An extended structure-based model based on a stochastic eddy-axis evolution equation

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1. Motivation and objectives

Engineering analysis of complex turbulent flows relies heavily on turbulence models. A good model should have a viscoelastic character, predicting turbulent stresses proportional to the mean strain rate for slow deformations and stresses determined by the amount of strain for rapid distortions. Current turbulence models work well only in near-equilibrium situations where the turbulent stresses can be predicted adequately using eddy viscosity representations. They do not perform well when the turbulence is subjected to strong or rapid deformations, which is the case in many engineering systems. More elaborate schemes in which the Reynolds Stress Transport (RST) equations are included in the PDE system have been used in an effort to rectify these problems. While RST models have enjoyed some success, they are not yet widely used in industry because they have not proven reliably better than simpler models in dealing with the more challenging types of complex flows.

We have shown that the Reynolds stresses do not always provide a complete description of the turbulence state and that this poses a fundamental problem for standard RST models that use the Reynolds stress tensor (along perhaps with the mean velocity gradient) as the unique tensorial base for the modeling of the unknown terms. The inadequacy of componentality information is more pronounced in flows with strong mean rotation. These ideas are described in detail by Kassinos and Reynolds (1994), hereafter denoted by KR.

Proper characterization of the state of the turbulence in non-equilibrium flows requires the inclusion of *structure* information to complement the *componentality* carried by the turbulent stresses. We have introduced a number of one-point turbulent tensors carrying non-local information about the turbulence structure and demonstrated how they could be used for the construction of one-point models. However, this approach would require the addition of one second-rank and one fully symmetric third-rank tensor in the PDE system, a considerable overhead for an engineering model.

These considerations motivated the *structure-based* model which incorporates the key structure information in a simple phenomenological approach. The goal is to construct an engineering model with proper viscoelastic character that will reduce to the form of a k-ϵ model when the mean deformation is weak, and will match rapid distortion theory (RDT) when the mean deformation is strong.

The backbone of the structure-based model is a one-point, structure-based model of RDT for homogeneous turbulence. The development of this RDT model has been completed successfully and reported in great in detail in KR, and for that reason is not discussed here. This preliminary report focuses on the extensions of the structure-based to flows with weak mean deformation rates. If the structure of the turbulence is assumed to be in equilibrium with the mean field and weakly anisotropic, the structure-based model reduces to the form of a k-ϵ model. Hence, we should be able to extend the model so that it spans between an eddy viscosity model, appropriate for weak mean strain rates, and RDT appropriate for high mean strain rates.

2. Accomplishments

2.1 Overview of the structure-based model

2.1.1 Algebraic equations

In a standard k-ϵ model, the turbulent stress tensor R_{ij} is related to the mean strain rate tensor S_{ij} through an eddy viscosity

$$R_{ij} = \frac{1}{3}q^2\delta_{ij} - 2\nu_T S_{ij} \qquad \nu_T = C_\mu k^2/\epsilon \tag{1}$$

where $R_{ii} = q^2 = 2k$. Transport equations are used for k and ϵ but not for R_{ij} itself. In the structure-based model, we also determine the Reynolds stresses through an algebraic constitutive equation. The difference is that we relate the turbulent stresses to parameters of the turbulence structure instead of the mean strain rate:

$$R_{ij} = q^2[(1-\phi)\frac{1}{2}(\delta_{ij} - a_{ij}) + \phi a_{ij} + \gamma \frac{1}{2}\frac{\Omega_k}{\Omega}(\epsilon_{iky}a_{yj} + \epsilon_{jky}a_{yi})]\,. \tag{2}$$

Here Ω_i is the mean vorticity vector and $\Omega^2 = \Omega_i\Omega_i$. The eddy-axis tensor a_{ij} carries information on the orientation and shape of large-scale eddies. The two scalar parameters (ϕ and γ) determine the character of the turbulence structure: ϕ is the fraction of the energy in the *jetal mode* (motion along the eddy axes), $1-\phi$ is the fraction of the energy in the *vortical mode* (motion in the plane normal to the eddy axes), and γ is the jet-vortex correlation parameter. In the RDT model, we only carry the transport equations for the structure parameters. For weak mean deformations (small Sk/ϵ), we need to add the transport equations for the turbulence scales k and ϵ.

The derivation of the algebraic constitutive Eq. (2) for r_{ij} is based on a representation the turbulence as a superposition of two-dimensional eddy fields. The motivation is to account for the effects of the mean deformation on the energy-containing eddies. The normalized eddy-axis tensor a_{ij} represents an energy-weighted direction cosine tensor of the large eddies,

$$A_{ij} \equiv \langle V^2 a_i a_j \rangle = A_{kk} a_{ij}, \tag{3}$$

where $\langle\ \rangle$ denotes an averaging, V^2 is twice the kinetic energy of the basis field, and a_i is a unit vector aligned with the axis of independence of the field of 2D eddies. Note that $A_{ii} = \langle V^2 \rangle = q^2$, and so the eddy-axis tensor scales on the turbulent kinetic energy, as does R_{ij}. When the turbulence structure is isotropic, all the eddies are randomly distributed and $a_{ij} = \delta_{ij}/3$. For turbulence consisting of 2D vortices aligned with the x_1 direction, $a_{11} = 1$ and all other components vanish. Hence, the eddy-axis tensor carries the dimensionality information needed by a turbulence model. In fact, a_{ij} is related to the structure *dimensionality* tensor (see KR) through the model algebraic equation

$$D_{ij} = \tfrac{1}{2}q^2(\delta_{ij} - a_{ij}). \tag{4}$$

2.1.2 Transport equations

We use PDE for the structure parameters but not for the turbulent stresses themselves. The evolution equation for the normalized eddy-axis tensor a_{ij} is determined from definition (3) and the kinematics of the eddy axis vector a_i. In the RDT limit a_i satisfies the simple equation

$$\frac{da_i}{dt} = G_{ik}a_k - G_{nm}a_m a_n a_i \tag{5}$$

where $G_{ij} = U_{i,j}$ is mean gradient tensor. Using (5), definition (3) and some analysis, one can show that in the RDT limit the evolution of a_{ij} is given by

