\bar{U}, 0)^T + \underline{\mathbf{u}}_{00} + \sum_{j=1}^{\infty}\left\{\underline{\mathbf{u}}_{0j}e^{ij\theta} + \underline{\mathbf{u}}_{0j}^{(c)}e^{-ij\theta}\right\}, \quad \underline{\mathbf{u}}_{0j} = (u_{0j}, v_{0j})^T\,, \tag{19}$$

where $\bar{U}$ denotes the non-dimensionalized basic flow, $G = -\frac{2}{Re}$ the basic driving pressure gradient and $^{(c)}$ a complex conjugation. At next order we obtain

$$\frac{1}{Re}\frac{\partial^2 u_{10}}{\partial y^2} = \frac{\partial p_{00}}{\partial X} + \frac{\partial\Omega_0}{\partial\alpha}\frac{\partial u_{00}}{\partial\alpha}\alpha_X + \bar{U}\frac{\partial u_{00}}{\partial X} + \frac{\partial\bar{U}}{\partial y}v_{10} + \sum_{j=-\infty}^{\infty}\left(u_{0j}\frac{\partial u_{0j}^{(c)}}{\partial X} + v_{0j}\frac{\partial u_{1j}^{(c)}}{\partial y} + v_{1j}\frac{\partial u_{0j}^{(c)}}{\partial y}\right) + \frac{\partial q_0}{\partial X}\,, \tag{20}$$

where p_{00} is determined from the $O(1)$ problem and

$$\underline{\mathbf{u}}_1 = \underline{\mathbf{u}}_{10} + \sum_{j=1}^{\infty}\left\{\underline{\mathbf{u}}_{1j}e^{ij\theta} + \underline{\mathbf{u}}_{1j}^{(c)}e^{-ij\theta}\right\}, \quad \underline{\mathbf{u}}_{1i} = (u_{1i}, v_{1i})^T\,; \tag{21}$$

the summation term in (20) is performed with negative subscripts denoting a complex conjugation. The equation of continuity at this order, $O(\delta)$, is

$$\alpha\frac{\partial u_1}{\partial\theta}+\frac{\partial u_0}{\partial X}+\frac{\partial v_1}{\partial y}=0\,, \tag{22}$$

which, when considering the mean flow terms only, reduces to

$$\frac{\partial v_{10}}{\partial y}=-\frac{\partial u_{00}}{\partial X}\,, \tag{23}$$

a first order equation for the mean flow correction to v, required to satisfy the impermeability conditions at *both* walls. This difficulty was anticipated earlier and is resolved by the introduction of a further, slow scale dependent, pressure expansion

$$q_{-1}(X,T)\delta^{-1}+q_0(X,T)+\ldots\,, \tag{24}$$

producing the extra term $\partial q_{-1}/\partial X$ in (16) which is chosen to satisfy (23). Thus, for impermeability at both boundaries, we must satisfy

$$\int_{-1}^{+1}\frac{\partial u_{00}}{\partial X}\,\mathrm{d}y=0\,, \tag{25}$$

which fixes the streamwise flux through the channel and hence determines q_{-1} as a function of X at given T. This is equivalent to solving the vorticity equation for the mean flow correction with the same boundary conditions $\psi_{00}=\partial\psi_{00}/\partial y=0$ at $y=\pm1$, where ψ_{00} is the part of the stream function having zero mean with respect to the phase, θ.

Now we have an $O(\delta^0)$ problem that can be solved numerically to give $\Omega_0(\alpha,Re)$ and an expansion in terms of δ giving a further system (13), which is used to compute Ω_1 at a given neutral surface point. The homogeneous form of (13) is solved by $\psi_{0\theta}$ (corresponding to the existence of a translationally invariant solution, since any arbitrary constant may be added to the phase) and so Ω_1 is determined by a solvability condition at $O(\delta)$. The form of (13) is

$$\alpha\frac{\partial}{\partial\theta}(A\underline{\mathbf{q}})+\frac{\partial}{\partial y}(B\underline{\mathbf{q}})+C\underline{\mathbf{q}}=\underline{\mathbf{H}}\,, \tag{26}$$

so premultiplying by the adjoint vector, $\underline{\mathbf{r}}^T=(r_1,\ldots,r_6)$, and integrating by parts gives

$$-\alpha\frac{\partial}{\partial\theta}(A^T\underline{\mathbf{r}})-\frac{\partial}{\partial y}(B^T\underline{\mathbf{r}})+C^T\underline{\mathbf{r}}=\underline{\mathbf{0}} \tag{27}$$

as the adjoint equation for the homogeneous form of (26), with

$$\underline{\mathbf{q}}=(\psi_1,\alpha\psi_{1\theta},\psi_{1y},\hat{\nabla}^2\psi_1,\alpha\hat{\nabla}^2\psi_{1\theta},\hat{\nabla}^2\psi_{1y})^T\,, \tag{28}$$

$$A=\begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&-\frac{1}{Re}&0\end{bmatrix}\,, \tag{29}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{Re} \end{bmatrix}, \tag{30}$$

and

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\hat{\nabla}^2\hat{\psi}_{0y} & \alpha\hat{\nabla}^2\hat{\psi}_{0\theta} & 0 & (\hat{\psi}_{0y} - \frac{\Omega_0}{\alpha}) & -\alpha\hat{\psi}_{0\theta} \end{bmatrix}. \tag{31}$$

The boundary conditions are also determined by the above process and are easily shown to be

$$r_5 = r_6 = 0 \text{ at } y = \pm 1\,, \tag{32}$$

plus a periodicity condition. Thus the solvability condition is

$$\int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} r_6 H_6 \, \mathrm{d}y \, \mathrm{d}\theta = 0\,, \tag{33}$$

giving

$$\Omega_1 = \frac{\partial\alpha}{\partial X}\left\{\int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} M(\theta, y)\, \mathrm{d}y\, \mathrm{d}\theta\right\}\left\{\int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} -\hat{\nabla}^2\psi_{0\theta}\, \mathrm{d}y\, \mathrm{d}\theta\right\}^{-1}, \tag{34}$$

where M is the term in the square brackets in (13) and $\underline{\mathbf{H}} = (0, \ldots, 0, H_6)^T$.

Given Ω_1 we can now write down the phase conservation condition upto $O(\delta)$ in the form

$$\frac{\partial\alpha}{\partial T} + \frac{\partial\Omega_0}{\partial\alpha}\frac{\partial\alpha}{\partial X} = \delta\frac{\partial}{\partial X}(-\Omega_1) + \ldots\,. \tag{35}$$

This is an evolution equation for $\alpha = \alpha(X, T)$, which we could *in principle* continue to any order through this expansion scheme; rather than solving this directly we shall restrict our attention to a somewhat simpler problem.

2.1 Stability of Uniform Wavetrains

Here we wish to discuss the stability of a fully nonlinear uniform wavetrain to a small, slow scale, wavenumber perturbation. Hence, following the above expansion scheme, we obtain an $O(\delta^0)$ problem with a phase function of leading order form

$$\theta_0 = \alpha_0 x - \Omega_0(\alpha_0)t\,, \tag{36}$$

where $\{\alpha_0, Re\}$ defines a point upon the neutral surface at $O(1)$ disturbance energy, with associated frequency Ω_0. If we now perturb the wavenumber of this uniform solution with a slowly varying function Δ,

$$\alpha = \alpha_0 + \Delta(X, T)\,, \tag{37}$$

and perform the above analysis, the evolution equation reduces to

$$\frac{\partial}{\partial T}\Delta(X,T)+\frac{\partial\Omega_0}{\partial\alpha}\frac{\partial}{\partial X}\Delta(X,T)+\delta\frac{\partial\Omega_1}{\partial X}+\delta^2\frac{\partial\Omega_2}{\partial X}+\ldots=0\,. \tag{38}$$

Now we introduce a Taylor series expansion for Ω_i about the uniform solution, and a transformation of the streamwise coordinate to a new frame of reference, moving with speed $\Omega_0'(\alpha_0)$, to obtain

$$\eta\frac{\partial\Delta}{\partial\tau}+\frac{\partial^2\Omega_0(\alpha_0)}{\partial\alpha^2}\Delta\frac{\partial\Delta}{\partial\xi}+\delta\Phi(\alpha_0)\frac{\partial^2\Delta}{\partial\xi^2}=O\Big(\delta\Delta^2\frac{\partial\Delta}{\partial\xi},\delta\Delta\frac{\partial^2\Delta}{\partial\xi^2},\delta\frac{\partial\Delta^2}{\partial\xi}\Big)\,. \tag{39}$$

Here we have defined τ,ξ and $\Phi(\alpha)$ by

$$\tau = \eta T\,, \tag{40}$$

$$\xi = X-\Omega_0'(\alpha_0)T\,, \tag{41}$$

$$\Omega_1 = \frac{\partial\alpha}{\partial X}\Phi(\alpha)\,, \tag{42}$$

so that

$$\frac{\partial}{\partial t} \rightarrow \eta\frac{\partial}{\partial\tau}-\Omega_0'(\alpha_0)\frac{\partial}{\partial\xi}\,, \tag{43}$$

$$\frac{\partial}{\partial X} \rightarrow \frac{\partial}{\partial\xi}\,. \tag{44}$$

We therefore obtain a leading order balance for $\eta=\delta$ and $\Delta\sim O(\delta)$.

Now an appropriate rescaling will reduce this evolution equation to the standard form for Burgers equation, namely

$$\Delta_\tau+\Delta\Delta_\xi=\mu\Delta_{\xi\xi}\,, \tag{45}$$

which has well known properties. This approach again follows that given by both Hall [8] and Bernoff [1], where Burgers equation has been derived by applying this technique to both the asymptotic suction boundary layer and the Ginzburg-Landau equation. Whitham [16] has discussed the dynamics of this equation in detail, it has an exact solution via the Cole-Hopf transformation,

$$\Delta=-2\mu\frac{\partial}{\partial\xi}\log(\psi)\,, \tag{46}$$

which removes the nonlinear term reducing the evolution equation to the form of the heat diffusion equation,

$$\psi_\tau=\mu\psi_{\xi\xi}+C(\tau)\psi\,, \tag{47}$$

where $C(\tau)$ is set to be identically zero since this merely corresponds to a scaling of the dependent variable. Now for the initial value problem, with known $\Delta(\xi,0)=F(\xi)$, it is possible to obtain the analytical solution

$$\psi(\xi,\tau)=\frac{1}{\sqrt{4\pi\mu\tau}}\int_{-\infty}^{+\infty}\exp\Big\{-\frac{(\xi-\eta)^2}{4\mu\tau}-\frac{1}{2\mu}\int_0^\eta F(\bar\eta)\,\mathrm{d}\bar\eta\Big\}\,\mathrm{d}\eta\ , \tag{48}$$

through an application of Laplace transform methods. Therefore the solution is

$$\Delta(\xi,\tau) = \left\{\int_{-\infty}^{+\infty} \frac{\xi-\eta}{\tau} e^{-D/2\mu}\,\mathrm{d}\eta\right\}\left\{\int_{-\infty}^{+\infty} e^{-D/2\mu}\,\mathrm{d}\eta\right\}^{-1}, \tag{49}$$

where

$$D = \int_0^{\eta} F(\bar{\eta})\,\mathrm{d}\bar{\eta} + \frac{(\xi-\eta)^2}{2t}. \tag{50}$$

This can support both shock structures and singularities at finite times. For a positive diffusive term on the right of (45) we have a bounded solution which will decay for a localized/periodic disturbance. For a negative right hand side the solution is diffusively unstable and will become singular at finite time, indicating that the slow variation assumption is no longer appropriate and a return to the full equations of motion is required. Weak shock structures are discussed by Bernoff [1] who notes that for a small monotonic wavenumber variation such that

$$\lim_{x\to-\infty} \alpha = \alpha_- \tag{51}$$

and

$$\lim_{x\to+\infty} \alpha = \alpha_+\,, \tag{52}$$

then Burgers equation applies as a leading order form for the evolution equation if

$$\Delta\alpha = \alpha_+ - \alpha_- \ll 1\,. \tag{53}$$

Here we also require that the unmodulated wavetrain corresponds to a distinct point on either the upper or lower branch of the neutral surface at $O(1)$ amplitude. Now this wavenumber variation will eventually become concentrated into a weak shock structure of width $O(\frac{1}{\Delta\alpha})$ in the unscaled streamwise coordinate, if

$$\Omega_0'(\alpha_0)(\alpha_+ - \alpha_-) < 0\,, \tag{54}$$

moving with speed

$$c = \frac{\Omega(\alpha_+) - \Omega(\alpha_-)}{\Delta\alpha}\,; \tag{55}$$

a discretized form of the group velocity. If this variation in wavenumber ($\Delta\alpha$) increases then the slow scale assumptions are eventually lost (as with the finite time singularity case) and the evolution of the wave system is governed by the full equations, namely the Navier–Stokes.

In this case (stability of uniform wavetrains) we should also note that the leading order problem is simplified since there is no slow scale dependency for the wavenumber. Thus the effect of the extra pressure term $\partial q_{-1}/\partial X$ (now a constant) is to induce an additional parabolic velocity profile into the mean flow correction and corresponds to a scaling of the Reynolds number at fixed amplitude/wavenumber. So the condition (25), which fixes the flux through the channel, effectively determines a unique parameterization of the problem (as discussed by Pugh & Saffman [14]) in a self-consistent manner.

