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Abstract. The geomagnetic field can be represented mathematically by spherical harmonic expansions of scalar magnetic potentials. The mean square value of the magnetic induction represented by potential harmonics of degree n averaged over a sphere gives the spatial magnetic power spectrum at degree n on the sphere. McLeod's Rule for the magnetic field generated by Earth's core says that the internal spatial geomagnetic power spectrum of the core field at the core surface, $R_{nc}(c)$, is expected to be inversely proportional to (2n + 1) for finite degrees $1 < n \leq N_E$. We verify McLeod's Rule by using it to locate the core-mantle boundary with single epoch main field models of satellite geomagnetic data. This method is found to be more accurate than other core magneto-location methods, with the estimated core radius of 3485 km being very close to the seismologic value of 3480 km.

With the core radius fixed at 3480 km, we then calibrate McLeod's Rule and similar spectral forms against main field model values of R_n for degrees 3 through 12. By extrapolation to the degree 1 dipole, we predict the expectation value of Earth's dipole moment to be about 5.89×10^{22} Am^2 rms (74.5% of its 1980 value) and the expectation value of geomagnetic intensity at Earth's surface to be about 35.6 μT rms. Archeo- and paleomagnetic intensity data show these and related predictions to be reasonably accurate.

The distribution χ^2 with 2n+1 degrees of freedom is assigned to $(2n+1)R_n(c)/\{R_n(c)\}$, where $\{R_n(c)\}$ is the expected power of the core field at degree n on the core surface. We extend this even to the first degree, arguing that the small tilt of Earth's magnetic dipole moment relative to its rotation axis is mainly a geometric, rather than an energetic, effect of the Coriolis pseudo-force on outer core field and flow; moreover, (i) small tilt need not imply excess dipole power, (ii) normalized dipole power can be distributed as χ^2 with three degrees of freedom when the axial dipole is not normally distributed, and (iii) examples include the composite bi-Maxwellian–Gaussian distribution for the axial dipole complementing weaker, normally distributed equatorial dipole moments. According to the χ^2 dipole power distribution, an exceptionally weak absolute dipole moment (≤20% of the 1980 value) will occur during 2.5% of geologic time. We estimate the mean duration for such major geomagnetic excursions, one quarter of which feature axial dipole reversal, using the dipole power time-scale from modern geomagnetic field models and a statistical model of geomagnetic dipole power excursions. The resulting mean excursion duration of 2767 years forces us to predict an average of 9.04 excursions per million years, 2.26 axial dipole reversals per million years, and a mean reversal duration of 5533 years. Paleomagnetic data show these purely geomagnetic predictions to be quite accurate.

McLeod's Rule for the core field, even when extrapolated to the first degree, led to (1) very accurate magneto-location of the core-mantle boundary; (2) fairly accurate prediction of paleomagnetic field intensity; and (3) accurate prediction of the mean frequency of major absolute geomagnetic dipole excursions and axial dipole reversals. We conclude that McLeod's Rule serves to unify geomagnetism and paleomagnetism, correctly relates theoretically predictable statistical properties of the core geodynamo to magnetic observation, and provides *bona fide a priori* information required for stochastic inversion of paleo-, archeo-, and/or historical magnetic measurements.

1. Introduction

In a geomagnetically source-free region, such as the region just above Earth's surface, the geomagnetic induction of internal origin **B** is the negative gradient of a scalar potential V_{int} that satifies Laplace's equation. At time t and position **r** in geocentric spherical polar coordinates (radius r, colatitude θ, and east longitude φ), the spherical harmonic expansion of (zero mean) V_{int} is well-known to be

$$V_{int}(\mathbf{r},t) = a \sum_{n=1}^{\infty} (a/r)^{n+1} \sum_{m=0}^{n} [g_n^m(t)\cos m\phi + h_n^m(t)\sin m\phi]P_n^m(\cos\theta) , \quad (1)$$

where P_n^m is the Schmidt-normalized associated Legendre function of degree n and order m and $[g_n^m(t), h_n^m(t)]$ are the Gauss coefficients at reference radius a = 6371.2 km. Gauss coefficients can be estimated by a weighted least squares fit to geomagnetic data; this usually requires truncating the sum over n at finite degree N (see, e.g., Langel [1987]). Accurate statistical prior information about the Gauss coefficients may supplement such regularization, increase the accuracy of estimated Gauss coefficients, and improve the reliability of associated uncertainty estimates [McLeod, 1986; Backus, 1988]. The accuracy of a statistical hypothesis about Gauss coefficients may be established by theoretical development and empirical testing of its predictions.