$$\begin{aligned} \frac{da_{ij}}{dt} =& G^*_{ik}a_{kj} + G^*_{jk}a_{ki} - [3\phi + 1]G^*_{km}Z_{kmij} + (3\phi - 1)G^*_{nm}a_{nm}a_{ij} \\ & - 2\gamma S^*_{nm}\frac{\Omega_k}{\Omega}\epsilon_{nkt}(Z_{tmij} - a_{tm}a_{ij}) \end{aligned} \tag{6}$$

where $G^*_{ij} = G_{ij} - G_{kk}\delta_{ij}/3$ and $S^*_{ij} = S_{ij} - S_{kk}\delta_{ij}/3$. Closure of (6) requires modeling of the energy weighted fourth-moment

$$Z_{ijnm} = \langle V^2 a_i a_j a_n a_m \rangle / q^2 \tag{7}$$

in terms of the second moments a_{ij}. A fully realizable accurate model for $\mathbf{Z}$ has been developed (see KR).

The evolution equations for the two scalar parameters

$$\frac{d\phi}{dt} = \ldots \qquad\qquad \frac{d\gamma}{dt} = \ldots \tag{8}$$

are derived from the Navier-Stokes equations with some modeling to account for information lost in conditionally averaging over the eddies. The exact form of these equations is given in KR (see Eqs. 5.10.4 and 5.10.5 therein).

2.2 Blending of RDT and k-ε modeling for homogeneous turbulence

The simplest approach in extending the RDT structure-based model to slow deformations is the addition of terms in the evolution equations for a_{ij} that model the restoration of isotropy as a result of turbulence-turbulence interactions. Similar terms in the evolution equations for ϕ and γ will restore the vortical turbulence ($\phi = 0, \gamma = 0$) appropriate for isotropy.

For slow deformations we must add equations for the turbulence scales. We are currently investigating the use of the familiar k and ϵ equations, with minor modifications to take advantage of the structure information provided by the eddy-axis tensor. An algebraic relationship is used to obtain the turbulence time scale τ in terms of k and ϵ.

The modeling of the return to isotropy terms in the eddy-axis tensor Eq. (6) is perhaps the most sensitive step in implementing these extensions since the simple kinematic basis of this equation is critical for full realizability in the RDT limit. The added return-to-isotropy terms must capture the key physics without disturbing the realizability of the model. For this reason, we next discuss in detail a method of extending the a_{ij} equation that guarantees maintaining realizability.

2.3 A stochastic eddy-axis evolution equation

In the RDT limit, the eddy axis vector a_i evolves according to the simple kinematic Eq. (5). When the mean deformation is weak, this equation must also involve return to isotropy terms accounting for the eddy-eddy (or turbulence-turbulence) interactions. Guidance on the form of these isotropization terms can be obtained by considering a generalization of the eddy-axis kinematic equation that includes stochastic forcing terms, in analogy to the Langevin equation (Arnold, 1974). This approach offers the advantage that the realizability of the resulting eddy-axis transport equation is guaranteed (Durbin and Speziale, 1994). We work with the energy-scaled eddy-axis vector

$$A_i = Va_i \tag{9}$$

where $V = \sqrt{V_k V_k}$ and a_i is the unit eddy-axis vector. The RDT evolution equation for A_i is simply [see (5) and (9)]:

$$\frac{dA_i}{dt} = G_{ik}A_k - G_{nm}a_n a_m A_i - G_{nm}v_n v_m A_i \tag{10}$$

where $v_i = V_i/V$. Next we consider a stochastic generalization of (10) given by

$$\begin{aligned} dA_i =& [G_{ik}A_k - G_{nm}a_n a_m A_i - G_{nm}v_n v_m A_i]dt \\ &+ C_1 A_i dt + C_2 d\mathcal{W}_i + C_3 d\mathcal{W}_p \Gamma_p A_i + C_4 \epsilon_{ipq} d\mathcal{W}_p A_q . \end{aligned} \tag{11}$$

The stochastic forcing in (11) is provided by the Wiener process $d\mathcal{W}_i(t)$, which has increments that are steps of the random walk and provide Guassian white-noise forcing (Arnold 1974). The properties of these increments are

$$\overline{d\mathcal{W}_i} = 0 \qquad \overline{d\mathcal{W}_i d\mathcal{W}_j} = dt\delta_{ij} \qquad \overline{d\mathcal{W}_i} = 0\,. \tag{12}$$

The second property in (12) shows that the Wiener process has magnitude $d\mathcal{W} = O(dt)^{1/2}$; therefore, $d\mathcal{W}_i/dt$ is not defined as $dt \to 0$. Hence, in order to evaluate $dA_{ij}/dt = d\overline{A_i A_j}/dt$, we first form the product

$$d(A_i A_j) = (A_i + dA_i)(A_j + dA_j) - A_i A_j \tag{13}$$

retaining terms to $O(dt)$, then average over all eddies, and finally divide by dt. Note that the coefficients in (11) are not necessarily constants and are assumed to have the appropriate functional forms (in terms of deterministic functions like k, ϵ, etc.) that give the correct dimensions in each term. The C_1 term is introduced by analogy to the Langevin equation and, as will be shown, is needed in order to ensure realizability. The C_2 term provides isotropic stochastic forcing that tends to randomize the orientation of the eddy axes. The deterministic vector Γ_i acts as an organizing vector for the stochastic forcing in the C_3 term; for example, Γ_i could represent an organizing effect for the non-linear interactions provided by the structure of the larger scales or the mean field. Finally, the C_4 term assumes that the non-linear turbulence-turbulence interactions can provide an effective random rotation acting on an individual eddy axis. Substituting (11) in (13), one obtains

$$\begin{aligned} d(A_iA_j) = {} & [G_{ik}A_kA_j + G_{jk}A_kA_i - 2G_{nm}(a_na_m + v_nv_m)A_iA_j + 2C_1A_iA_j]dt \\ & + C_2^2 dW_i dW_j + C_2C_3(dW_i dW_q \Gamma_q A_j + dW_p\Gamma_p A_i dW_j) \\ & + C_2C_4(dW_i\epsilon_{jqr}dW_qA_r + \epsilon_{ipt}dW_pA_tdW_j) \\ & + C_3^2 dW_p\Gamma_pA_idW_q\Gamma_qA_j + C_3C_4(\epsilon_{jqr}A_rA_i + \epsilon_{iqt}A_tA_j)dW_qdW_p\Gamma_p \\ & + C_4^2\epsilon_{ipt}dW_pA_t\epsilon_{jqr}dW_qA_r\,. \end{aligned} \tag{14}$$