3 An Amplitude Perturbation Approach

We now show how the phase equation method described previously is distinct from a more typical amplitude perturbation approach. In this method we solve the same leading order problem for the uniform wavetrain, but it is now perturbed by an eigenfunction with a slowly varying amplitude $B = B(X,T)$. Again we introduce the slow scales

$$X = \delta x\,, \tag{56}$$

and

$$T = \delta t\,, \tag{57}$$

together with

$$\hat{X} = (X - c_g T)\,, \tag{58}$$

a new moving coordinate system, and a further timescale

$$\hat{T} = \delta T\,. \tag{59}$$

This slower timescale is known from the previous section but could otherwise be determined from the final solvability condition. Here c_g is a group velocity and given values for α and Ω we can expand in terms of δ and a phase variable $\theta = \alpha x - \Omega t$. Now, seeking a solution analogous to that in Hall [8], we expand the stream function as

$$\psi = \hat{\psi}_0 + \delta\psi_1 + \ldots\,, \tag{60}$$

where

$$\hat{\psi}_0 = \bar{\Psi} + \psi_0\,, \tag{61}$$

$$\psi_1 = B(\hat{X},\hat{T})\alpha\frac{\partial\psi_0}{\partial\theta}\,, \tag{62}$$

with $\bar{\Psi}$ corresponding to the basic parallel flow, and

$$\psi_n = \frac{[B(\hat{X},\hat{T})\alpha]^n}{n!}\frac{\partial^n\psi_0}{\partial\theta^n} + \tilde{\psi}_n\ ,\ n \geq 2\,. \tag{63}$$

We now return to the vorticity equation (8) and substitute the above expansions along with

$$\frac{\partial}{\partial x} \rightarrow \alpha\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial\hat{X}}\,, \tag{64}$$

$$\frac{\partial}{\partial t} \rightarrow -\Omega\frac{\partial}{\partial\theta} - \delta c_g\frac{\partial}{\partial\hat{X}} + \delta^2\frac{\partial}{\partial\hat{T}}\,, \tag{65}$$

$$\hat{\nabla}^2 \equiv \frac{\partial^2}{\partial y^2} + \alpha^2\frac{\partial^2}{\partial\theta^2}\,, \tag{66}$$

to give a leading order the form of the vorticity equation

$$O(\delta^0): \quad -\Omega\hat{\nabla}^2\hat{\psi}_{0\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\hat{\psi}_{0\theta} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\hat{\psi}_{0y} - \frac{1}{Re}\hat{\nabla}^4\hat{\psi}_0 = 0\,. \tag{67}$$

At next order

$$O(\delta): \quad -\Omega\hat{\nabla}^2\psi_{1\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\psi_{1\theta} + \alpha\psi_{1y}\hat{\nabla}^2\hat{\psi}_{0\theta} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\psi_{1y} - \alpha\psi_{1\theta}\hat{\nabla}^2\hat{\psi}_{0y} - \frac{1}{Re}\hat{\nabla}^4\psi_1 = 0\,, \tag{68}$$

with solution (62), which is an amplitude perturbation of the underlying periodic flow; in this formulation ψ_0 is *independent* of the slow scales $\hat{X}, \hat{T}$. The required group velocity is now determined from the next order system, which can be rearranged more clearly as

$$L_{O(\delta)}\{\tilde{\psi}_2\} = \alpha\Big[2\alpha\Omega\hat{\psi}_{0\theta\theta\theta} + c_g\hat{\nabla}^2\hat{\psi}_{0\theta} - \hat{\psi}_{0y}(2\alpha^2\hat{\psi}_{0\theta\theta\theta} + \hat{\nabla}^2\hat{\psi}_{0\theta}) + \hat{\psi}_{0\theta}(2\alpha^2\hat{\psi}_{0\theta\theta y} + \hat{\nabla}^2\hat{\psi}_{0y}) + \frac{4\alpha}{Re}\hat{\nabla}^2\hat{\psi}_{0\theta\theta}\Big]B_{\hat{X}}\,, \tag{69}$$

with the $L_{O(\delta)}$ operator defined by the $O(\delta)$ equation and once terms proportional to B^2 have been eliminated by taking $\partial/\partial\theta$ of (67). We observe that $\tilde{\psi}_2 = \alpha B_{\hat{X}}\psi_{0\alpha}$ and as expected c_g corresponds to $\partial\Omega_0(\alpha)/\partial\alpha$ in the phase equation approach, this follows by taking $\partial/\partial\alpha$ of (67) . Note that an additional multiple of the homogeneous solution to $\tilde{\psi}_2$ will not alter the solvability condition at next order, but will contribute to an amplitude equation at higher order.

We also must remember that the additional pressure term (q_{-1}, discussed in §2) is still required in the leading order mean flow problem. Obviously a similar pressure term, $\partial q_{-1}/\partial\alpha$, is necessary at $O(\delta^2)$ but it is not until $O(\delta^3)$ that the condition determining q_{-1} is obtained. Now continuity of mass, at $O(\delta^3)$, requires the same constant mass flow condition to be satisfied for impermeability of the boundaries. Thus we determine q_{-1}, appearing in the leading order problem, in the same manner as discussed in the phase equation approach.

The same process can be repeated for the problem at next order which becomes

$$L_{O(\delta)}\{\tilde{\psi}_3\} = BB_{\hat{X}}\Big\{K(\theta,y)\Big\} + B_{\hat{T}}\Big\{-\alpha\hat{\nabla}^2\psi_{0\theta}\Big\} + B_{\hat{X}\hat{X}}\Big\{M(\theta,y)\alpha\Big\}\,, \tag{70}$$

after again eliminating the B^3 terms by taking $\partial^2/\partial\theta^2$ of (67). The M expression, in (70), is as defined previously in the phase equation analysis with $\{\Omega_0, \partial_\alpha\Omega_0\}$ replaced by $\{\Omega, c_g\}$ and K is given by

$$K = 2\alpha^3\Omega\psi_{0\theta\theta\theta\theta} + c_g\alpha^2\hat{\nabla}^2\psi_{0\theta\theta} + 2\alpha^4\frac{\partial(\psi_0, \psi_{0\theta\theta\theta})}{\partial(\theta,y)} + \alpha^2\frac{\partial(\psi_0, \hat{\nabla}^2\psi_{0\theta})}{\partial(\theta,y)} + \alpha^3\frac{\partial(\psi_{0\theta}, \hat{\nabla}^2\psi_{0\alpha})}{\partial(\theta,y)} + 2\alpha^4\frac{\partial(\psi_{0\theta}, \psi_{0\theta\theta})}{\partial(\theta,y)} + \alpha^2\frac{\partial(\psi_{0\theta}, \hat{\nabla}^2\psi_0)}{\partial(\theta,y)} + \alpha^3\frac{\partial(\psi_{0\alpha}, \hat{\nabla}^2\psi_{0\theta})}{\partial(\theta,y)} - \frac{1}{Re}\hat{\nabla}^2\psi_{0\theta\theta\theta}\,. \tag{71}$$

Although at first sight (70) appears to be Burgers equation, reproduced through an amplitude perturbation approach, it does in fact reduce to a simpler form once we have observed that a particular solution is available to remove the nonlinear term from the solvability condition;

$$L_{O(\delta)}\Big\{\alpha^2\frac{\partial\psi_{0\alpha}}{\partial\theta}\Big\} = K\,. \tag{72}$$

Similarly by an inductive method we can show that a rescaling does not reintroduce nonlinear terms (which will be of the form $B^n \partial B/\partial \hat{X}$) into the amplitude equation, since we can develop a general particular solution;

$$\frac{\alpha^{n+1}}{n!}\frac{\partial^n \psi_{0\alpha}}{\partial \theta^n}\,. \tag{73}$$

If we now follow the same method outlined in section two we can obtain a solvability condition at this order, namely

$$B_{\hat{T}} \int_y \int_\theta -r_6 \hat{\nabla}^2 \psi_{0\theta}\, \mathrm{d}\theta\, \mathrm{d}y + B_{\hat{X}\hat{X}} \int_y \int_\theta M r_6\, \mathrm{d}\theta\, \mathrm{d}y = 0\,. \tag{74}$$

This is the heat transfer equation; the solution is characterized by the sign of the diffusive term, given by

$$\left\{\int_{y=-1}^{+1} \int_{\theta=0}^{2\pi} M r_6\, \mathrm{d}\theta\, \mathrm{d}y\right\} \left\{\int_{y=-1}^{+1} \int_{\theta=0}^{2\pi} -r_6 \hat{\nabla}^2 \hat{\psi}_{0\theta}\, \mathrm{d}\theta \mathrm{d}y\right\}^{-1} \equiv \Phi(\alpha) \tag{75}$$

in the previous notation. So we note that these amplitude perturbations are of less interest than the corresponding phase instabilities; since they are governed by the heat equation the solutions will simply decay exponentially to zero or grow becoming singular in a finite time. Thus we have stable decaying solutions for $\Phi(\alpha) < 0$ but when $\Phi(\alpha) > 0$ we must at some stage return to the full evolution equations to determine the development as higher order spatial and temporal derivatives are reintroduced.

4 Numerical Methods

4.1 The leading order problem

In solving the leading order form of the vorticity equation we look for solutions that expand as

$$\hat{\psi}_0 = \bar{\Psi} + \sum_{n=-\infty}^{\infty} \psi_{0n} e^{in\theta}\,, \tag{76}$$

where $\hat{\psi}_0$ is the leading order term in the expansion of the stream function ψ, (6), and $\bar{U} \equiv D\bar{\Psi}$. The vorticity equation (8) reduces to, after substitution of the above form, to

$$\begin{aligned}\frac{1}{Re}(D^2 - n^2\alpha^2)^2 \psi_{0n} - i\alpha n \left\{\left[\bar{U} - \frac{\Omega_0}{\alpha}\right](D^2 - n^2\alpha^2)\psi_{0n} - \psi_{0n} D^2 \bar{U}\right\} \\ + i\alpha \sum_{m=-\infty}^{\infty} \Big\{(n-m)\psi_{0n-m}(D^3 - m^2\alpha^2 D)\psi_{0m} \\ - mD\psi_{0n-m}(D^2 - m^2\alpha^2)\psi_{0m}\Big\} = 0\,,\end{aligned} \tag{77}$$

for $n = 0, \pm 1, \pm 2, \ldots$ with boundary conditions $\psi_{0n} = D\psi_{0n} = 0$; where $D \equiv d/dy$. Now we solve this with a truncation of the Fourier modes and a Chebyshev expansion in the y–direction for each harmonic,

$$\psi_{0n}(y) = \frac{a_{n0}}{2} + \sum_{r=1}^{\infty} a_{nr} T_r(y)\,, \tag{78}$$

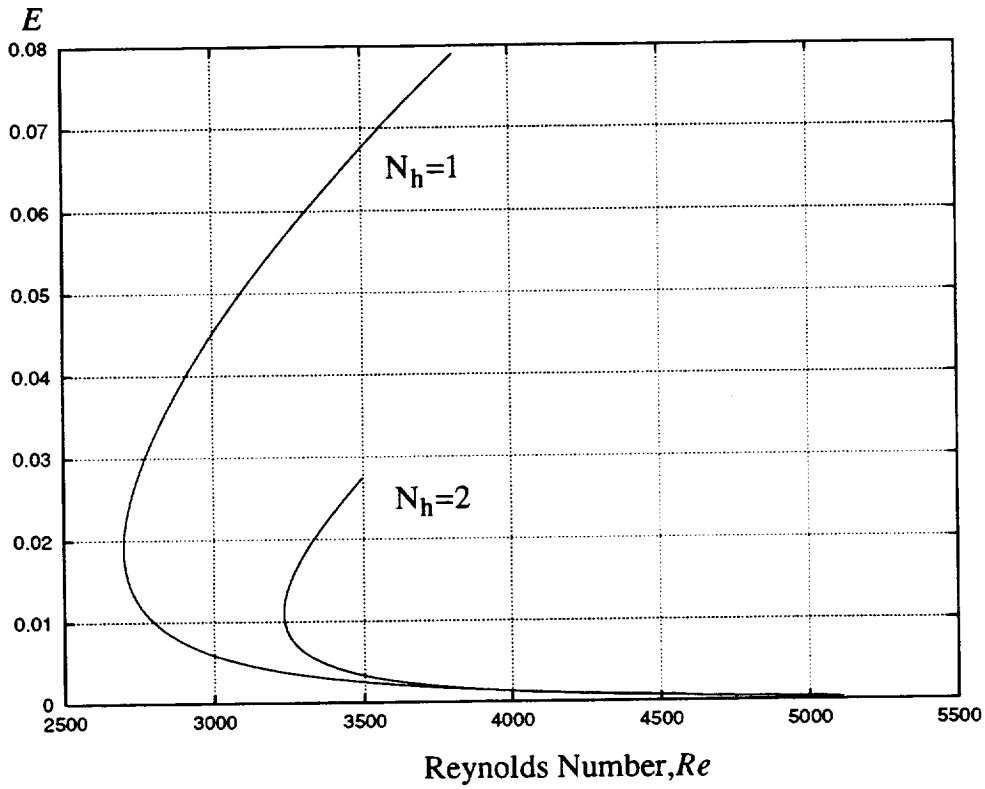

Figure 1: A cross section of the neutral surface at $\alpha = 1.1$ $N_h = 1, 2$.

along with $\bar{U} = \frac{1}{2}(1 - T_2)$. This formulation has been applied previously by Herbert [9] to plane Poiseuille flow with a constant pressure gradient rather than the constant mass flux condition we apply in this case.