The main source of the main geomagnetic field is electric current flowing in Earth's roughly spherical, electrically conducting, ferro-metallic liquid outer core and solid inner core. So here we use $\mathbf{B} = -\nabla V$ and (1) to describe this field outside the core at radii $r \geq c$ = 3480 km [Dziewonski and Anderson, 1981; Kennett et al., 1995]. Fields due to weaker currents in the resistive, ferro-magnesian silicate and oxide mantle, magnetization in the colder crust, and currents in the ionosphere above and comet-shaped magnetosphere beyond are of but secondary interest here.

1.1 The Core Multipolar Power Spectrum

Lowes [1966, 1974] and others (see Cain et al. [1989]) show the mean square magnetic induction configured in potential harmonics of degree n averaged over a sphere of radius r enclosing its sources to be

$$R_n(r,t) = (n+1)(a/r)^{2n+4} \sum_{m=0}^{n} [(g_n^m)^2 + (h_n^m)^2] . \quad (2)$$

The values of R_n form the internal spatial geomagnetic power spectrum. If core and other, mainly crustal, contributions to the Gauss coefficients are uncorrelated, then $R_n(a)$ is the sum of the core multipolar spectrum $R_{nc}(a)$ and the crustal spectrum $R_{nx}(a)$.

1.2 Previous Exponential Forms

Following Lowes [1974], lower degrees of $R_n(a)$ are approximated by the exponential $A^*(a/c^*)^{2n+4}$; linear regression through values of $\ln[R_n(a)]$ calculated from a geomagnetic field model gives

$$\ln[R_n^*(a)] = n[\ln(a/c^*)^2] + [\ln(A^*) + 4\ln(a/c^*)] . \quad (3)$$

The slope of this line implies the radius c* at which the model spectrum $R_n^*(c^*)$ becomes independent of n or "flat". The dipole power $R_1(a)$ seems large compared with other terms and has often been excluded from the regression. If R_n^* is extrapolated to arbitrarily high degree, then the sum over n of $R_n^*(r)$ diverges for r

$\leq c^*$; therefore, the extrapolation is not valid below c*. This might be due to failure of the potential field representation, so c* might be the minimum radius of a sphere containing the sources.

Lowes' [1974] estimate ($R^*_n(a) = 4.0\times10^{10}(0.222)^n$ nT^2) becomes flat about 480 km below c = 3480 km. The exponential fall-off of $R^*_n(c)$ with n is faster than needed for finite mean magnetic energy density ($n^{-1-\varepsilon}$ with $\varepsilon > 0$). Indeed, the finitude of Ohmic dissipation within the core and Gubbins' [1975] expression for the minimum value thereof imply

$$R_n(a) \leq An^{-2-\delta}(0.2983)^n \quad \text{for } \delta > 0 \text{ and } n \geq N_D, \qquad (4)$$

where N_D is the minimum degree of the magnetic dissipation range.

Langel and Estes [1982] interpreted the R_n spectrum from the degree 23 model MGST 10/81 of Magsat data as a core dipole, a non-dipole core field ($2 \leq n \leq 12$), and a crustal field ($n \geq 16$). Their core spectrum ($R^*_{nc}(a) = 1.349\times10^9(0.270)^n$ nT^2) becomes flat 174 km below c. Their crustal spectrum ($R^*_{nx}(a) = 37.1(0.974)^n$ nT^2) becomes flat 83 km below a.

Rather than assume zero toroidal magnetic field in the imperfectly insulating mantle, Voorhies [1984] considered the mean square radial field component alone as given by MGST 10/81. His core spectrum ($(n+1)R_n(a)/(2n+1) \approx B^{*}_{rc}{}^2(n) = 8.175\times10^8\,(0.2649)^n$ nT^2) becomes flat 200 km below c and was used to argue against narrow scale, intensely magnetized core spots on the core-mantle boundary (CMB); his crustal spectrum ($B_{rx}{}^2(n) = 11.91(0.9969)^n$ nT^2) becomes flat 9.9 km below a and indicates upper crustal, rather than deep lithospheric, sources.

Cain et al. [1989] used their degree 63 model M07AV6 of Magsat data to obtain a core spectrum ($R^*_{nc}(a) = 9.66\times10^8(0.286)^n$ nT^2) that becomes flat 73 km below c; a crustal spectrum ($R^*_{nx}(a) = 19.1(0.996)^n$ nT^2) that becomes flat 14 km below a; and an estimated noise level.