Averaging over the ensemble of eddies and simplifying, one obtains

$$\begin{aligned} \frac{dA_{ij}}{dt} = {} & G_{ik}A_{kj} + G_{jk}A_{ki} - 2G_{nm}q^2Z_{nmij} - 2G_{nm}\langle v_nv_mA_iA_j\rangle + 2C_1A_{ij} \\ & + C_2^2\delta_{ij} + C_3^2\Gamma^2A_{ij} + C_3C_4\Gamma_p(\epsilon_{jpt}A_{ti} + \epsilon_{ipt}A_{tj}) + C_4^2(q^2\delta_{ij} - A_{ij})\,. \end{aligned} \tag{15}$$

Note that we have no control over the sign of the terms involving C_2^2, C_3^2 and C_4^2, which must be positive for realizability, but we have a choice over the sign of the terms involving C_1 and C_3C_4. Taking the trace of (15), one finds

$$\frac{dA_{ii}}{dt} = \frac{dq^2}{dt} = 2P - 2\epsilon = -2G_{nm}R_{nm} + 2C_1q^2 + 3C_2^2 + C_3^2\Gamma^2q^2 + 2C_4^2q^2\,. \tag{16}$$

Therefore we must have

$$2C_1q^2 + 3C_2^2 + C_3^2\Gamma^2q^2 + 2C_4^2q^2 = -2\epsilon\,. \tag{17}$$

Based on dimensional considerations, we let

$$C_1 = \tilde{C}_1/\tau \qquad C_2 = \tilde{C}_2\sqrt{\epsilon} \qquad C_3 = \tilde{C}_3/\sqrt{\tau} \qquad C_4 = \tilde{C}_4/\sqrt{\tau}\,. \tag{18}$$

Then condition (17) becomes

$$2\tilde{C}_1q^2/\tau + 3\tilde{C}_2\epsilon + \tilde{C}_3\Gamma^2q^2/\tau + 2\tilde{C}_4^2q^2/\tau = -2\epsilon\,, \tag{19}$$

and if we assume $\tau = q^2/2\epsilon$ then

$$2\tilde{C}_1 + \frac{3}{2}\tilde{C}_2^2 + \tilde{C}_3^2\Gamma^2 + 2\tilde{C}_4^2 = -1 \,. \tag{20}$$

Note that realizability requires $\tilde{C}_1 \leq -1/2$. Using (3), (15), and (18), one finds that the evolution equation for a_{ij} is given by

$$\begin{aligned}\frac{da_{ij}}{dt} &= G_{ik}a_{kj} + G_{jk}a_{ki} - 2G_{nm}Z_{nmij} - 2G_{nm}\langle v_n v_m A_i A_j\rangle/q^2 + 2S_{nm}r_{nm}a_{ij} \\ &+ \frac{1}{\tau}\left[(1 + 2\tilde{C}_1 + \tilde{C}_3^2\Gamma^2 - \tilde{C}_4^2)a_{ij} + \frac{1}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2)\delta_{ij} + \tilde{C}_3\tilde{C}_4\Gamma_p(\epsilon_{jpt}a_{ti} + \epsilon_{ipt}a_{tj})\right] .\end{aligned} \tag{21}$$

Note that the trace condition $da_{ii}/dt = 0$ is satisfied if (20) is assumed. Next we require that the terms involving $\tilde{C}_1$, $\tilde{C}_2^2$, $\tilde{C}_3^2$, and $\tilde{C}_4^2$ in (21) combine to produce a return to anisotropy term of the form $C_*^2(\frac{1}{3}\delta_{ij} - a_{ij})$. This can be accomplished if

$$-1 = 2\tilde{C}_1 + \frac{3}{2}\tilde{C}_2^2 + \tilde{C}_3^2\Gamma^2 + 2\tilde{C}_4^2 \,. \tag{22}$$

Note that (22) is identical to (20), and hence this additional requirement is automatically satisfied if the trace condition (20) is imposed. Solving (20) for $\tilde{C}_1$ and substituting back into (21), one finds

$$\begin{aligned}\frac{da_{ij}}{dt} &= G_{ik}a_{kj} + G_{jk}a_{ki} - 2G_{nm}Z_{nmij} - 2G_{nm}\langle v_n v_m A_i A_j\rangle/q^2 + 2S_{nm}r_{nm}a_{ij} \\ &+ \frac{1}{\tau}\left[\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2)(\frac{1}{3}\delta_{ij} - a_{ij}) + \tilde{C}_3\tilde{C}_4\Gamma_p(\epsilon_{jpt}a_{ti} + \epsilon_{ipt}a_{tj})\right] .\end{aligned} \tag{23}$$

The fourth term on the RHS of (23) can be evaluated using the conditional averaging procedure described in KR (see pp. 85-95). Substituting the resulting expression in (23) produces

$$\begin{aligned}\frac{da_{ij}}{dt} &= G^*_{ik}a_{kj} + G^*_{jk}a_{ki} - [3\phi + 1]G^*_{km}Z_{kmij} + (3\phi - 1)G^*_{nm}a_{nm}a_{ij} \\ &- 2\gamma S^*_{nm}\frac{\Omega_k}{\Omega}\epsilon_{nkt}(Z_{tmij} - a_{tm}a_{ij}) \\ &+ \frac{1}{\tau}\left[\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2)(\frac{1}{3}\delta_{ij} - a_{ij}) + \tilde{C}_3\tilde{C}_4\Gamma_p(\epsilon_{jpt}a_{ti} + \epsilon_{ipt}a_{tj})\right] .\end{aligned} \tag{24}$$