If the harmonics are truncated at N_h, and the Chebyshev series at N_c, this yields a numerical problem that can be solved using Lanczo's τ–method with $N_h(N_c + 7)$ nonlinear equations plus a coupled mean flow problem. We also note that the computational task can be simplified by assuming the symmetry condition

$$\psi_n(y) = (-1)^{n+1}\psi_n(-y)\,, \tag{79}$$

together with requiring a real solution,

$$\psi_{-n} = \psi_{+n}^{(c)}\,. \tag{80}$$

The τ–method is essentially equivalent to determining the higher Chebyshev coefficients through the boundary conditions, however we can only replace two of the dynamical equations with boundary conditions and therefore have to retain at least one τ element. Various methods were investigated for the solution of these simultaneous nonlinear equations and finally a Newton iteration technique was found to provide the best convergence over large amplitude ranges. We write the nonlinear system of equations as

$$\underline{\mathbf{f}}(\underline{\mathbf{x}}) = \underline{\mathbf{0}}\,, \tag{81}$$

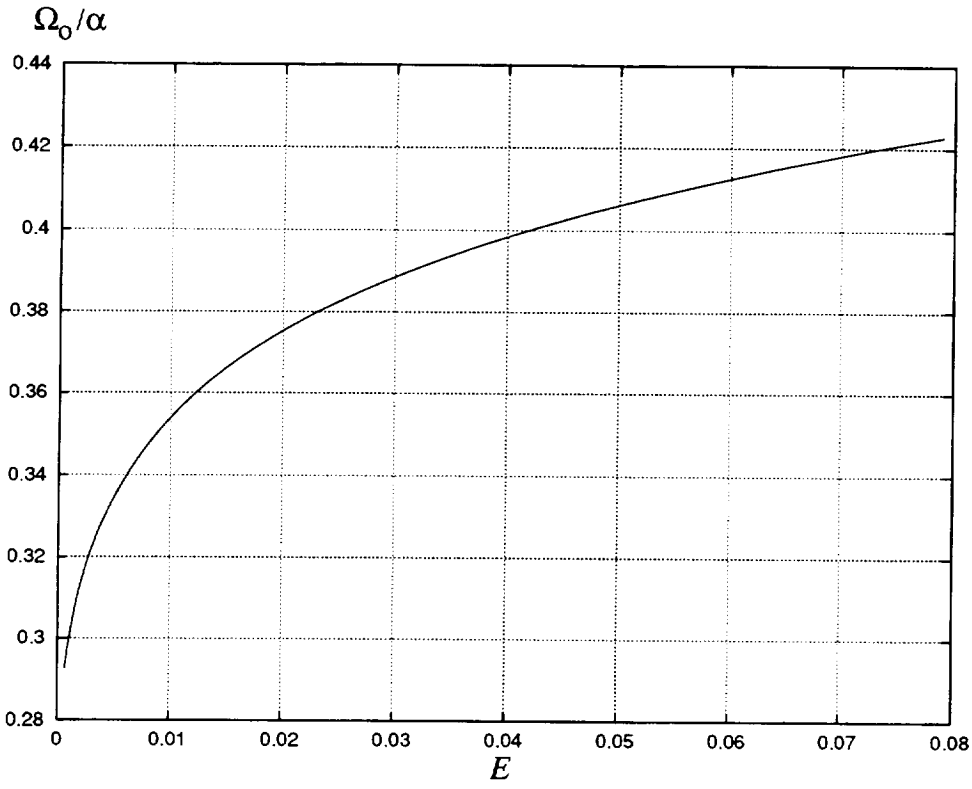

Figure 2: Variation of frequency with E at $\alpha = 1.1$.

where

$$\underline{\mathbf{x}} = (\Omega_0, a_{11}, a_{12}, \ldots, a_{1N_c}, a_{20}, a_{21}, \ldots, a_{2N_c}, a_{N_h 0}, \ldots, a_{N_h N_c})^T , \tag{82}$$

and at each iteration level solve

$$J\underline{\mathbf{z}} = -\underline{\mathbf{f}}(\underline{\mathbf{x}}_k) , \tag{83}$$

to give

$$\underline{\mathbf{x}}_{k+1} = \underline{\mathbf{x}}_k + \underline{\mathbf{z}} , \tag{84}$$

where J is the Jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} . \tag{85}$$

The number of unknowns in this solution procedure can be effectively halved by applying the symmetry assumptions discussed above. From this nonlinear eigenvalue problem we determine the relationship between Ω_0 and $\{\alpha, Re\}$, to do this we specify an amplitude of disturbance by choosing a value for the first Chebyshev coefficient, a_{10}, in the expansion of the fundamental mode. Without loss of generality we can assume that this amplitude measure is real since this corresponds to a unique determination of the phase. Here the initial guess for a solution of the

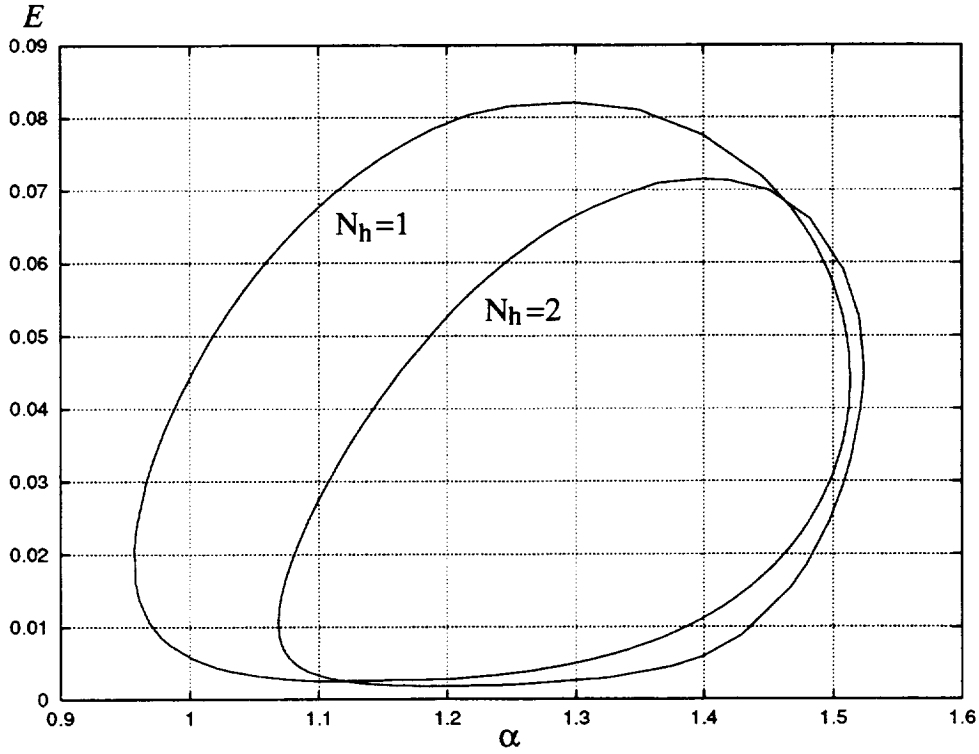

Figure 3: A cross section at $Re = 3500$ for $N_h = 1, 2$.

system, at a general neutral surface point, is derived from an interpolation of previous "nearby" solutions. Since a value for one of the Chebyshev coefficients is specified we must replace it by some other unknown in the Newton iteration scheme, typically the frequency.

As discussed in §2 we must also iterate upon the additional pressure constant, $\partial q_{-1}/\partial_X$ in the mean flow equation (16), to satisfy the constant mass flux condition. At each level of the iteration scheme described above we compute $\underline{\mathbf{x}}_k$, then the corresponding mean flow problem is solved directly, including the constant $\partial q_{-1}/\partial_X$, from (16). We continue the iteration, as described above, until some measure of convergence is satisfied, which in this case we choose to be that the frequency correction and largest change in Chebyshev coefficients are less than preset tolerances.

In figure 1 we show a typical cross section of the neutral surface at a fixed wavenumber, for which there are no linearly unstable modes. These results are consistent with those obtained by Pugh & Saffman [14]. The disturbance energy here is defined as

$$E = \sum_{n=1}^{N_h} E_n \,, \tag{86}$$

where

$$E_n = \frac{15}{8} \int_{-1}^{+1} (|D\psi_{0n}|^2 + |n\alpha\psi_{0n}|^2) \, \mathrm{d}y \,, \tag{87}$$

a normalized energy measure for each harmonic.

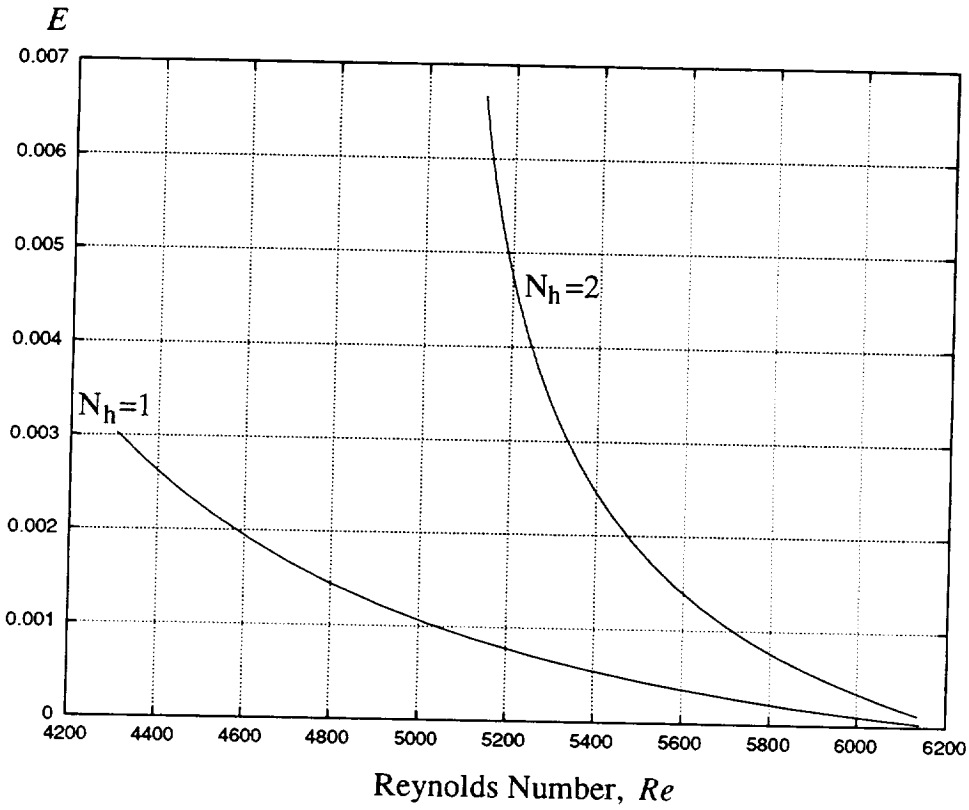

Figure 4: The neutral surface at $\alpha = 0.95$

In figure 2 we show the variation of the frequency for the same cross section, and figure 3 displays the form of the cross section for a fixed subcritical Reynolds number. Finally in figures 4 and 5 we show several cross sections for separate wavenumber choices of $\{0.95, 1.08\}$, showing the effect of higher harmonics.

We found that $N_c = 40$ was generally sufficient to give converged results for a chosen number of harmonics in this Reynolds number range. Although the inclusion of higher harmonics still altered the mid-range amplitude results the qualitative behaviour remained the same, as noted in Herbert's investigation [9] of the slightly different problem of a constant pressure gradient flow.

4.2 The $O(\delta)$ problem

To determine the sign of the diffusive coefficient on the right of Burgers equation we need to solve the next order problem. In §2.1 the phase equation analysis of the uniform wavetrain problem showed that the frequency correction term was determined by the solvability condition

$$\Omega_1 = \frac{\partial \alpha}{\partial X}\left\{\int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} -\hat{\nabla}^2\hat{\psi}_{0\theta} r_6 \, \mathrm{d}y \, \mathrm{d}\theta\right\}^{-1}\left\{\int_{\theta=0}^{2\pi}\int_{y=-1}^{+1}\left[\Omega_0(2\alpha\hat{\psi}_{0\theta\theta\alpha} + \hat{\psi}_{0\theta\theta})\right.\right.$$

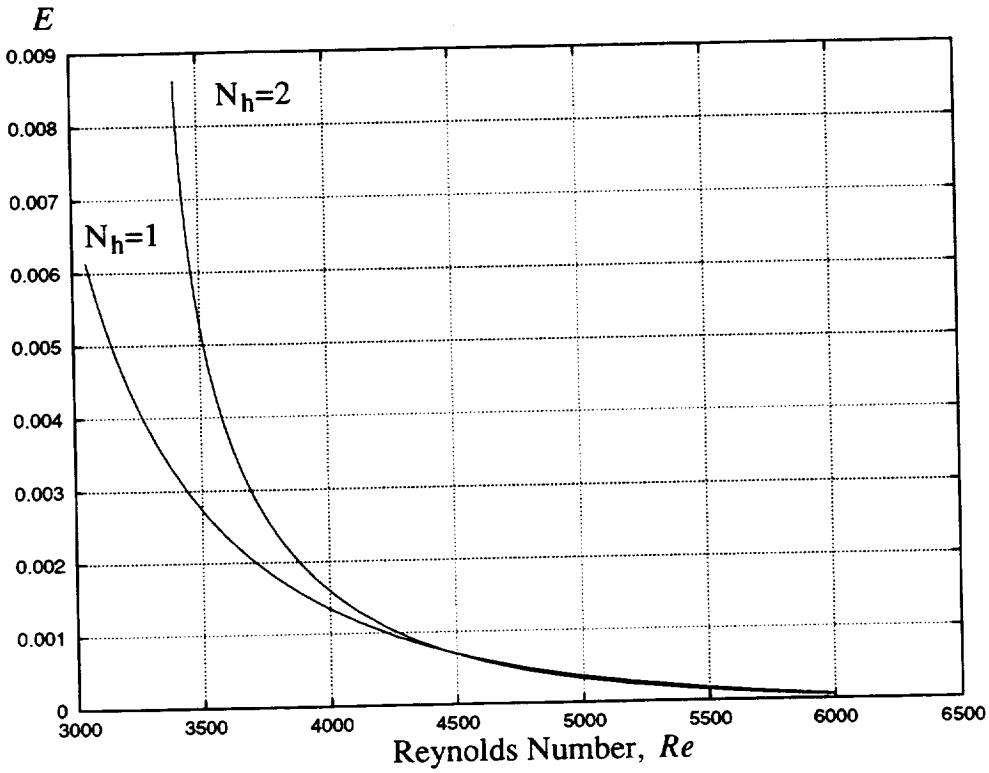

Figure 5: The neutral surface at $\alpha = 1.08$

$$+\Omega_{0\alpha}\frac{\partial\hat{\nabla}^2\hat{\psi}_0}{\partial\alpha}+\alpha\frac{\partial(\hat{\psi}_0,2\alpha\hat{\psi}_{0\theta\alpha}+\hat{\psi}_{0\theta})}{\partial(\theta,y)}+\frac{\partial(\hat{\psi}_0,\hat{\nabla}^2\hat{\psi}_0)}{\partial(\alpha,y)}$$
$$+Re^{-1}\{4\alpha\hat{\nabla}^2\hat{\psi}_{0\theta\alpha}+4\alpha^2\hat{\psi}_{0\theta\theta\theta}+2\hat{\nabla}^2\hat{\psi}_{0\theta}\}\Big]r_6\;\mathrm{d}y\,\mathrm{d}\theta\Big\}. \tag{88}$$