1.3 A New View of Power Law Forms

We questioned the omission of the degree 1 dipole from estimation of R^*_{nc}; the existence of physical sources giving a flat R_n spectrum; and popular, but unrealized, expectations that the dissipation range should begin below degree 14. Examination of R_n from the degree 60 model M102189 of Cain et al. [1990] suggested that not only was the dipole power R_1 "too large", but that the quadrupole power R_2 was "too small". Comparatively large R_1 and small R_2 are thought to be artifacts of undersampling as follows: the 1980 dipolar and quadrupolar powers, being derived from but 3 and 5 Gauss coefficients respectively, might be fairly far from values obtained by either averaging over, or taking many more random samples from, a geologically long time interval.

So we set c to 3480 km and tried power laws, $R_{nc\alpha}(c) = K_P n^{-\alpha}$, by fitting $\ln[R_n(c)]$ with a linear function of $\ln(n)$ for $3 \leq n \leq 12$. We found α to be 0.94. This seemed close to 1, particularly as crustal contributions may lead to a slight underestimate of α. We also vaguely recalled k^{-1} kinetic energy spectra for large-scale, energy containing modes of hydrodynamic turbulence (wherein wavenumber k is less than in the inertial sub-range or dissipation range). So we then set α to 1, estimated K_P alone, and found that the sum of the squared residuals per degree of freedom was less than when both α and K_P were estimated together. We thus learned to expect a core spectrum of the form

$$R_{nc}(r) \approx \{R_{nc1}(r)\} = K_P\, n^{-1}\,(r/c)^{2n+4} \quad \text{for } 1 \leq n \leq N_E, \qquad (5)$$

where the curly brackets represent the expectation value and N_E is the finite, maximum degree of the magnetic energy range ($1 < N_E \leq N_D$).

If expectations akin to (5) accurately represent statistical properties of the core geodynamo, then one must expect deviations $[R_n - \{R_{nc}\}]$ eventually relax to zero and perhaps change sign (barring all but the slowest changes in K_P, α and c due to planetary evolution and omitting weak crustal fields). Finding R_1 substantially higher, and R_2 substantially lower, than expected thus forced us to predict relaxation of these deviations. So we examined the main field and secular variation coefficients of model GSFC 12/83 [Langel and Estes, 1985] and found R_1 decreasing and R_2 increasing as predicted; moreover, for all orders m, $(\partial_t g_1{}^m)/g_1{}^m < 0$ and $(\partial_t g_2{}^m)/g_2{}^m > 0$. The plain chance of such perfect (anti-) correlation of signs is 1/8 for the dipoles, 1/32 for the quadrupoles, and 1/256 for all eight coefficients together. This otherwise remarkable coincidence is here viewed as merely an efficient relaxation of the geomagnetic field towards expectation values like (5).

It turns out that spectral forms similar to (5) were advanced and used over a decade ago by McLeod [1985] and were derived from theoretical consideration of the dynamo generated core field by McLeod [1994, 1996].

2. McLeod's Rule

McLeod's Rule [McLeod, 1985; 1994, 1996, equation (20a)] for finite degrees n > 1 is

$$R_{nc}(a) \approx R_{nc}{}^M(a) = K_M (n + 1/2)^{-1} (c/a)^{2n}. \qquad (6)$$

where both c/a and McLeod's constant $K_M = 5\times10^9$ nT^2 must be determined empirically. The late M. G. McLeod derived this form from a spatial power spectrum for the secular variation F_n (obtained using first time derivatives of the Gauss coefficients in (2) instead of the coefficients themselves) and a temporal geomagetic power spectrum $P_n(\omega)$ that depends upon spatial degree n and temporal frequency ω [McLeod, 1994, 1996]. The form of F_n is that of horizontal dipoles induced by lateral advection of magnetic field line footpoints at the top of a high conductivity liquid core; the form of $P_n(\omega)$ is appropriate to a two time-scale model of the two processes, magnetic flux diffusion and fluid motion, that change the core field and are the basis of core geodynamo theory.

2.1 Remarks on the Theoretical Derivation

This paper is mainly concerned with the empirical accuracy of predictions based on (6) and ancillary hypotheses; however, we must stress that the theoretical foundation of McLeod's Rule bears carefull scrutiny. Firstly, the functional form of F_n advanced by McLeod [1994, 1996, equation (11)] is appropriate to secular variation induced by random fluid motion at the top of a free-streaming, high conductivity core mantled by a rigid insulator. The advection of magnetic field line footpoints is mainly horizontal by the kinematic boundary condition, as is the locally induced differential dipole moment. Spatial correlations of the induced differential moments at well separated points may be either positive or negative for any particular realization of core surface field and flow, but we have no kinematical reason to expect such correlations survive either an ensemble average or a geologic time average. So the expected form of F_n for broad-scale secular variation is that introduced by McLeod.