The algebraic k-ϵ Eq. (1), expressed in terms of the Reynolds stress anisotropy $\tilde{r}_{ij} = r_{ij} - \delta_{ij}/3$, is given by

$$\tilde{r}_{ij} = -\frac{2\nu_T}{q^2}S^*_{ij} = -2C_\mu\frac{q^2}{4\epsilon}S^*_{ij} = -C_\mu\frac{q^2}{2\epsilon}S^*_{ij} \tag{25}$$

where $\nu_T = C_\mu k^2/\epsilon = C_\mu (q^2)^2/(4\epsilon)$ is the turbulent viscosity. For irrotational mean strain, the algebraic constitutive Eq. (2) for the structure-based model produces (with $\phi = \gamma = 0$)

$$\tilde{r}_{ij} = -\frac{1}{2}\tilde{a}_{ij} \tag{26}$$

where $\tilde{a}_{ij} = a_{ij} - \frac{1}{3}\delta_{ij}$ is the anisotropy of a_{ij}. From (25) and (26) we see that consistency with k-ϵ modeling (in the weak strain limit) would require

$$\tilde{a}_{ij} = C_\mu \frac{q^2}{\epsilon} S^*_{ij}\,. \tag{27}$$

Now if we assume equilibrium under weak strain rates in (23), we obtain

$$\frac{1}{\tau}[\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2)\tilde{a}_{ij}] = \frac{4}{15}S^*_{ij}\,. \tag{28}$$

Substituting (27) in (28), one finds that consistency between the structure-based model and k-ϵ modeling in the limit of equilibrium structure under weak deformation requires

$$\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2) = \frac{4}{15C_\mu}\,. \tag{29}$$

Next we consider two limiting cases where there are no non-linear eddy-eddy interactions, and hence the coefficient $\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2)$ should vanish. The first case is that of a 2C-field of jets having $a_{22} = 0$ and $\phi = 1$, corresponding to the type of structure one might expect to find at the wall in a boundary layer. The jets in this 2C field have no way of re-orienting each other towards a more isotropic distribution. The second limiting case is that of a 2D-field of vortices with $a_{11} = 1$ and $\phi = 0$, corresponding to the RDT limiting state in the irrotational axisymmetric contraction flow. Again these vortices have no means of re-orienting each other, and the return to isotropy must shut off. Both of these limiting cases can be accounted for by the postulating the functional form

$$\frac{3}{2}(\tilde{C}_2^2 + 2\tilde{C}_4^2) = \alpha(1-\phi)(1-a^2) \tag{30}$$

where $a^2 = a_{ik}a_{ki}$. Then the k-ϵ consistency requirement (29) for equilibrium under weak strain rates produces

$$\alpha = \frac{2}{5C_\mu}\,. \tag{31}$$

Based on this analysis, we propose using

$$\begin{aligned}\frac{da_{ij}}{dt} &= G^*_{ik}a_{kj} + G^*_{jk}a_{ki} - [3\phi+1]G^*_{km}Z_{kmij} + (3\phi-1)G^*_{nm}a_{nm}a_{ij} \\ &- 2\gamma S^*_{nm}\frac{\Omega_k}{\Omega}\epsilon_{nkt}(Z_{tmij} - a_{tm}a_{ij}) \\ &+ \frac{1}{\tau}\Big[\alpha(1-\phi)(1-a^2)(\frac{1}{3}\delta_{ij} - a_{ij}) + C_e\frac{\Omega_p}{\Omega}(\epsilon_{jpt}a_{ti} + \epsilon_{ipt}a_{tj})\Big]\,.\end{aligned} \tag{32}$$

Considering equilibrium in homogeneous shear suggests the values using $\alpha \approx 1.8$ and $C_\epsilon \approx -0.35$, which we have adopted. Note that we have taken the deterministic vector Γ_i to be a unit vector aligned with the mean vorticity vector Ω_i; this choice was suggested by looking at a number of homogeneous flows, including homogeneous shear, irrotational axisymmetric strain, and plane stain.

2.4 Extension of the scalar transport equations

We are currently using one simple term in each of the two scalar equations that tend to restore vortical turbulence ($\phi = 0, \gamma = 0$), appropriate for isotropy. The form of the extended equations is as follows:

$$\frac{d\phi}{dt} = \ldots \text{RDT} \ldots - C_\phi \phi/\tau \qquad \text{with} \qquad C_\phi = 1.3 \tag{33}$$

$$\frac{d\gamma}{dt} = \ldots \text{RDT} \ldots - C_\gamma \gamma/\tau \qquad \text{with} \qquad C_\gamma = 2.8\,. \tag{34}$$

The numerical values for the model constants were calibrated for homogeneous shear. Here RDT stands for the RHS of these equations in the RDT limit as given in KR [see Eqs. (5.10.4) and (5.10.5)].

2.5 Evolution of the turbulence scales

The choice of turbulence scales to be used in a turbulence model is not unique. For example, standard k-ϵ models use transport equations for k and ϵ, and determine the turbulence time scale through an algebraic equation. Another possibility is to use the evolution equation for the time scale τ (or the reciprocal time scale ω) along with the equation for k, and then evaluate ϵ from an algebraic equation. Each of these approaches has some problems. We are currently investigating the use of the k and ϵ equations in the form shown below.

$$\frac{dk}{dt} = P - \epsilon \tag{35}$$

$$\frac{d\epsilon}{dt} = [-C_d S_{kk} - C_s S_{ij} r_{ij} - C_0/\tau - C_\Omega \sqrt{\Omega_{ij}\Omega_{jk}a_{ik}}]\epsilon \tag{36}$$

with

$$C_d = \frac{4}{3} \qquad C_s = 3.0 \qquad C_0 = \frac{11}{6} \qquad C_\Omega = 0.01\,. \tag{37}$$

Note that the ϵ equation has the standard form except for the last term involving C_Ω. This term takes advantage of the structure information in a_{ij} and allows for a decrease in the dissipation rate in the presence of mean rotation, except when the turbulence becomes two-dimensional, as observed in direct numerical simulations.