Here $\hat{\psi}_0$ is the basic parallel flow plus two dimensional periodic flow and r_6 solves the adjoint problem, which can be shown to be

$$-Re^{-1}\hat{\nabla}^4 r_6-(\alpha\hat{\psi}_{0y}-\Omega_0)\hat{\nabla}^2 r_{6\theta}+\alpha\hat{\psi}_{0\theta}\hat{\nabla}^2 r_{6y}$$
$$-2\alpha\hat{\psi}_{0y\theta}\Big(\alpha^2\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial y^2}\Big)r_6+2\alpha r_{6\theta y}\Big(\alpha^2\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial y^2}\Big)\hat{\psi}_0=0\,, \tag{89}$$

with boundary conditions $r_6 = r_{6\theta} = 0$ at $y = \pm 1$. However in our calculations we do not solve the linear homogeneous problem (89) but rather define a new composite nonlinear problem that is an inhomogeneous form of the order one system, this can be computed directly after the leading order solution using the same routine. We solve

$$N\{\tilde{\psi}_0;\tilde{\Omega}_0\}=\delta M(\theta,y)+O(\delta^2)\,, \tag{90}$$

where the operator N and the new parameters are given by

$$N \;=\; L_{O(1)}+\delta L_{O(\delta)}\,, \tag{91}$$

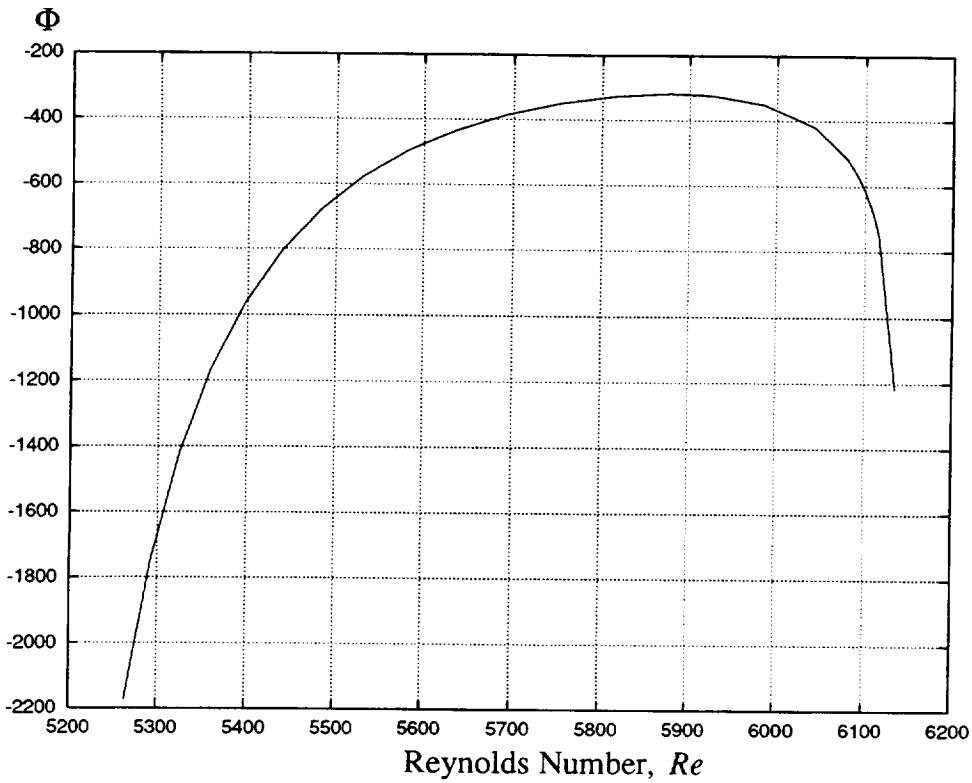

Figure 6: The variation of Φ at $\alpha = 0.95$ & $N_h = 2$.

$$\tilde{\Omega}_0 = \Omega_0 + \delta\Omega_1 , \tag{92}$$

$$\tilde{\psi}_0 = \hat{\psi}_0 + \delta\psi_1 , \tag{93}$$

with M and $L_{O(.)}$ as defined previously. Hence after solving the $O(1)$ problem we compute the inhomogeneous terms produced by M then iterate on the amplitude of the new composite solution at fixed $\{\alpha, Re\}$. The frequency correction term is then determined by an application of the condition $\Omega_1 = \partial\tilde{\Omega}_0/\partial\delta$ at $\delta = 0$.

A point to note is that we divide the neutral surface into two sections which are referred to as the upper and lower branches. We can see from (88) that we should expect a singular behaviour for the coefficient of the diffusive term as we approach two other regions, namely where the lower branch connects with the $E = 0$ plane and where the upper and lower branches join at finite amplitude. The singularities are due to the way in which the amplitude and frequency vary with α at these points and further work is required near these regions.

We present in figures 6 and 7 the behaviour of the diffusion coefficient in the evolution equation (as the $O(1)$ solution varies along the lower branches shown in figures 4 and 5) since this essentially characterizes the solution; we plot $\Phi(\alpha_0)$ where $\Omega_1 = \Phi(\alpha)\Delta_X$. Since the linear neutral curve exists for each of these wavenumbers we will find a singular behaviour of Φ at a higher Reynolds number where the neutral surface intersects the $E = 0$ plane as discussed above.

We have carried out calculations (of Φ), with more harmonics, for a number of typical

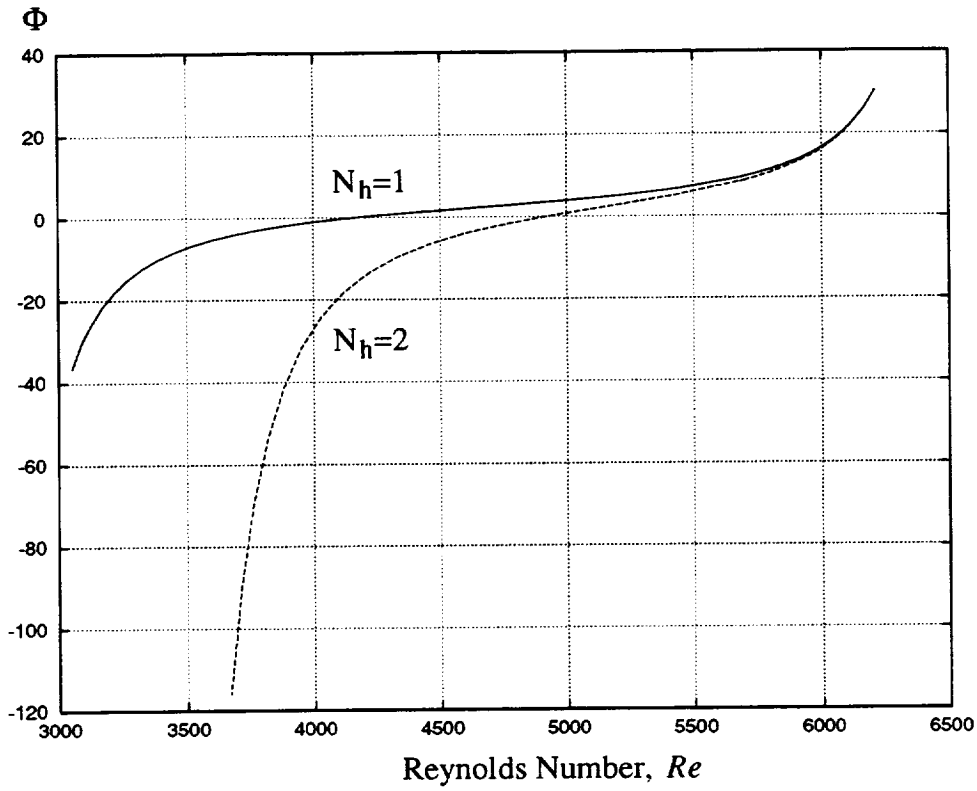

Figure 7: The variation of Φ at $\alpha = 1.08$.

parameter values and find that such results have the same qualitative behaviour. In fact the main effect of the higher harmonics is to alter the cross section shape as shown in figure 3 thus altering the position of the singularities when evaluating Φ for varying Reynolds number, wavenumber or amplitude. In figures 8 and 9 we show the behaviour of the coefficient as a function of the wavenumber α, the vertical lines in figure 8 show the critical wavenumbers predicted by linear theory. Finally figures 10 and 11 show the behaviour for solutions that correspond to points upon the upper branch of the neutral surface at $\alpha = 1.1$, $N_h = 1$ and $\alpha = 0.9707$, $N_h = 1, 2$ respectively.

5 The Evolution Equation as $E \to 0$

As noted in the previous section the phase equation method is only valid for finite amplitude periodic states; if we allow the leading order solution to approach the linear limit then the coefficient Φ, in Burgers equation, becomes singular. A simplistic approach would be to return to the uniform wavetrain problem and allow the $O(\delta^0)$ system to be a small amplitude solution with a phase function of leading order form

$$\theta_0 = \alpha_0 x - \Omega_0(\alpha_0) t \,. \tag{94}$$

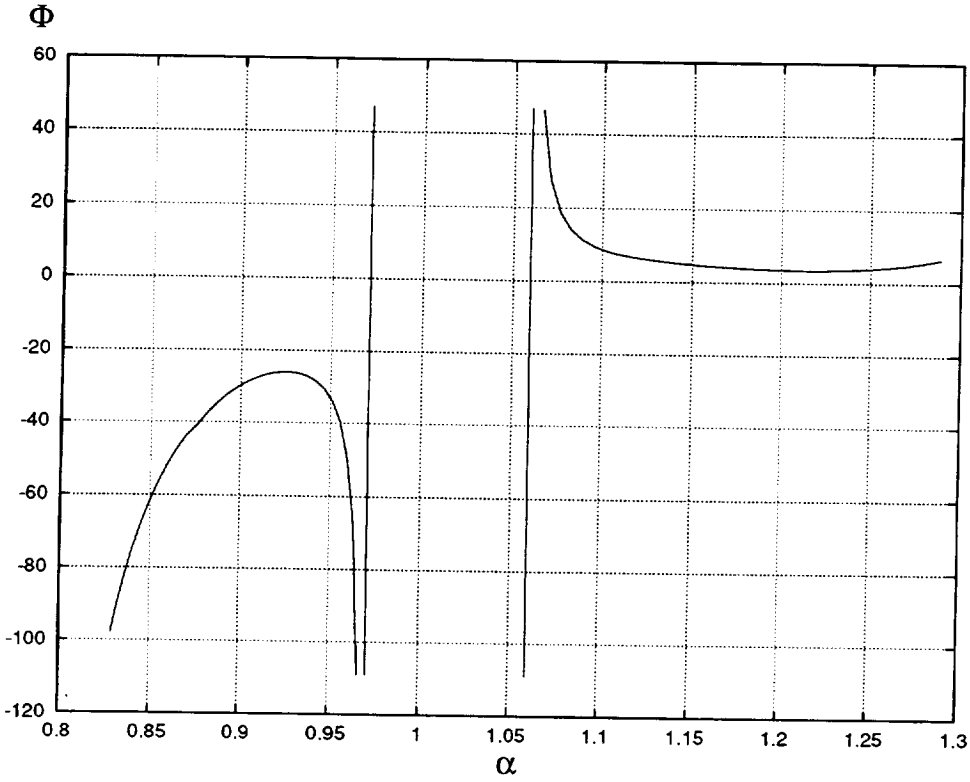

Figure 8: The variation of Φ for constant supercritical Re with $N_h = 1$.

