A family of dynamic models
for large-eddy simulation

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1. Motivation and objectives

Since its first application, the dynamic procedure has been recognized as an effective means to compute rather than prescribe the unknown coefficients that appear in a subgrid-scale model for Large-Eddy Simulation (LES). The dynamic procedure (Germano *et al.* 1991; Ghosal *et al.* 1995) is usually used to determine the non-dimensional coefficient in the Smagorinsky (1963) model. In reality the procedure is quite general and it is not limited to the Smagorinsky model by any theoretical or practical constraints. The purpose of this note is to consider a generalized family of dynamic eddy viscosity models that do not necessarily rely on the local equilibrium assumption built into the Smagorinsky model. By invoking an inertial range assumption, it will be shown that the coefficients in the new models need not be non-dimensional. This additional degree of freedom allows the use of models that are scaled on traditionally unknown quantities such as the dissipation rate. In certain cases, the dynamic models with dimensional coefficients are simpler to implement, and allow for a 30% reduction in the number of required filtering operations.

2. Accomplishments

2.1 A new family of dynamic eddy viscosity models

The LES equations are obtained from the Navier-Stokes equations by applying a filter, denoted by an overline, which is assumed to damp scales smaller than Δ. In the context of eddy viscosity models, the unknown subgrid-scale stress generated by this operation, $\tau_{ij} = \overline{u_i\,u_j} - \overline{u}_i\,\overline{u}_j$, is assumed to be proportional to the strain tensor $\overline{S}_{ij} = (\partial_i\overline{u}_j + \partial_j\overline{u}_i)/2$:

$$\tau_{ij} = -2\nu_e\overline{S}_{ij}. \tag{1}$$

The eddy viscosity, ν_e, has dimensions L^2/T, where L is length and T is time. The characteristic length in the problem is obviously $L_c = \Delta$. Following the Kolmogorov (1941) dimensional analysis, the characteristic time may be expressed as a function of the rate of energy transfer within the inertial range $\mathcal{E}$: $T_c = (\Delta^2/\mathcal{E})^{1/3}$. The "Kolmogorov expression" for the eddy viscosity is thus:

$$\nu_e = c_k\mathcal{E}^{1/3}\Delta^{4/3}, \tag{2}$$

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where c_k is a non-dimensional constant. The rate of energy transfer is usually not directly accessible in LES, and thus Smagorinsky proposed to identify the rate energy transfer within the inertial range with the subgrid-scale dissipation:

$$\mathcal{E} \approx -\tau_{ij}\overline{S}_{ij} = \nu_e|\overline{S}|^2, \tag{3}$$

where $|\overline{S}|^2 = 2\overline{S}_{ij}\overline{S}_{ij}$. When integrated over the volume, the above relation becomes a good approximation since nearly all the dissipation will be carried by the subgrid-scale model when the cutoff is in the inertial range. In the Smagorinsky model, this equality is assumed to be valid at every point in space by invoking a local-equilibrium assumption between production and dissipation of energy. Inserting relation (3) into the Kolmogorov scaling for the eddy viscosity (2) gives the Smagorinsky model

$$\nu_e = c_s|\overline{S}|\Delta^2, \tag{4}$$

where $c_s = c_k^{(3/2)}$ is the non-dimensional Smagorinsky constant. In the Smagorinsky model, the time scale is seen to be $|\overline{S}|^{-1}$. Thus, if local equilibrium is assumed, two expressions are available for the time scale in the eddy viscosity. By dimensional analysis, the eddy viscosity can depend on the ratio of these two time scales as well as on the fundamental scaling in Eq. (2). The most general model can therefore be written as

$$\nu_e = F\left(\frac{|\overline{S}|^3\Delta^2}{\mathcal{E}}\right) \mathcal{E}^{1/3} \Delta^{4/3}, \tag{5}$$

where F is an arbitrary function. In particular, we may focus on a series representation for F:

$$\nu_e = \sum_{l=1}^{n} c_l \, |\overline{S}|^{\zeta_l} \, \mathcal{E}^{(1-\zeta_l)/3} \, \Delta^{(4+2\zeta_l)/3}. \tag{6}$$

Here ζ_l are a sequence of numbers that define the exponents for the various terms in the series. They need not be integers. The parameters c_l are non-dimensional coefficients. As important special cases, note that $n = 1, \zeta_1 = 0$ leads to the Kolmogorov scaling with $c_1 = c_k$, whereas $n = 1, \zeta_1 = 1$ leads to the Smagorinsky model with $c_1 = c_s$.