3. Evaluation of the proposed extensions

In this section, the extended structure-model given by (2), (4), and (32)-(37) is tested for four independent homogeneous flows. First we summarize the values of the constants that we will be using. For clarity we also include the values for the constants in the RDT model [Eqs. (5.9.6)-(5.9.11) in KR] for which we use the notation of KR:

- Rapid model : $C_1 = 5.9 \quad C_2 = 2.0 \quad C_3 = 7.0 \quad C_4 = 2.5$ (38)

- Slow model : $$\begin{array}{llll} \alpha = 1.8 & C_\epsilon = -0.35 & C_\phi = 1.3 & C_\gamma = 2.8 \\ C_d = \frac{4}{3} & C_s = 3.0 & C_0 = \frac{11}{6} & C_\Omega = 0.01\,. \end{array} \tag{39}$$

3.1 Homogeneous shear in a rotating frame

We first consider the problem of homogeneous shear in a rotating frame. The mean velocity gradient tensor G_{ij}, the frame vorticity Ω_i^f, and frame rotation rate Ω_i are defined by

$$G_{ij} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad 2\Omega_i = \Omega_i^f = (0, 0, \Omega^f)\,. \tag{40}$$

We consider initially isotropic turbulence

$$r_{ij} = \tfrac{1}{3}\delta_{ij}, \qquad k = k_0, \qquad \epsilon = \epsilon_o\,. \tag{41}$$

First, we consider the case of homogeneous shear in a stationary frame ($\Omega^f = 0$) with an initial $Sk_0/\epsilon_0 = 2.36$. Figure 1 shows the model predictions for the components of the normalized Reynolds stress tensor $r_{ij} = R_{ij}/q^2$. The symbols are from the direct numerical simulation of Rogers *et. al.* (1986), which also had $Sk_0/\epsilon_0 = 2.36$. The agreement between the model predictions and the direct numerical simulation is good. As shown in Table 1, the equilibrium state predicted by the model is in good agreement with the experiments of Tavoularis & Karnik (1989).

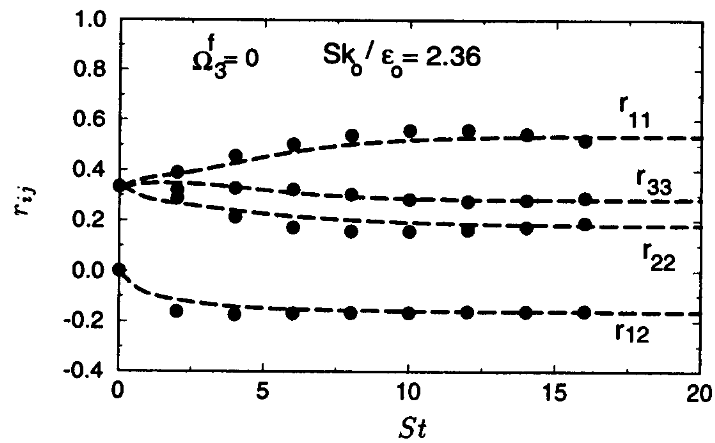

FIGURE 1. Time evolution of the normalized Reynolds stress tensor in homogeneous shear for $Sk_0/\epsilon_0 = 2.36$. Comparison of the predictions of the structure-based model (----) with the direct numerical simulations of Rogers *et. al.* (1986) (•).

The solution in the case of homogeneous shear in a rotating frame depends on the initial conditions only through the dimensionless parameter Sk_0/ϵ_0, and on the frame vorticity through the dimensionless parameter Ω^f/S (Speziale *et. al,* 1991). The value of Ω^f/S determines whether the flow is stable, in which case k and ϵ decay in time, or unstable, in which case both k and ϵ grow exponentially in time. In the stable regime $(\epsilon/Sk)_\infty = 0$, and in the unstable regime $(\epsilon/Sk)_\infty > 0$.

Equilibrium Values	Structure Model	Experiments
r_{11}	0.53	0.51 ± 0.04
r_{22}	0.18	0.22 ± 0.02
r_{33}	0.29	0.27 ± 0.03
r_{12}	-0.16	-0.16 ± 0.01
Sk/ϵ	5.30	4.60 ± 0.50
P/ϵ	1.70	1.47 ± 0.14

TABLE 1. Equilibrium results for homogeneous shear: comparison with the experiments of Tavoularis & Karnik (1989).

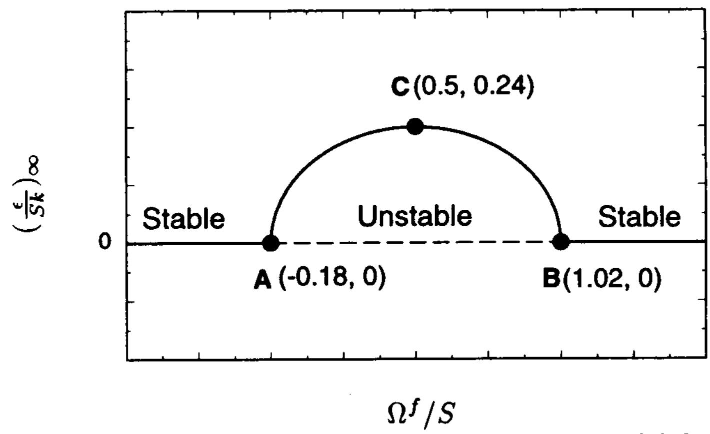

FIGURE 2. Bifurcation diagram of the structure-based model for homogeneous shear in a rotating frame.

Linear analysis and LES show that the flow is unstable for $-0.21 \lesssim \Omega^f/S \leq 1$ and stable outside these bounds. The most unstable case, having the highest growth rate for k and ϵ and the largest $(\epsilon/Sk)_\infty$, corresponds to $\Omega^f/S = 0.5$. Figure 2 shows the bifurcation diagram for the structure-based model. The structure-based model does an excellent job predicting the location of the bifurcation points A and B, and that of the most energetic state C (largest growth rate for k).