In this formulation $\{\alpha_0, Re\}$ is a point on the lower branch of the neutral surface in the weakly nonlinear region with

$$\alpha_0 = \alpha_n + \tilde{\alpha}\,, \quad |\tilde{\alpha}| \ll 1\,, \tag{95}$$

where α_n is a wavenumber corresponding to a point on the linear neutral curve. If we perturb this solution with a slowly varying function Δ,

$$\alpha = \alpha_0 + \Delta(X,T)\,, \tag{96}$$

then, as we approach the linear neutral curve, we know that the neutral surface has the scalings

$$A = O(|\alpha - \alpha_n|^{\frac{1}{2}})\,, \tag{97}$$

$$\Omega = \Omega_n + O(\alpha - \alpha_n)\,, \tag{98}$$

for some amplitude measure A and where Ω_n is a frequency associated with the linear solution. The full form of the evolution equation is

$$\frac{\partial \Delta}{\partial T} + \Big(\Omega_0(\alpha_0) + \Delta\Omega_0'(\alpha_0) + \ldots\Big)\frac{\partial \Delta}{\partial X} = \delta\frac{\partial}{\partial X}\Big\{\Phi(\alpha)\frac{\partial \Delta}{\partial X} + \delta\Omega_2(\alpha) + \ldots\Big\}\,, \tag{99}$$

or equivalently,

$$\eta\frac{\partial \Delta}{\partial \tau} + \Omega_0'(\alpha_0)\Delta\frac{\partial \Delta}{\partial \xi} + \Big\{\Omega_0''(\alpha_0)\Delta^2\frac{\partial \Delta}{\partial \xi} + \ldots\Big\} = \delta\Phi(\alpha_0)\frac{\partial^2 \Delta}{\partial \xi^2}$$

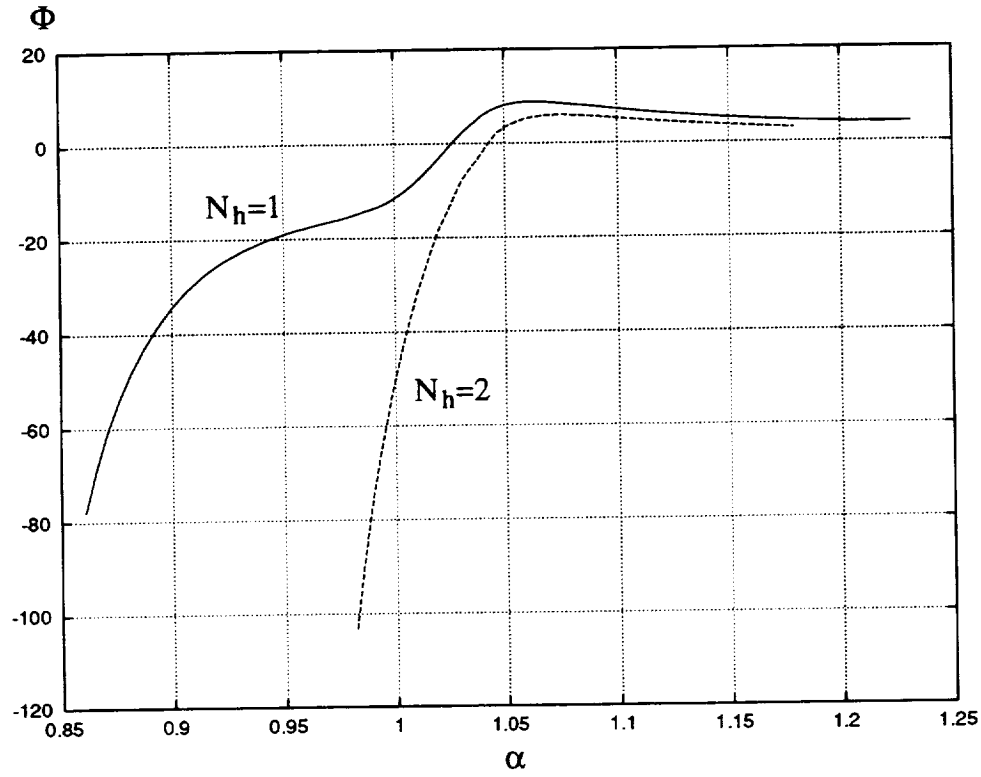

Figure 9: The variation of Φ for constant subcritical Re.

$$+\delta\left\{\Phi'(\alpha_0)\left(\frac{\partial\Delta}{\partial\xi}\right)^2+\Phi'(\alpha_0)\Delta\frac{\partial^2\Delta}{\partial\xi^2}+\ldots\right\}+\delta^2\frac{\partial\Omega_2}{\partial\xi}+\ldots, \quad (100)$$

after applying the substitutions (40)–(42).

Therefore approaching the linear neutral curve may reintroduce other terms from the expression

$$\frac{\partial\Omega_1}{\partial\xi}=\frac{\partial\Phi}{\partial\alpha_0}\left(\frac{\partial\Delta}{\partial\xi}\right)^2+\frac{\partial\Phi}{\partial\alpha_0}\Delta\frac{\partial^2\Delta}{\partial\xi^2}+\ldots, \quad (101)$$

since applying the scalings (97)–(98) to equation (88) we see that

$$\Phi(\alpha_0)\sim\frac{1}{\tilde{\alpha}}, \quad (102)$$

and the above terms can be of the same order of magnitude when

$$\eta\sim\Delta\sim\delta^{\frac{1}{2}}\sim\tilde{\alpha}. \quad (103)$$

However the theory in this limit is more complex, as we approach the linear neutral curve the amplitude will no longer be determined explicitly from the leading order eigenvalue problem. So instead of assuming the weakly nonlinear scalings (97)–(98) we must determine the real amplitude of the weakly nonlinear uniform wavetrain by solving an appropriate nonlinear partial

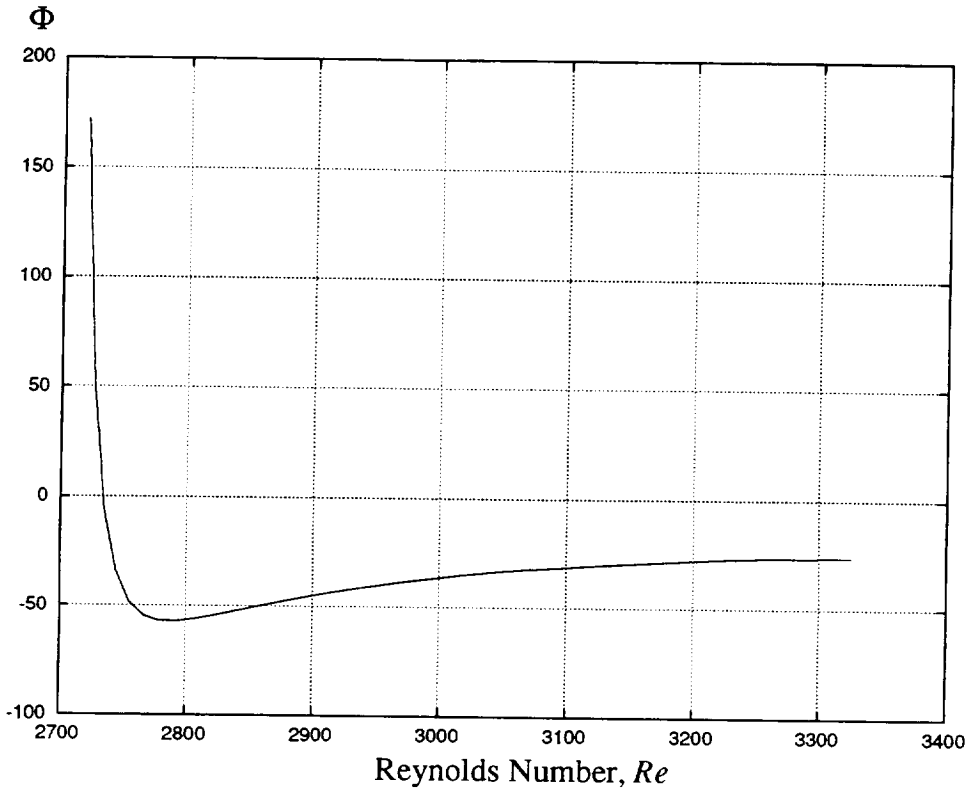

Figure 10: The variation of Φ for upper branch solutions at $\alpha = 1.1$.

differential equation. This amplitude equation will in general be coupled to a limiting form of the phase equation. Thus as we move from a fully nonlinear to a weakly nonlinear uniform wavetrain solution (for the leading order problem) we obtain two eigenfunctions for the problem and so must satisfy two, coupled, real partial differential equations (for the two–dimensional case). These two real equations may then be combined to recover a single complex evolution equation equivalent to that obtained by an application of the Stuart–Watson approach to the plane Poiseuille flow problem. Furthermore, in this limit, we no longer have a small parameter, δ, that is arbitrary, it will now be related to the magnitude of the wavenumber displacement from the linear neutral value, α_n.

The connection of the phase equation method to the appropriate weakly nonlinear theory proves difficult in general, indeed the analysis still remains incomplete for convection problems in which phase–equation methods have been applied for some years. The recent works of Newell, Passot & Lega [12], Newell & Passot [11] and Cross & Newell [3] have discussed the points above and gone some way towards resolving the difficulties, although they restrict attention to much simplified model equations for convection (e.g the Swift–Hohenburg equation). In particular Cross and Newell [3] show how the evolution equation for such a model system can be matched with the Newell–Whitehead–Segel (NWS) equation in the weakly nonlinear limit. In their analysis they demonstrate that the limiting forms of the phase evolution equation and the amplitude equation (required in this limit) form the imaginary and real parts of the NWS

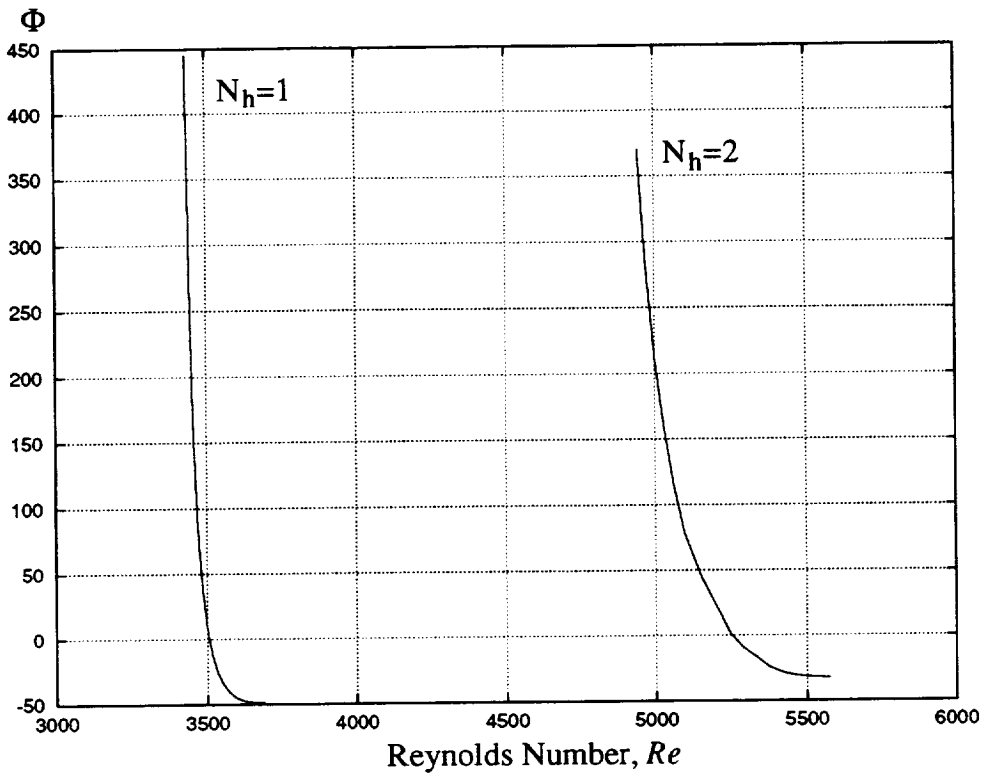

Figure 11: The variation of Φ for upper branch solutions at $\alpha = 0.9707$.

evolution equation respectively, hence they can be combined to match with previous results.

6 A 3-D Phase Equation Theory

We now apply a generalized (following Howard and Kopell [10]) form of the two dimensional phase equation method to a spanwise dependent problem. To achieve this we first introduce a further spanwise scale

$$Z = \delta z\,, \tag{104}$$

and generalize the phase as a function of three variables, $\theta(x,z,t) = \Theta(X,Z,T)/\delta$, now defining a spanwise wavenumber, β, by

$$\beta = \Theta_Z\,. \tag{105}$$

The consistency conditions in this approach are

$$\alpha_T + \Omega_X = 0\,, \tag{106}$$

$$\alpha_Z - \beta_X = 0\,, \tag{107}$$

$$\beta_T + \Omega_Z = 0\,, \tag{108}$$

where both wavenumbers and frequency are functions of the slow scales X, Z, and T. We restrict attention to the stability of uniform wavetrains, considering a leading order problem that is an

oblique travelling wave solution at finite amplitude. We now perturb the wavenumbers as

$$\alpha = \alpha_0 + \delta\Delta_1(X,Z,T)\,, \tag{109}$$
$$\beta = \beta_0 + \delta\Delta_2(X,Z,T)\,, \tag{110}$$

and hence transform the partial derivatives in the following manner

$$\frac{\partial}{\partial x} \rightarrow \alpha_0\frac{\partial}{\partial\theta} + \delta\Big(\Delta_1\frac{\partial}{\partial\theta} + \frac{\partial}{\partial X}\Big)\,, \tag{111}$$
$$\frac{\partial}{\partial z} \rightarrow \beta_0\frac{\partial}{\partial\theta} + \delta\Big(\Delta_2\frac{\partial}{\partial\theta} + \frac{\partial}{\partial Z}\Big)\,, \tag{112}$$
$$\frac{\partial}{\partial t} \rightarrow -\Omega_0\frac{\partial}{\partial\theta} + \delta\Big(-\Omega_1\frac{\partial}{\partial\theta} + \frac{\partial}{\partial T}\Big) + \ldots\,. \tag{113}$$

We also expand the velocity field and pressure function as

$$\underline{\mathbf{u}} = (1-y^2,0,0)^T + \underline{\mathbf{u}}_0(\theta,y) + \delta\underline{\mathbf{u}}_1(X,Z,T,\theta,y) + \ldots\,, \tag{114}$$
$$p = \Big[-\frac{2}{Re}X + q_{-1}(X,Z)\Big]\delta^{-1} + [p_0(\theta,y) + q_0(X,Z,T)] + \ldots\,, \tag{115}$$

where q_i are the additional pressure terms required to satisfy a constant mass flux through the channel.