While Eq. (6) is rather general, it has the apparent drawback that the unknown dissipation rate, $\mathcal{E}$, appears as a model parameter for $\zeta_l \neq 1$. Historically this defect has effectively excluded all models encompassed by Eq. (6) except for the Smagorinsky model. The situation has changed with the introduction of the dynamic procedure, however, and it is possible to use Eq. (6) generally if it is recast in a slightly different form. If we assume that the test and grid filters are in the inertial range, then the dissipation rate as well as each of the model coefficients, c_l, should be the same at two filtering levels. The product of the dissipation rate (raised to some power) and a model coefficient should also be invariant with filtering

scale, and thus the dynamic procedure may be used to determine the *dimensional* parameters $\tilde{c}_l = c_l \mathcal{E}^{(1-\zeta_l)/3}$. Thus when Eq. (6) is recast in terms of $\tilde{c}_l$, we can make use of Eq. (1) and write the subgrid-scale models at the grid and test level as

$$\tau_{ij} = -2 \sum_{l=1}^{n} \tilde{c}_l |\overline{S}|^{\zeta_l} \Delta^{(4+2\zeta_l)/3)} \; \overline{S}_{ij}, \tag{7a}$$

$$T_{ij} = -2 \sum_{l=1}^{n} \tilde{c}_l |\widehat{\overline{S}}|^{\zeta_l} \hat{\Delta}^{(4+2\zeta_l)/3} \; \widehat{\overline{S}}_{ij}, \tag{7b}$$

where $\hat{\Delta}$ is the test-filter width and $\widehat{\overline{S}}_{ij}$ is the test-filtered strain rate. When Eqs. (7a) and (7b) are substituted into the Germano identity (Germano *et al.* 1991), a set of integral equations for the $\tilde{c}_l$ are obtained. Following Ghosal *et al.* (1995) we can reduce the integral equations to algebraic relations if we constrain the coefficients to have no spatial variation over the directions in which the test filter is applied. The end result is

$$\langle \dot{M}_{lk} \rangle \tilde{c}_k = -\langle L_{ij} m_{ij}^{(l)} \rangle, \tag{8}$$

where the Leonard tensor is given by $L_{ij} = \widehat{\overline{u}_i \overline{u}_j} - \widehat{\overline{u}}_i \widehat{\overline{u}}_j$. The lth model tensor is defined as

$$m_{ij}^{(l)} = -2 \left(\Delta^{(4+2\zeta_l)/3} \widehat{|\overline{S}|^{\zeta_l} \overline{S}_{ij}} - \hat{\Delta}^{(4+2\zeta_l)/3} |\widehat{\overline{S}}|^{\zeta_l} \widehat{\overline{S}}_{ij} \right). \tag{9}$$

The left hand side of Eq. (8) is a matrix of products of these tensors: $M_{lk} = m_{ij}^{(l)} m_{ij}^{(k)}$. Finally, $\langle \rangle$ denotes a spatial average taken over the directions in which the test filter is applied*. Note that when $n \neq 1$, a linear system must be solved in order to determine the dynamic model coefficients. When the pure Kolmogorov scaling ($n = 1, \zeta_1 = 0$) is used, the dynamic estimation for the eddy viscosity reduces to:

$$\nu_e \approx -\frac{1}{2(\alpha^{4/3} - 1)} \frac{\langle L_{ij} \widehat{\overline{S}}_{ij} \rangle}{\langle \widehat{\overline{S}}_{ij} \widehat{\overline{S}}_{ij} \rangle}, \tag{10}$$

where $\alpha = \hat{\Delta}/\Delta$. This relation was derived earlier by Wong & Lilly, (1994). This model has the advantage that knowledge of the Smagorinsky time scale $|\overline{S}|$ is not required, and thus the model is independent of the local equilibrium assumption. The Kolmogorov model also has the practical advantage that fewer filtering operations are required as compared with the Smagorinsky model. This is true since the term $\widehat{|\overline{S}|\overline{S}}_{ij}$ does not appear in the Kolmogorov model. Finally, it should