In the absence of DNS or experimental data, we evaluate the model performance using the large-eddy simulations of Bardina *et. al* (1983). Figure 3 shows the evolution of the normalized kinetic energy k/k_0 with non-dimensional time St. Note

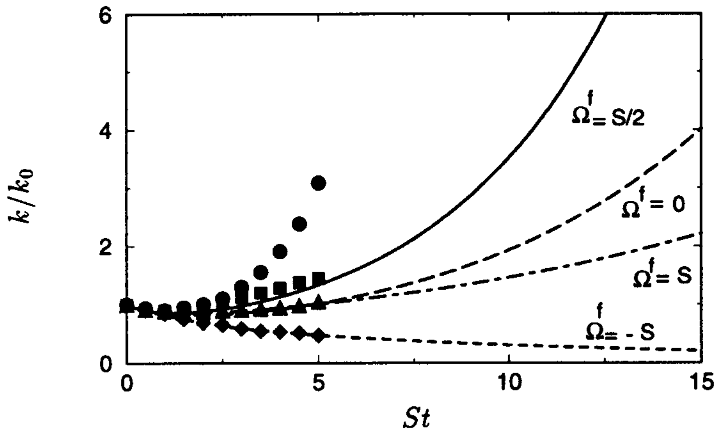

FIGURE 3. Time evolution of the turbulent kinetic energy in rotating shear flows. Comparison of the predictions of the structure-based model (lines) with the large-eddy simulations of Bardina *et. al* (1983) shown as symbols: $\Omega^f = 0$, (---- , ■); $\Omega^f = 0.5S$, (—— , •); $\Omega^f = S$, (—·— , ▴); $\Omega^f = -S$, (······· , ♦).

that the model captures the general trends correctly. For example, it correctly predicts that the highest rate of growth (for both k and ϵ) should occur for $\Omega^f = S/2$, which RDT shows is the most unstable case. It also predicts a weak rate of growth for the case $\Omega^f = S$ and a decay (relaminarization) for $\Omega^f = -S$. The numerical agreement with the LES is reasonable, but the model tends to predict somewhat lower rates of growth, particularly so in the case $\Omega^f = 0.5S$. This problem is common to all the currently available second-order closures as noted by Speziale *et. al.* (1989).

3.2 Axisymmetric strain

Next, we consider the performance of the extended structure-based model for the cases of axisymmetric contraction and expansion in homogeneous turbulence. The mean velocity gradient tensor is given by

$$S_{ij} = \begin{pmatrix} S & 0 & 0 \\ 0 & -S/2 & 0 \\ 0 & 0 & -S/2 \end{pmatrix} \tag{42}$$

with $S > 0$ for contraction and $S < 0$ for expansion. We consider an initially isotropic state as specified in (41). The solution depends on these conditions through the non-dimensional parameter Sk_0/ϵ_o. Comparisons are made with the DNS of Lee & Reynolds (1985). In both cases, we compare with the slowest runs from these simulations, which correspond to $Sk_0/\epsilon_0 = 0.56$ (contraction case AXK) and $Sk_0/\epsilon_0 = 0.41$ (expansion case EXO).

In Fig. 4(a), we consider the time evolution of the components of the Reynolds stress anisotropy $\tilde{r}_{ij}$. The total strain

$$C^* = \exp\left(\int_0^t |S_{\max}(t')|\, dt'\right) \tag{43}$$

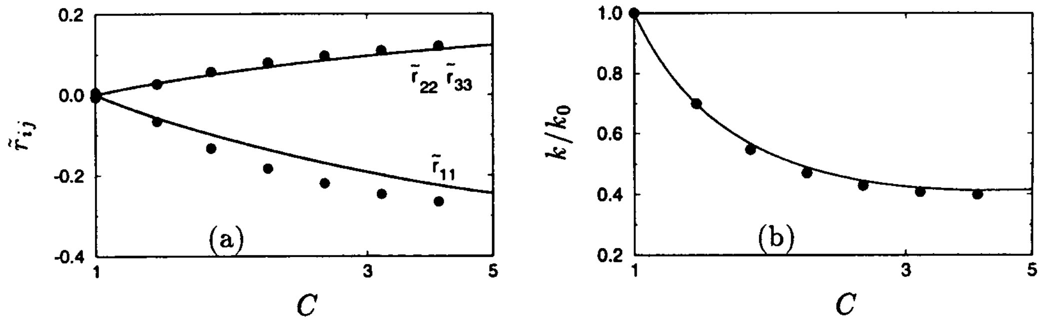

FIGURE 4. Comparison of the model predictions (——) with the direct numerical simulations of Lee & Reynolds (1985) (•) for irrotational axisymmetric contraction with $Sk_0/\epsilon_0 = 0.56$. (a) Evolution of the Reynolds stress anisotropy tensor $\tilde{r}_{ij}$. (b) Evolution of the normalized turbulent kinetic energy k/k_0.

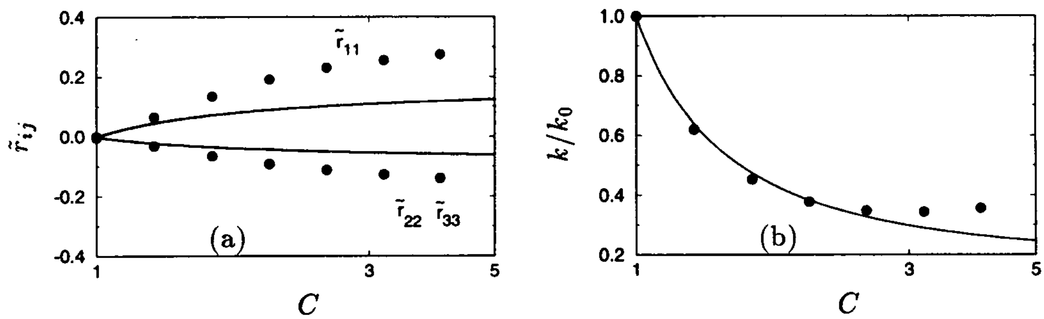

FIGURE 5. Comparison of the model predictions (——) with the direct numerical simulations of Lee & Reynolds (1985) (•) for irrotational axisymmetric expansion with $Sk_0/\epsilon_0 = 0.41$. (a) Evolution of the Reynolds stress anisotropy tensor $\tilde{r}_{ij}$. (b) Evolution of the normalized turbulent kinetic energy k/k_0.