Now as before we can expand the Navier–Stokes equations in terms of the slow scale parameter, giving a leading order system

$$-\Omega_0\hat{u}_{0\theta} + \alpha_0\hat{u}_0\hat{u}_{0\theta} + v_0\hat{u}_{0y} + \beta_0 w_0\hat{u}_{0\theta} - \frac{1}{Re}\hat{\nabla}_3^2\hat{u}_0 + \alpha_0 p_{0\theta} = -\frac{\partial q_{-1}}{\partial X}\,, \tag{116}$$
$$-\Omega_0 v_{0\theta} + \alpha_0\hat{u}_0 v_{0\theta} + v_0 v_{0y} + \beta_0 w_0 v_{0\theta} - \frac{1}{Re}\hat{\nabla}_3^2 v_0 + p_{0y} = 0\,, \tag{117}$$
$$-\Omega_0 w_{0\theta} + \alpha_0\hat{u}_0 w_{0\theta} + v_0 w_{0y} + \beta_0 w_0 w_{0\theta} - \frac{1}{Re}\hat{\nabla}_3^2 w_0 + \beta_0 p_{0\theta} = -\frac{\partial q_{-1}}{\partial Z}\,, \tag{118}$$
$$\alpha_0\hat{u}_{0\theta} + v_{0y} + \beta_0 w_{0\theta} = 0\,, \tag{119}$$

$$\hat{\nabla}_3^2 \equiv (\alpha_0^2+\beta_0^2)\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial y^2}\,, \tag{120}$$

where $\hat{u}_0 = \bar{U} + u_0$. At this order we let $q_{-1}(X,Z) = \kappa_1 X + \kappa_2 Z$ and choose κ_1, κ_2 to satisfy zero mass flux through the channel for the disturbance. Continuing further yields an $O(\delta)$ system of the form

$$\begin{aligned} L_u\{u_1,v_1,w_1,p_1\} &= \Delta_1\Big[-\hat{u}_0\hat{u}_{0\theta} - p_{0\theta} + \frac{2}{Re}\alpha_0\hat{u}_{0\theta\theta}\Big] \\ &\quad + \Delta_2\Big[-w_0\hat{u}_{0\theta} + \frac{2}{Re}\beta_0\hat{u}_{0\theta\theta}\Big] + \Omega_1\hat{u}_{0\theta} - \frac{\partial q_0}{\partial X}\,, \\ L_v\{u_1,v_1,w_1,p_1\} &= \Delta_1\Big[-\hat{u}_0 v_{0\theta} + \frac{2}{Re}\alpha_0 v_{0\theta\theta}\Big] \end{aligned} \tag{121}$$

$$+\Delta_2\left[-w_0 v_{0\theta}+\frac{2}{Re}\beta_0 v_{0\theta\theta}\right]+\Omega_1 v_{0\theta}\,, \tag{122}$$

$$L_w\{u_1,v_1,w_1,p_1\}=\Delta_1\left[-\hat{u}_0 w_{0\theta}+\frac{2}{Re}\alpha_0 w_{0\theta\theta}\right]$$
$$+\Delta_2\left[-w_0 w_{0\theta}-p_{0\theta}+\frac{2}{Re}\beta_0 w_{0\theta\theta}\right]+\Omega_1 w_{0\theta}-\frac{\partial q_0}{\partial Z}\,, \tag{123}$$

$$\alpha_0 u_{1\theta}+v_{1y}+\beta_0 w_{1\theta}=\Delta_1\left[-\hat{u}_{0\theta}\right]+\Delta_2\left[-w_{0\theta}\right], \tag{124}$$

where $L_u\{u_1,v_1,w_1,p_1\}$ is given by

$$L_u\{u_1,v_1,w_1,p_1\}\equiv -\Omega u_{1\theta}+\alpha_0\hat{u}_0 u_{1\theta}+\alpha_0 u_1\hat{u}_{0\theta}+v_0 u_{1y}+v_1\hat{u}_{0y}$$
$$+\beta_0 w_0 u_{1\theta}+\beta_0 w_1\hat{u}_{0\theta}-\frac{1}{Re}\hat{\nabla}_3^2 u_1+\alpha_0 p_{1\theta}\,, \tag{125}$$

with analogous forms for L_v and L_w.

After some work we can see that the solution is of the form

$$(u_1,v_1,w_1,p_1)^T=\underline{\phi}_{11}\Delta_1(X,Z,T)+\underline{\phi}_{12}\Delta_2(X,Z,T) \tag{126}$$
$$+\underline{\phi}_{13}\frac{\partial q_0}{\partial X}(X,Z,T)+\underline{\phi}_{14}\frac{\partial q_0}{\partial Z}(X,Z,T)\,, \tag{127}$$

and the solvability condition determines Ω_1 as

$$\Omega_1=\frac{\partial\Omega_0}{\partial\alpha_0}\Delta_1(X,Z,T)+\frac{\partial\Omega_0}{\partial\beta_0}\Delta_2(X,Z,T)\,; \tag{128}$$

where we have used

$$\underline{\phi}_{11}=\frac{\partial\underline{\phi}_0}{\partial\alpha_0}-\frac{\partial\kappa_1}{\partial\alpha_0}\underline{\phi}_{13}-\frac{\partial\kappa_2}{\partial\alpha_0}\underline{\phi}_{14}\,, \tag{129}$$

$$\underline{\phi}_{12}=\frac{\partial\underline{\phi}_0}{\partial\beta_0}-\frac{\partial\kappa_1}{\partial\beta_0}\underline{\phi}_{13}-\frac{\partial\kappa_2}{\partial\beta_0}\underline{\phi}_{14}\,, \tag{130}$$

$$\underline{\phi}_{13}=\left.\frac{\partial\underline{\phi}_0}{\partial\kappa_1}\right|_{\{\alpha_0,\beta_0,\Omega_0\}\,\text{fixed}}, \tag{131}$$

$$\underline{\phi}_{14}=\left.\frac{\partial\underline{\phi}_0}{\partial\kappa_2}\right|_{\{\alpha_0,\beta_0,\Omega_0\}\,\text{fixed}}, \tag{132}$$

with $\underline{\phi}_0$ replaced by $\{u_0,v_0,w_0,p_0\}$.

In the usual two dimensional iteration scheme (Re fixed) we specify $\{\alpha_0,\text{Amplitude}\}$ and iterate on $\{\Omega_0,\kappa_1\}$. Now for given wavenumber and amplitude, we can think of Ω_0 as a function of κ_1 with the required frequency given by $\Omega_0(\kappa_1^*)$, where $\kappa_1=\kappa_1^*$ is determined by the zero mass flux condition for the disturbance. Obviously we could equally fix $\{\alpha_0,\Omega_0\}$ and iterate on $\{\text{Amplitude},\kappa_1\}$ to define the amplitude as a function of the pressure constant. We can apply this same argument to the three dimensional iteration scheme and so define the solutions $\{\underline{\phi}_{13},\underline{\phi}_{14}\}$.

If we continue the expansion scheme to next order we obtain a similar system of equations with inhomogeneous terms of the form $\{\Delta_1^2, \Delta_2^2, \Delta_{1X}, \Delta_{2X}, \Delta_1 q_{0X}, \ldots\}$; we state the results for the fully three–dimensional problem in Appendix A but to simplify this problem (and outline the basic method) we shall consider the weakly three dimensional limit of $\beta_0 \to 0$. In this limit the leading order system will be the same two dimensional problem computed in previous sections but the solutions to the $O(\delta)$ equations are

$$(u_1, v_1, w_1, p_1)^T = (u_{11}, v_{11}, 0, p_{11})^T \Delta_1 + (0, 0, w_{12}, 0)^T \Delta_2 \tag{133}$$

$$+ (u_{13}, v_{13}, 0, p_{13})^T \frac{\partial q_0}{\partial X} + (0, 0, w_{14}, 0)^T \frac{\partial q_0}{\partial Z}\,, \tag{134}$$

where

$$(u_{11}, v_{11}, p_{11})^T = \frac{\partial}{\partial \alpha_0}(u_0, v_0, p_0)^T - \frac{\partial \kappa_1}{\partial \alpha_0}(u_{13}, v_{13}, p_{13})^T\,, \tag{135}$$

$$(u_{13}, v_{13}, p_{13})^T = \frac{\partial}{\partial \kappa_1}(u_0, v_0, p_0)^T\Big|_{\{\alpha_0, \beta_0, \Omega_0\}\,\text{fixed}}, \tag{136}$$

$$w_{12} = \frac{\partial w_0}{\partial \beta_0} - \frac{\partial \kappa_2}{\partial \beta_0} w_{14}\,, \tag{137}$$

$$w_{14} = \frac{\partial w_0}{\partial \kappa_2}\Big|_{\{\alpha_0, \beta_0, \Omega_0\}\,\text{fixed}}. \tag{138}$$

The solvability condition at this order reduces to

$$\Omega_1 = \frac{\partial \Omega_0}{\partial \alpha_0} \Delta_1(X, Z, T)\,, \tag{139}$$

since for $\beta_0 \sim O(\gamma)$, $\gamma \ll 1$,

$$\begin{aligned} &\frac{\partial}{\partial \alpha_0}\{\hat{u}_0, v_0, p_0, \kappa_1\},\ \{p_{13}, u_{13}, v_{13}\} \sim O(1)\,, \quad \frac{\partial}{\partial \alpha_0}\{w_0, \kappa_2\},\ w_{13} \sim O(\gamma)\,, \\ &\frac{\partial}{\partial \beta_0}\{\hat{u}_0, v_0, p_0, \kappa_1\},\ \{p_{14}, u_{14}, v_{14}\} \sim O(\gamma)\,, \quad \frac{\partial}{\partial \beta_0}\{w_0, \kappa_2\},\ w_{14} \sim O(1)\,. \end{aligned} \tag{140}$$

The group velocity obtained in this limit is equivalent to the two–dimensional case, as expected. Continuing the expansion scheme further, to $O(\delta^2)$, yields the following after some simplification,

$$\begin{aligned} \bar{L}_u\{u_2, v_2, p_2\} = &\Big[\frac{\partial \Omega_0}{\partial \alpha_0} u_{11\theta} - \hat{u}_0 u_{11\theta} - \alpha_0 u_{11} u_{11\theta} - u_{11}\hat{u}_{0\theta} \\ &- v_{11} u_{11y} - p_{11\theta} + \frac{1}{Re}(2\alpha_0 u_{11\theta\theta} + \hat{u}_{0\theta\theta})\Big]\Delta_1^2 \\ &+ \Big[-w_{12}\hat{u}_{0\theta} + \frac{1}{Re}(\hat{u}_{0\theta\theta})\Big]\Delta_2^2 \\ &+ \Big[-\alpha_0 u_{13} u_{13\theta} - v_{13} u_{13y}\Big]\Big(\frac{\partial q_0}{\partial X}\Big)^2 + \Big[-w_{13}\hat{u}_{0\theta}\Big]\Delta_2 \frac{\partial q_0}{\partial Z} \end{aligned}$$

$$
\begin{aligned}
&+\Big[\frac{\partial\Omega_0}{\partial\alpha_0}u_{13\theta}-\hat{u}_0u_{13\theta}-\alpha_0u_{11}u_{13\theta}-\alpha_0u_{13}u_{11\theta}-u_{13}\hat{u}_{0\theta}\\
&\quad-v_{11}u_{13y}-v_{13}u_{11y}-p_{13\theta}+\frac{1}{Re}(2\alpha_0u_{13\theta\theta})\Big]\Delta_1\frac{\partial q_0}{\partial X}\\
&+\Big[\frac{\partial\Omega_0}{\partial\alpha_0}u_{13}-\hat{u}_0u_{13}-p_{13}+\frac{1}{Re}(2\alpha_0u_{13\theta})\Big]\frac{\partial^2 q_0}{\partial X^2}\\
&+\Big[\frac{\partial\Omega_0}{\partial\alpha_0}u_{11}-\hat{u}_0u_{11}-p_{11}+\frac{1}{Re}(2\alpha_0u_{11\theta}+\hat{u}_{0\theta})\Big]\frac{\partial\Delta_1}{\partial X}\\
&+\Big[\frac{1}{Re}\hat{u}_{0\theta}\Big]\frac{\partial\Delta_2}{\partial Z}+\Omega_2\hat{u}_{0\theta}-\frac{\partial q_1}{\partial X}\,,
\end{aligned}
\tag{141}
$$

with an analogous form for the v–momentum equation and a decoupled w–momentum problem. Continuity of mass at this order gives the further equation

$$
\begin{aligned}
\alpha_0\frac{\partial u_2}{\partial\theta}+\frac{\partial v_2}{\partial y}=&-u_{11}\frac{\partial\Delta_1}{\partial X}-u_{13}\frac{\partial^2 q_0}{\partial X^2}-w_{12}\frac{\partial\Delta_2}{\partial Z}-w_{14}\frac{\partial^2 q_0}{\partial Z^2}\\
&-u_{11\theta}\Delta_1^2-u_{13\theta}\Delta_1\frac{\partial q_0}{\partial X}-w_{12\theta}\Delta_2^2-w_{14\theta}\Delta_2\frac{\partial q_0}{\partial Z}\,.
\end{aligned}
\tag{142}
$$

We have used $\bar{L}_u$ (in 141) to indicate the operator L_u, defined by (125), with $\beta_0 = 0$, and the boundary conditions for the systems discussed above are no-slip and impermeability at the fixed planes $y = \pm 1$. In this three dimensional formulation the additional pressure term, q_0, will contribute explicitly to the solvability condition for Ω_2; whereas in the previous theory this term was formally eliminated using the stream function and vorticity equation approach. This is analogous to the work of Davey, Hocking and Stewartson [4], concerning the weakly nonlinear evolution of three dimensional disturbances in plane Poiseuille flow, where they found that a secular pressure term contributed to the final form of the evolution equation.