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* In practice averaging is usually not performed in inhomogeneous directions even if these are included in the test filter. This inconsistency introduces an error that has been found to have a negligible impact on the simulation results (Ghosal *et al.*, 1995).
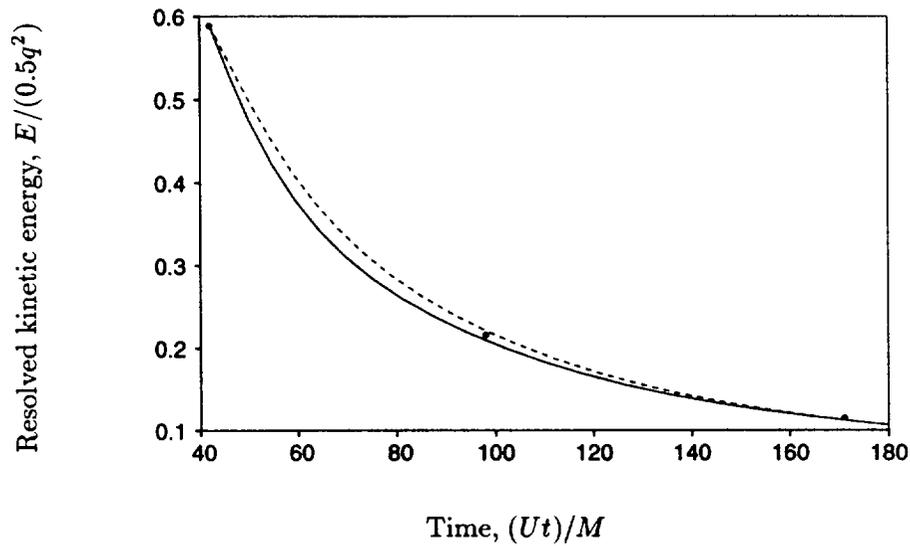
FIGURE 1.   Decay of resolved turbulent kinetic energy. ——— : Dynamic Smagorin-sky model; ———— : Dynamic Kolmogorov model; • : filtered experimental data of Comte-Bellot and Corrsin (1971). U is the mean advection speed in the wind tunnel experiments, M is the spacing between the bars in the turbulence-generating grid, and $0.5q^2$ is the total turbulent kinetic energy at the first measurement station.
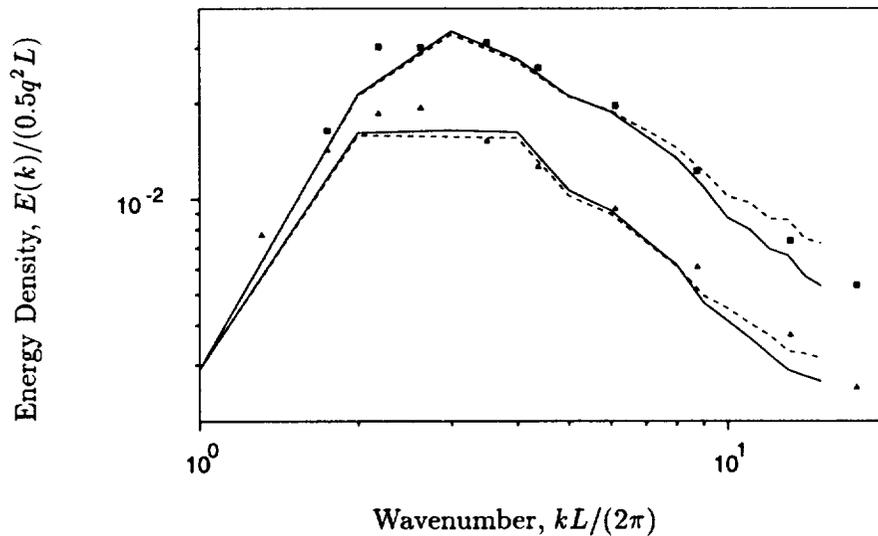

FIGURE 2.   Velocity spectra. ——— : Dynamic Smagorinsky model; ———— : Dynamic Kolmogorov model; ■ , ▲ : experimental data of Comte-Bellot and Corrsin (1971) for $Ut/M = 98$ and 171 respectively. $L = 10.8M$ is the length of a side of the computational box. The other scaling parameters are defined in Fig. 1.

be mentioned that models mixing the Kolmogorov and the Smagorinsky scalings ($n \geq 2, \zeta_1 = 0, \zeta_2 = 1$) could be investigated for situations with poorly developed inertial ranges. Indeed, in that case both Kolmogorov and Smagorinsky time scales might play independent roles and the dynamic procedure could determine the relative weighting of these two scalings.

2.2 Numerical tests

As a first step in evaluating the new class of models, the Kolmogorov model (Eq. (10)) is tested in simulations of decaying isotropic turbulence. The simulations target the experimental measurements of Comte-Bellot and Corrsin (1971) and are performed with a pseudo-spectral code (Rogallo, 1981) using 32^3 mesh points. The equation for the model coefficient is averaged over the volume so that the coefficient is a function of time only. The simulations are initialized so that the 3-D energy spectrum agrees with the experimental data (up to the mesh wavenumber) at the first measuring station. The initial field is obtained by simulating the decay from an earlier time where the velocity phases are set at random. By iteratively adjusting the energy spectrum at the earlier time, it is possible to construct a field that has the desired energy spectrum as well as realistic phase information. The objective of the simulation is to predict the energy decay rate and the 3-D spectrum at the two subsequent experimental measurement stations.

Figure 1 shows the kinetic energy decay history for the dynamic Kolmogorov and Smagorinsky models. There is little difference between the results of the two models and both agree quite well with the experimental data. Near the starting point, the Kolmogorov model is seen to be slightly less dissipative than the Smagorinsky model. This could have to do with the fact that the initial field is generated with the Smagorinsky model and thus a transient is introduced when the model is suddenly switched to the Kolmogorov scaling. Three-dimensional velocity spectra are shown in Fig. 2. Again there is very little difference between the two models. The spectra are seen to be slightly less damped at high wavenumbers in the case of the Kolmogorov model. This difference actually makes the Kolmogorov model agree slightly better with the experimental data at the final measurement station.

The results of these tests suggest that the dynamic Kolmogorov model may work just as well as the Smagorinsky model. This is significant since comparable accuracy can be expected with 30% fewer filtering operations. The fact that the Kolmogorov scaling works also suggests that other terms in Eq. (6) may be useful as well.

3. Future plans

The Kolmogorov model will be tested next in turbulent channel flow. If is proves successful there it will be incorporated in the CTR complex geometry codes. Once these results are interpreted, we will study models that include more terms with the obvious first choice being a blend of Smagorinsky and Kolmogorov scaling ($n = 2, \zeta_1 = 0, \zeta_2 = 1$).

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