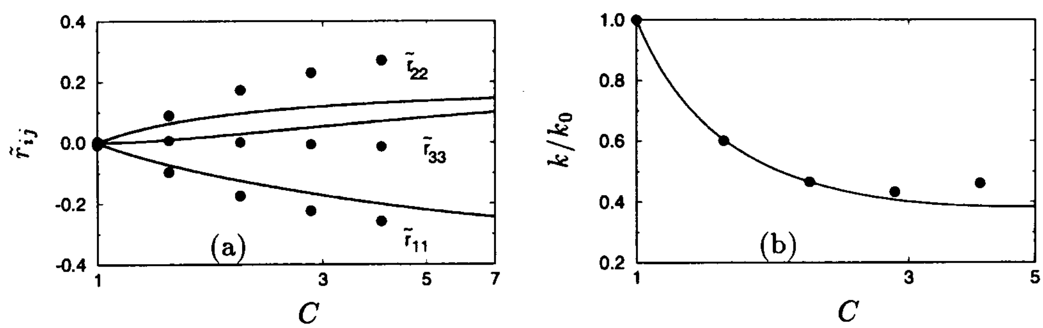

FIGURE 6. Comparison of the model predictions (——) with the direct numerical simulations of Lee & Reynolds (1985) (•) for irrotational plane strain with $Sk_0/\epsilon_0 = 0.50$. (a) Evolution of the Reynolds stress anisotropy tensor $\tilde{r}_{ij}$. (b) Evolution of the normalized turbulent kinetic energy k/k_0.

serves as the time coordinate in this plot. The model predictions, shown in solid lines, are in very good agreement with the DNS of Lee & Reynolds (1985) shown in symbols. The same good agreement between model and DNS is obtained in Fig. 4(b), where we consider the time evolution of the normalized turbulent kinetic energy k/k_0.

The evolution of the Reynolds stress anisotropy in the case of axisymmetric expansion is considered in Fig. 5(a). Note that the model underpredicts significantly the level of anisotropy as compared to the DNS of Lee & Reynolds (1985). The model prediction for the evolution of the normalized turbulent kinetic energy [shown in Fig. 5(b)] is accurate up to $C \approx 2.6$, but eventually it also degrades. This deficiency of structure-based model is discussed shortly.

3.3 Plane strain

We now turn to the case of homogeneous turbulence subjected to plane strain. The mean velocity gradient tensor is given by

$$S_{ij} = \begin{pmatrix} S & 0 & 0 \\ 0 & -S & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{44}$$

We consider initially isotropic conditions corresponding to (41) with $Sk_0/\epsilon_o = 0.5$. These conditions correspond to the slowest run (case PXA) reported by Lee & Reynolds (1985).

Figure 6(a) shows the time evolution of the Reynolds stress anisotropy. Note that the model predictions are accurate only for very small total strain and quickly degrade, particularly for $\tilde{r}_{11}$ and $\tilde{r}_{33}$. As in the axisymmetric expansion case, the model prediction for the rate of decay of k/k_0 remains accurate for a somewhat larger total strain, but eventually it degrades also [see Fig. 6(b)].

3.4 Some problems with the current approach

The relatively poor performance of the structure-based model in the axisymmetric expansion and plane strain flows prompted us to take a closer look at both the physics of these flows and at our model. What we have learned helped us understand better these flows and also provided us with a solution to the problems faced by the structure-based model in these flows.

Rapid distortion analysis (RDT) shows that under irrotational mean strain $\tilde{r}_{ij} = \tilde{d}_{ij}$. This result is clearly exhibited in the most rapid runs from the DNS of Lee & Reynolds (1985), including the rapid expansion and plane strain runs corresponding to $Sk_0/\epsilon_0 = 41.0$ and $Sk_0/\epsilon_0 = 50.0$ respectively. However, when the slowest runs for these two flows are considered, corresponding to cases EXO and PXA discussed above, one finds that $\tilde{\mathbf{r}} \gg \tilde{\mathbf{d}}$ (see Fig. 7). These observations become even more interesting if one notices that the level of stress anisotropy $\tilde{r}_{ij}$ in the slow axisymmetric expansion and plane strain runs exceeds the level of stress anisotropy in the corresponding rapid runs! This effect is demonstrated in Fig. 8, where we show plots of $|II_r| = \tilde{r}_{ij}\tilde{r}_{ji}/2$ versus $|II_d| = \tilde{d}_{ij}\tilde{d}_{ji}/2$. The open symbols correspond

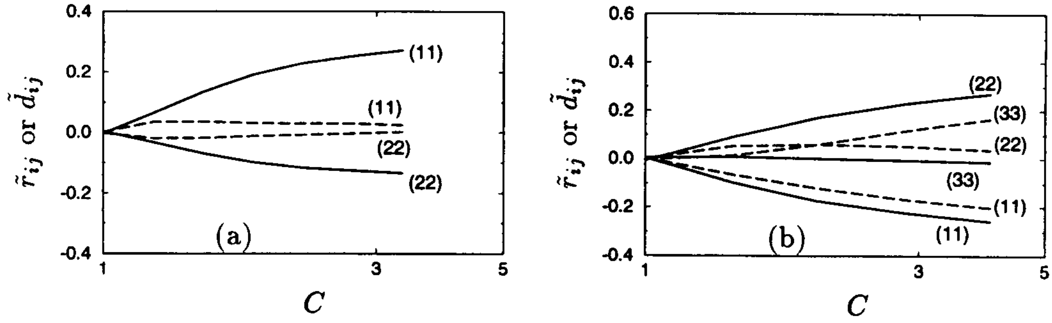

FIGURE 7. Comparison of the Reynolds stress anisotropy $\tilde{r}_{ij}$ (———) with the dimensionality anisotropy $\tilde{d}_{ij}$ (- - - -) from the direct numerical simulations of Lee & Reynolds (1985). (a) Irrotational axisymmetric expansion with $Sk_0/\epsilon_0 = 0.41$. (b) Plane strain with $Sk_0/\epsilon_0 = 0.50$.