If we consider the continuity equation for those terms which have zero mean with respect to the phase variable θ, we obtain

$$
\frac{\partial\bar{v}_2}{\partial y}=-\bar{u}_{11}\frac{\partial\Delta_1}{\partial X}-\bar{u}_{13}\frac{\partial^2 q_0}{\partial X^2}-\bar{w}_{12}\frac{\partial\Delta_2}{\partial Z}-\bar{w}_{14}\frac{\partial^2 q_0}{\partial Z^2}\,,
\tag{143}
$$

where the bar notation indicates the mean part of the relevant expression. Integration of this equation shows that, for impermeability at both boundaries, we must satisfy the linear partial differential equation

$$
\frac{\partial^2 q_0}{\partial X^2}I_1+\frac{\partial^2 q_0}{\partial Z^2}I_2=\frac{\partial\Delta_1}{\partial X}\frac{\partial\kappa_1}{\partial\alpha_0}I_1+\frac{\partial\Delta_2}{\partial Z}\frac{\partial\kappa_2}{\partial\beta_0}I_2\,,
\tag{144}
$$

where

$$
I_1=\int_{y=-1}^{+1}\bar{u}_{13}\,\mathrm{d}y\quad\text{and}\quad I_2=\int_{y=-1}^{+1}\bar{w}_{14}\,\mathrm{d}y\,.
\tag{145}
$$

Now returning to the solvability condition determining Ω_2, we can see that further simplifications are possible by introducing some particular solutions; for example,

$$
\bar{L}_u\Big\{\frac{\partial u_{11}}{\partial\kappa_1},\frac{\partial v_{11}}{\partial\kappa_1},\frac{\partial p_{11}}{\partial\kappa_1}\Big\}=S_1\,,
\tag{146}
$$

$$\bar{L}_u\left\{\frac{\partial u_{13}}{\partial \kappa_1}, \frac{\partial v_{13}}{\partial \kappa_1}, \frac{\partial p_{13}}{\partial \kappa_1}\right\} = S_2\,, \tag{147}$$

$$\bar{L}_u\left\{\frac{\partial u_{12}}{\partial \kappa_2}, \frac{\partial v_{12}}{\partial \kappa_2}, \frac{\partial p_{12}}{\partial \kappa_2}\right\} = S_3\,, \tag{148}$$

where S_1, S_2, S_3 are the $[\ldots]$ coefficients of the $\Delta_1 q_{0X}, (q_{0X})^2$ and $\Delta_2 q_{0Z}$ terms respectively in (141). These particular solutions can be applied to the system as a whole, reducing the form of the frequency correction term to

$$\Omega_2 = \Omega_{21}\Delta_1^2 + \Omega_{22}\Delta_2^2 + \Phi_1\frac{\partial \Delta_1}{\partial X} + \Phi_2\frac{\partial \Delta_2}{\partial Z} + \Phi_3\frac{\partial^2 q_0}{\partial X^2}\,, \tag{149}$$

where $\{\Omega_{2i}, \Phi_i\}$ are constants determined from the appropriate integrals contained in the solvability condition. Furthermore a perturbation of $\{\alpha_0, \beta_0\}$ in the leading order problem shows that

$$\Omega_{21} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \alpha_0^2} \quad , \quad \Omega_{22} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \beta_0^2}\,. \tag{150}$$

The consistency conditions now determine the evolution of the wavenumber perturbations, and applying the above results we obtain

$$\delta\frac{\partial \Delta_1}{\partial T} + \delta\frac{\partial \Omega_0}{\partial \alpha_0}\frac{\partial \Delta_1}{\partial X} + \delta^2\left\{\frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\Delta_1\frac{\partial \Delta_1}{\partial X} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\Delta_2\frac{\partial \Delta_2}{\partial X}\right.$$
$$\left. + \Phi_1\frac{\partial^2 \Delta_1}{\partial X^2} + \Phi_2\frac{\partial^2 \Delta_2}{\partial X \partial Z} + \Phi_3\frac{\partial^3 q_0}{\partial X^3}\right\} + O(\delta^3) = 0\,, \tag{151}$$

$$\frac{\partial \Delta_1}{\partial Z} = \frac{\partial \Delta_2}{\partial X}\,. \tag{152}$$

Now for a leading order balance we again introduce the slower timescale τ (defined by 40) and make the Galilean transformation (42). Now by defining $\tilde{\Theta}$ as

$$\frac{\partial \tilde{\Theta}}{\partial \xi} = \Delta_1 \quad , \quad \frac{\partial \tilde{\Theta}}{\partial Z} = \Delta_2\,, \tag{153}$$

we obtain

$$\tilde{\Theta}_{\xi\tau} + \frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\tilde{\Theta}_\xi\tilde{\Theta}_{\xi\xi} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\tilde{\Theta}_Z\tilde{\Theta}_{\xi Z} = -\Phi_1\tilde{\Theta}_{\xi\xi\xi} - \Phi_2\tilde{\Theta}_{\xi ZZ} - \Phi_3 q_{0\xi\xi\xi}\,, \tag{154}$$

and

$$q_{0\xi\xi} + \lambda_1 q_{0ZZ} = -\lambda_2\tilde{\Theta}_{\xi\xi} - \lambda_3\tilde{\Theta}_{ZZ}\,. \tag{155}$$

As mentioned above the terms $\{\Phi_1, \Phi_2, \Phi_3\}$ can be calculated numerically (although it is not a trivial computation) from the integrals derived in the solvability condition, and $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the coefficients $\{I_2/I_1, \partial\kappa_1/\partial\alpha_0, (I_2/I_1)\,\partial\kappa_2/\partial\beta_0\}$ in (144). We observe that if $\partial_Z \equiv 0$ then $q_{0\xi\xi\xi}$ reduces to a constant multiple of $\Delta_{1\xi\xi}$, the $O(\delta)$ problem is simplified, and we return to Burgers equation once more.

There is little we can say about the solution of this equation without a detailed numerical investigation and computation of the coefficients from their integral representations. The form of the system (154)–(155) can be simplified by rescaling appropriately but only to the form

$$A_{\chi\tau} + A_\chi A_{\chi\chi} \pm A_\zeta A_{\zeta\chi} = \pm A_{\chi\chi\chi} + c_1 A_{\chi\zeta\zeta} + B_{\chi\chi\chi}\,, \tag{156}$$

$$B_{\chi\chi} + c_2 B_{\zeta\zeta} = c_2 A_{\chi\chi} + c_3 A_{\zeta\zeta}\,, \tag{157}$$

where c_j are constants, which can be written in terms of those appearing previously. We note that the first equation, (156), can be integrated with respect to χ if we redefine the dependent variable to include an arbitrary function of τ. We also note (as in Davey, Hocking and Stewartson [4]) that a return to Burgers equation (after a further rescaling) is achieved for skewed two–dimensional perturbations, i.e. for wavenumber perturbations which may be written solely in terms of a skewed variable

$$\Xi = l\chi + m\zeta\,, \tag{158}$$

for some real values l and m. Obviously the sign of the diffusive term will determine the type of solution obtained, but this must again be calculated from the coefficients introduced above and any chosen values of $\{l, m\}$.

7 Conclusions

We have applied a phase equation technique to develop a perturbation theory for slowly varying finite amplitude wavetrains, at finite Reynolds number, in plane Poiseuille flow. When the method is applied to uniform wavetrains we have shown that a small wavenumber perturbation evolves according to Burgers equation. We can obtain an exact solution to Burgers equation via the Cole–Hopf transformation and the solution is characterized by the sign of the diffusive coefficient. For a typical uniform wavetrain with $O(1)$ disturbance energy we have discussed how to compute the coefficients appearing in the evolution equation.

A number of numerical results, for solutions corresponding to cross sections of the neutral surface, have been presented showing that both diffusively stable and unstable cases are possible in general. In the unstable case (Φ positive) we know that the wavenumber perturbation develops a singularity at finite time in the slow scale. For the diffusively stable case (Φ negative) it is possible for weak shock structures to appear for non-localized or non-periodic initial conditions. We make no claims that this instability of the finite amplitude wavetrain is in any way the "most unstable", but since an initial disturbance can always be found that does not decay, it suggests that the use of uniform wavetrains in theory and computation needs some justification for large scale problems.

As the shock structures in the wavenumber perturbation are approached we expect higher order spatial derivatives to become of equal importance in the evolution equation. We see (as noted by Bernoff [1]), by an induction argument, that higher order frequency corrections are of the form

$$\Omega_n = P_n\left(\frac{\partial\alpha}{\partial X}, \frac{\partial^2\alpha}{\partial X^2}, \ldots, \frac{\partial^n\alpha}{\partial X^n}\right)\,, \tag{159}$$

a polynomial in the slow scale derivatives of the wavenumber. Thus as the length scales shorten all the higher order terms will become of comparable magnitude simultaneously. When all the previously neglected terms return to the evolution equation we must return to the full unsteady 2–D Navier–Stokes equations to determine the development.

We have shown in §3 that the phase equation method is distinct from a multi-scale amplitude perturbation approach. A perturbation of the phase in the form

$$\theta = \theta_0(x,t) + \frac{\epsilon}{\delta}\Theta_1(\delta x, \delta t) + \dots , \tag{160}$$

may be expected to be equivalent to an amplitude perturbation for ϵ sufficiently small. For uniform wavetrains however we showed, in §2, that for a leading order balance we required that the wavenumber perturbation be of comparable magnitude to the slow scale parameter, thus $\epsilon \sim \delta$ and the multi-scale approach must be distinct. The evolution equation obtained from the amplitude perturbation method is simply the heat transfer equation and no nonlinearity can be introduced. For the diffusively unstable case we must rely on linear higher order derivatives to become of comparable magnitude and alter the form of the evolution equation.

We have also shown that the evolution equation must breakdown to a different form as the leading order problem approaches the linear limit or the region where the upper and lower branches join at finite amplitude. This singularity is due to the behaviour of the neutral surface in these areas, and in the weakly nonlinear case we observed that a regularization of the evolution equation is nontrivial. Indeed this has to be the case if we are to be able to reconcile the phase equation approach with the well known results of weakly nonlinear theory, namely the Stuart-Watson equation.

To investigate the effects of three dimensionality we also developed a form for the evolution equation by extending the definitions of the phase method applied in previous sections. To simplify the basic analysis a *weak* spanwise dependence is allowed for in the uniform wavetrain problem; the analogous results for a fully three–dimensional situation are stated in the Appendix. In this case we show that the evolution equation is more complex, and is coupled with a further linear partial differential equation. This coupled equation determines an additional pressure function (required in order satisfy an impermeability at the parallel boundaries) that in the two–dimensional formulation is formally eliminated using a stream function approach. The comments made above concerning the breakdown of the phase equation method, at the $E = 0$ plane of the neutral surface, will similarly apply here in the three-dimensional approach. We should expect to be able to, likewise, match the phase method, in the appropriate areas of the neutral surface, to the results of Davey, Hocking and Stewartson [4] concerning the weakly nonlinear development of three–dimensional disturbances. The multi-scale approach, of §3, can be similarly extended to three dimensions, giving the heat equation with diffusion on both slow scales although this is again linearly coupled with the same partial differential equation for the pressure term.

Thus generally the development of, and role played by, these uniform wavetrains in PPF is complex to determine. We have shown susceptibility to weak shocks and finite time singularities for both upper and lower branch solutions, together with a complex regularization problem in the $E \to 0$ limit. As well as these slow scale effects we also have the superharmonic instability

results of Pugh & Saffman [14], which occur on $O(1)$ scales, the apparent slow decay of three dimensional perturbations to the fully developed flow as discussed by Orzsag & Patera [13] and the bypass mechanisms of Gustavsson *et al*, to mention just some of the instability mechanisms present. Obviously it is difficult to say (in any given problem) how such mechanisms will interact but to assume that any one will dominate over others needs to be carefully justified.

There are a number of questions that remain unanswered in this discussion; *i.e.* matching the phase equation theory to a relevant approach in the weakly nonlinear case, and the effect of the weak three dimensionality through computation of the evolution equations (154-155). If in future we wish to produce a full numerical procedure, to give refined values for the neutral surface, or for computation of the evolution equation with weak spanwise dependence, we should perhaps consider replacing the numerical method described previously with a similar collocation method. Such techniques were used successfully at a later stage by Herbert [9] and have the advantage of reducing computation time by allowing for a simpler evaluation of the nonlinear terms in Newtons method.

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A The Fully 3–D Problem

In this appendix we present some of the more lengthy details involved with the application of phase equation methods to a fully three-dimensional problem; by which we mean (in this application) the modulation over long spanwise and streamwise scales of a fully oblique uniform wavetrain in PPF. We prefer to present these details as an appendix in order to keep the basic method (as presented in §6 for the case of a weak spanwise dependence) as simple as possible.