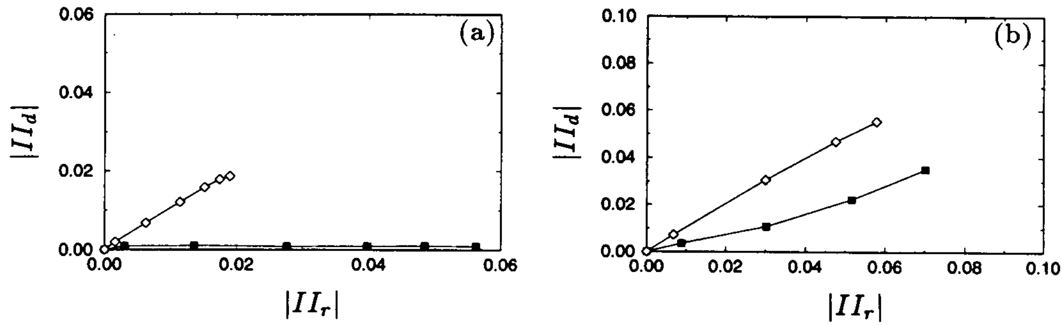

FIGURE 8. The second invariants of the stress anisotropy $-II_r$ vs. the second invariant of the dimensionality anisotropy $-II_d$. (a) Axisymmetric expansion at $Sk_0/\epsilon_0 = 0.41$ (■) and at $Sk_0/\epsilon_0 = 41.0$ (◇). (b) Plane strain at $Sk_0/\epsilon_0 = 0.50$ (■) and at $Sk_0/\epsilon_0 = 50.0$ (◇).

to the most rapid run and the closed symbols to the slowest run of Lee & Reynolds (1985) for each flow. Note that in the rapid runs $|II_d| \approx |II_r|$ whereas in the slow runs $|II_d| \ll |II_r|$. What is more, in each flow, the maximum level reached by $|II_r|$ is higher in the slow run that it is the rapid run.

By using a linearized version of the RDT evolution equations for $\tilde{r}_{ij}$ and $\tilde{d}_{ij}$, valid for small anisotropies, we have been able to show that these intriguing effects are primarily controlled by the rapid terms in the two evolution equations. In other words, RDT will maintain $\tilde{\mathbf{r}} = \tilde{\mathbf{d}}$ if it is initially true, but an arbitrarily small deviation $\tilde{\mathbf{\Delta}} = \tilde{\mathbf{r}} - \tilde{\mathbf{d}}$ will be amplified by the rapid terms. The initial conditions of the simulations of Lee & Reynolds imposed a very small initial $\tilde{\mathbf{\Delta}}_0 = \tilde{\mathbf{r}}(0) - \tilde{\mathbf{d}}(0)$. However, even in the absence of any initial $\tilde{\mathbf{\Delta}}_0$, such a deviation could be triggered by unequal rates of return to isotropy for the two tensors.

The fact that these unexpected effects (once triggered by the initial conditions or non-linear effects) seem to be dominated by the rapid terms prompted us to take a closer look at our rapid model. We believe that the difficulties encountered in these flows are related not to the slow model developed above, but rather to the form of the basic constitutive Eq. (2), which relates the Reynolds stresses to the eddy-axis tensor. The reason for this failure lies in the fact the current version of the structure model assumes that, in absence of mean rotation, $\phi = \gamma = 0$. This means that the principal axes of r_{ij} remain locked onto the principal axes of a_{ij}. This is appropriate for the RDT of initially isotropic turbulence, when the eddies do not have time to interact with each other. The cases examined above show that this is not appropriate for slower mean strain rates where the non-linear eddy-eddy interactions are important. These non-linear eddy-eddy interactions provide an effective eddy rotation acting on an individual eddy due to the circulation associated with the background sea of eddies. The effective eddy rotation tends to rotate the principal axes of the stresses associated with an individual eddy so that these become misaligned with the eddy axis, and some ϕ and γ are produced. But in order to capture these effects it is not enough to allow for non-zero ϕ and γ under irrotational strain; we also need to replace the mean vorticity Ω_i in (2) with the effective eddy rotation rate Ω_i^*; this will produce a contribution in the jet-vortex correlation term even in the absence of mean rotation. Simple kinematic analysis (see Appendix I in KR) shows that Ω_i^* is given by

$$\Omega_i^* = \epsilon_{irp} a_p \dot{a}_r \qquad \dot{a}_i = \frac{da_i}{dt} . \tag{45}$$

Note that because of (45) the effective eddy rotation rate will be sensitive to the slow model adopted in the a_i (or A_i) evolution equation. Some preliminary analysis suggests that these changes in the constitutive Eq. (2), coupled with an appropriate slow model in the A_i (and hence a_{ij}) equations, will allow the structure-based model to access states above the RDT limit on the axisymmetric expansion line of the anisotropy invariant map.

4. Summary and future plans

We have proposed and implemented an extension of the structure-based model for weak deformations. It was shown that the extended model will correctly reduce to the form of standard k-ϵ models for the case of equilibrium under weak mean strain. The realizability of the extended model is guaranteed by the method of its construction. The predictions of the proposed model were very good for rotating homogeneous shear flows and for irrotational axisymmetric contraction, but were seriously deficient in the case of plane strain and axisymmetric expansion.

We have concluded that the problem behind these difficulties lies in the algebraic constitutive equation relating the Reynolds stresses to the structure parameters rather than in the slow model developed here. In its present form, this equation assumes that under irrotational strain the principal axes of the Reynolds stresses remain locked onto those of the eddy-axis tensor. This is correct in the RDT limit, but inappropriate under weaker mean strains, when the non-linear eddy-eddy interactions tend to misalign the two sets of principal axes and create some non-zero ϕ and γ.

We plan to modify the constitutive equation and the evolution equation for the eddy-axis tensor a_{ij} as necessary to reflect these effects. This will require replacing the mean vorticity vector Ω_i in the constitutive equation by an effective eddy rotation rate $\Omega_i^* = \epsilon_{irp} a_p \dot{a}_r$ that correctly accounts for the non-linear effects described above. The slow model in the eddy-axis equation may have to be adjusted accordingly since the effective eddy rotation rate Ω_i^* will be sensitive to it.

Once these modifications have been implemented and evaluated, we will focus in extending the structure-based model for inhomogeneous flows. This extension will require the addition of diffusion terms in the transport equations for the structure parameters and the turbulence scales. Some preliminary work in determining the form of the diffusion terms and appropriate boundary conditions for these equations has been carried out.

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