Substituting the form of solution for the $O(\delta)$ system, defined by (129–132), into the inhomogeneous terms at $O(\delta^2)$ yields

$$\begin{aligned}\underline{\mathbf{L}}\{u_2,v_2,w_2,p_2\} = &\underline{\mathbf{R}}_1\,\Delta_1^2+\underline{\mathbf{R}}_2\,\Delta_2^2+\underline{\mathbf{R}}_3\,\Delta_1\Delta_2\\ &+\underline{\mathbf{R}}_4\frac{\partial\Delta_1}{\partial X}+\underline{\mathbf{R}}_5\frac{\partial\Delta_2}{\partial X}+\underline{\mathbf{R}}_6\frac{\partial\Delta_1}{\partial Z}+\underline{\mathbf{R}}_7\frac{\partial\Delta_2}{\partial Z}\\ &+\underline{\mathbf{R}}_8\frac{\partial^2 q_0}{\partial X^2}+\underline{\mathbf{R}}_9\frac{\partial^2 q_0}{\partial X\partial Z}+\underline{\mathbf{R}}_{10}\frac{\partial^2 q_0}{\partial Z^2}\\ &+\underline{\mathbf{R}}_{11}\,\Delta_1\frac{\partial q_0}{\partial X}+\underline{\mathbf{R}}_{12}\,\Delta_2\frac{\partial q_0}{\partial X}+\underline{\mathbf{R}}_{13}\,\Delta_1\frac{\partial q_0}{\partial Z}+\underline{\mathbf{R}}_{14}\,\Delta_2\frac{\partial q_0}{\partial Z}\\ &+\underline{\mathbf{R}}_{15}\left(\frac{\partial q_0}{\partial X}\right)^2+\underline{\mathbf{R}}_{16}\frac{\partial q_0}{\partial X}\frac{\partial q_0}{\partial Z}+\underline{\mathbf{R}}_{17}\left(\frac{\partial q_0}{\partial Z}\right)^2\\ &+\Omega_2\frac{\partial\hat{\underline{\mathbf{u}}}_0}{\partial\theta}-\left(\frac{\partial q_1}{\partial X},0,\frac{\partial q_1}{\partial Z}\right)^T,\end{aligned}\tag{161}$$

as the vector form of the momentum equations, where $\underline{\mathbf{L}}\equiv(L_u,L_v,L_w)^T$, together with a continuity of mass condition

$$\alpha_0\frac{\partial u_2}{\partial\theta}+\frac{\partial v_2}{\partial y}+\beta_0\frac{\partial w_2}{\partial\theta}=-\left\{\frac{\partial u_{11}}{\partial\theta}\Delta_1^2+\frac{\partial w_{12}}{\partial\theta}\Delta_2^2+\frac{\partial}{\partial\theta}(u_{12}+w_{11})\Delta_1\Delta_2\right.$$

$$
\begin{aligned}
&+ u_{11}\frac{\partial \Delta_1}{\partial X} + u_{12}\frac{\partial \Delta_2}{\partial X} + w_{11}\frac{\partial \Delta_1}{\partial Z} + w_{12}\frac{\partial \Delta_2}{\partial Z} \\
&+ u_{13}\frac{\partial^2 q_0}{\partial X^2} + (u_{14} + w_{13})\frac{\partial^2 q_0}{\partial X \partial Z} + w_{14}\frac{\partial^2 q_0}{\partial Z^2} \\
&+ \frac{\partial u_{13}}{\partial \theta}\Delta_1\frac{\partial q_0}{\partial X} + \frac{\partial w_{13}}{\partial \theta}\Delta_2\frac{\partial q_0}{\partial X} \\
&+ \frac{\partial u_{14}}{\partial \theta}\Delta_1\frac{\partial q_0}{\partial Z} + \frac{\partial w_{14}}{\partial \theta}\Delta_2\frac{\partial q_0}{\partial Z}\Big\} . \qquad (162)
\end{aligned}
$$

The coefficients, denoted by $\underline{\mathbf{R}}_i$ above, are straightforward to determine; for example

$$
\begin{aligned}
\underline{\mathbf{R}}_1 = &\frac{\partial \Omega_0}{\partial \alpha_0}\frac{\partial \underline{\mathbf{u}}_{11}}{\partial \theta} - \hat{u}_0\frac{\partial \underline{\mathbf{u}}_{11}}{\partial \theta} - \alpha_0 u_{11}\frac{\partial \underline{\mathbf{u}}_{11}}{\partial \theta} - u_{11}\frac{\partial \hat{\underline{\mathbf{u}}}_0}{\partial \theta} \qquad (163) \\
&- v_{11}\frac{\partial \underline{\mathbf{u}}_{11}}{\partial y} - \beta_0 w_{11}\frac{\partial \underline{\mathbf{u}}_{11}}{\partial \theta} - \left(\frac{\partial p_{11}}{\partial \theta}, 0, 0\right)^T + \frac{1}{Re}\left\{2\alpha_0\frac{\partial^2 \underline{\mathbf{u}}_{11}}{\partial \theta^2} + \frac{\partial^2 \hat{\underline{\mathbf{u}}}_0}{\partial \theta^2}\right\} .
\end{aligned}
$$

This system requires a solvability condition to be satisfied and therefore determines Ω_2 for a given leading order solution. Before computing the integrals needed for the evaluation of Ω_2 we note that a number of the inhomogeneous terms given above can be removed from the solvability condition by introducing some particular solutions,

$$
\underline{\mathbf{L}}\left\{\frac{\partial u_{11}}{\partial \kappa_1}, \frac{\partial v_{11}}{\partial \kappa_1}, \frac{\partial w_{11}}{\partial \kappa_1}, \frac{\partial p_{11}}{\partial \kappa_1}\right\} = \underline{\mathbf{R}}_{11} , \quad \underline{\mathbf{L}}\left\{\frac{\partial u_{12}}{\partial \kappa_1}, \frac{\partial v_{12}}{\partial \kappa_1}, \frac{\partial w_{12}}{\partial \kappa_1}, \frac{\partial p_{12}}{\partial \kappa_1}\right\} = \underline{\mathbf{R}}_{12} , \qquad (164)
$$

$$
\underline{\mathbf{L}}\left\{\frac{\partial u_{11}}{\partial \kappa_2}, \frac{\partial v_{11}}{\partial \kappa_2}, \frac{\partial w_{11}}{\partial \kappa_2}, \frac{\partial p_{11}}{\partial \kappa_2}\right\} = \underline{\mathbf{R}}_{13} , \quad \underline{\mathbf{L}}\left\{\frac{\partial u_{12}}{\partial \kappa_2}, \frac{\partial v_{12}}{\partial \kappa_2}, \frac{\partial w_{12}}{\partial \kappa_2}, \frac{\partial p_{12}}{\partial \kappa_2}\right\} = \underline{\mathbf{R}}_{14} , \qquad (165)
$$

$$
\underline{\mathbf{L}}\left\{\frac{\partial u_{13}}{\partial \kappa_1}, \frac{\partial v_{13}}{\partial \kappa_1}, \frac{\partial w_{13}}{\partial \kappa_1}, \frac{\partial p_{13}}{\partial \kappa_1}\right\} = \underline{\mathbf{R}}_{15} , \quad \underline{\mathbf{L}}\left\{\frac{\partial u_{14}}{\partial \kappa_1}, \frac{\partial v_{14}}{\partial \kappa_1}, \frac{\partial w_{14}}{\partial \kappa_1}, \frac{\partial p_{14}}{\partial \kappa_1}\right\} = \underline{\mathbf{R}}_{16} , \qquad (166)
$$

$$
\underline{\mathbf{L}}\left\{\frac{\partial u_{14}}{\partial \kappa_2}, \frac{\partial v_{14}}{\partial \kappa_2}, \frac{\partial w_{14}}{\partial \kappa_2}, \frac{\partial p_{14}}{\partial \kappa_2}\right\} = \underline{\mathbf{R}}_{17} , \qquad (167)
$$

together with the previous solutions (129)–(132); obviously these solutions likewise apply to the appropriate inhomogeneous terms of the continuity equation.

The explicit content of the solvability condition can be determined by computing the adjoint and relevant integrals, although we do not give such details here it is noted that the final form of the frequency correction must be

$$
\begin{aligned}
\Omega_2 = &\hat{\Omega}_{21}\Delta_1^2 + \hat{\Omega}_{22}\Delta_1\Delta_2 + \hat{\Omega}_{23}\Delta_2^2 + \hat{\Phi}_1\frac{\partial \Delta_1}{\partial X} + \hat{\Phi}_2\frac{\partial \Delta_2}{\partial X} \\
&+ \hat{\Phi}_3\frac{\partial \Delta_1}{\partial Z} + \hat{\Phi}_4\frac{\partial \Delta_2}{\partial Z} + \hat{\Phi}_5\frac{\partial^2 q_0}{\partial X^2} + \hat{\Phi}_6\frac{\partial^2 q_0}{\partial X \partial Z} + \hat{\Phi}_7\frac{\partial^2 q_0}{\partial Z^2} , \qquad (168)
\end{aligned}
$$

for coefficients Ω_{2i} and $\hat{\Phi}_i$, which in general must be determined numerically. Although we can not remove the terms Δ_1^2, $\Delta_1\Delta_2$, Δ_2^2 from the solvability condition we can effectively remove

them from the computation by noting that a further wavenumber perturbation of the systems satisfied by $\{u_{11}, v_{11}, w_{11}, p_{11}\}$ and $\{u_{12}, v_{12}, w_{12}, p_{12}\}$ yields

$$\hat{\Omega}_{21} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \alpha_0^2}, \quad \hat{\Omega}_{22} = \frac{\partial^2 \Omega_0}{\partial \alpha_0 \partial \beta_0}, \quad \hat{\Omega}_{23} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \beta_0^2}. \tag{169}$$

The evolution of the wavenumber perturbations is then governed by the following system

$$\begin{aligned}\frac{\partial \Delta_1}{\partial \tau} + \frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\Delta_1 \frac{\partial \Delta_1}{\partial \xi} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\Delta_2 \frac{\partial \Delta_2}{\partial \xi} + \frac{\partial^2 \Omega_0}{\partial \alpha_0 \partial \beta_0}\frac{\partial}{\partial \xi}(\Delta_1 \Delta_2) = -\hat{\Phi}_1 \frac{\partial^2 \Delta_1}{\partial \xi^2} - \hat{\Phi}_2 \frac{\partial^2 \Delta_2}{\partial \xi^2} \\ -\hat{\Phi}_3 \frac{\partial^2 \Delta_1}{\partial \zeta \partial \xi} - \hat{\Phi}_4 \frac{\partial^2 \Delta_2}{\partial \zeta \partial \xi} - \hat{\Phi}_5 \frac{\partial^3 q_0}{\partial \xi^3} - \hat{\Phi}_6 \frac{\partial^3 q_0}{\partial \zeta \partial \xi^2} - \hat{\Phi}_7 \frac{\partial^3 q_0}{\partial \xi \partial \zeta^2},\end{aligned} \tag{170}$$

$$\frac{\partial \Delta_1}{\partial \zeta} = \frac{\partial \Delta_2}{\partial \xi} \tag{171}$$

$$\frac{\partial^2 q_0}{\partial \xi^2} + \hat{\lambda}_1 \frac{\partial^2 q_0}{\partial \xi \partial \zeta} + \hat{\lambda}_2 \frac{\partial^2 q_0}{\partial \zeta^2} = -\hat{\lambda}_3 \frac{\partial \Delta_1}{\partial \xi} - \hat{\lambda}_4 \frac{\partial \Delta_2}{\partial \xi} - \hat{\lambda}_5 \frac{\partial \Delta_1}{\partial \zeta} - \hat{\lambda}_6 \frac{\partial \Delta_2}{\partial \zeta}, \tag{172}$$

where we have introduced the constants $\hat{\lambda}_i$, the slower timescale τ and the new coordinate ζ defined by

$$\zeta = Z - \frac{\partial \Omega_0}{\partial \beta_0} T. \tag{173}$$

The equation (172) and constants $\hat{\lambda}_i$ are obtained from solving the continuity equation (162) for the mean flow term, $\bar{v}_2$, and applying the impermeability condition at the boundaries $y = \pm 1$.

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13. ABSTRACT *(Maximum 200 words)*

We consider a viscous incompressible fluid flow driven between two parallel plates by a constant pressure gradient. The flow is at a finite Reynolds number, with an 0(1) disturbance in the form of a traveling wave. A phase equation approach is used to discuss the evolution of slowly varying fully nonlinear two dimensional wavetrains. We consider uniform wavetrains in detail, showing that the development of a wavenumber perturbation is governed by Burgers equation in most cases. The wavenumber perturbation theory, constructed using the phase equation approach for a uniform wavetrain, is shown to be distinct from an amplitude perturbation expansion about the periodic flow. In fact we show that the amplitude equation contains only linear terms and is simply the heat equation. We review, briefly, the well known dynamics of Burgers equation, which imply that both shock structures and finite time singularities of the wavenumber perturbation can occur with respect to the slow scales. Numerical computations have been performed to identify areas of the (wavenumber, Reynolds number, energy) neutral surface for which each of these possibilities can occur. We note that the evolution equations will breakdown under certain circumstances, in particular for a weakly nonlinear secondary flow. Finally we extend the theory to three dimensions and discuss the limit of a weak spanwise dependence for uniform wavetrains, showing that two functions are required to describe the evolution. These unknowns are a phase and a pressure function which satisfy a pair of linearly coupled partial differential equations. The results obtained from applying the same analysis to the fully three dimensional problem are included as an appendix.

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