Secondly, the functional form for the diffusive time constants [McLeod, 1994, 1996 equation (15)] is here viewed as linking empirically unknown vertical length scales of the poloidal field within the core to the lateral length scales of the matching scaloidal field outside the core. This view eases the derivation as follows. At the top of the very thin viscous boundary layer separating the free-streaming liquid core from the mantle, the fluid motion vanishes by the no-slip boundary condition and secular variation is purely diffusive. There the radial component of magnetic diffusion is proportional to the jump in the second radial derivative of the radial field component across the core-mantle interface, denoted $\partial_r^2 B_r^{ns}$ [Voorhies, 1991]. This jump in field line curvature is due to core toroidal currents that are the source of the diffusing, "non-scaloidal" part of the poloidal field. Indeed, with core conductivity σ and magnetic permeability μ, the radial component of McLeod's [1994, 1996] diffusion equation (13) is just $\partial_t B_r = (\mu\sigma c)^{-1}\nabla^2 rB_r = (\mu\sigma)^{-1}\partial_r^2 B_r^{ns}$. The spherical harmonic coefficients of $\partial_t B_r$ are denoted $\partial_t B_n^m$; those of $\partial_r^2 B_r^{ns}$ are denoted $B_n^m(\pi k_{rn}^m)^2$. Qualitatively, $(\pi k_{rn}^m)^{-1}$ is an effective vertical scale height for a harmonic of the field within the core; quantitatively, the factor of π renders k_{rn}^m equal to 1/c for the dipole's gravest radial free-decay mode [Moffat, 1978, p39-40]. With this notation, each harmonic of the radial diffusion equation is $\partial_t B_n^m = (\mu\sigma)^{-1}B_n^m(\pi k_{rn}^m)^2$. The sum of $[(2n+1)/(n+1)](\partial_t B_n^m)^2$ over orders $0 \leq m \leq n$ yields $F_n = (\mu\sigma)^{-2}(\pi k_{rn})^4 R_n$, which defines the characteristic radial wavenumber k_{rn} for degree n. The characteristic time constant is $\tau_n = (R_n/F_n)^{1/2} = \mu\sigma/\pi^2 k_{rn}^2$.

We do not claim to know k_{rn} from theory; we do know that it should be compared with the horizontal wavenumber $k_{hn} \equiv [n(n+1)/c^2]^{1/2}$. So we introduce the aspect ratio $A \equiv k_{hn}^2/k_{rn}^2$ and obtain $\tau_n = A\mu\sigma c^2/\pi^2 n(n+1)$. Apart from the factor of A/π^2, this is McLeod's [1994, 1996] equation (15). We have no reason to prefer aspect ratios that either increase or decrease rapidly with n, or that are either many orders of magnitude greater or less than unity, especially when averaged over geologic time. So constant A is the natural selection, the postulate, or the theoretical null-hypothesis to be tested. Because the secular variation diffusing across the thin viscous sub-layer is essentially that induced by fluid motion at the top of the free-stream (with spectrum $F_n(c)$ given by McLeod [1994, 1996, equation (11)]), this selection implies $R_n(c) = \tau_n^2 F_n(c) \approx K_M/(n + 1/2)$, which is McLeod's Rule.

Empirically, we calculated, tabulated and graphed numerical values $\tau_n^{-2} = F_n/R_n$ from model GSFC 9/80 [Langel et al., 1982] evaluated at epochs 1960, 1970, and 1980 (Voorhies 1985, unpublished notes). These tabulations, akin to a dispersion relation for secular variation, are at long last of some use. The constant aspect ratio hypothesis implies $\{\tau_n^{-2}\} = C[n(n+1)]^2$, so $\ln\{\tau_n^{-2}\} = \ln C + 2\ln[n(n+1)]$. For each of the three epochs, we fitted the form $\ln C' + \beta\ln[n(n+1)]$ to the numerical values of $\ln(\tau_n^{-2})$ for degrees 3 through 12 by least squares. The three estimates of β average to 1.957 ± 0.156 (1σ); this is in excellent agreement with the constant aspect ratio prediction of $\beta = 2$. The mean and standard deviation of $\ln C'$ indicates $(C')^{-1/2}$ is within a factor of 1.94 of 26,400 years, so we write the dispersion relation as $\tau_n \approx 26{,}000/[n(n+1)]$ years.

Thirdly, an eligible theoretical temporal power spectrum should represent the two main dynamo processes of magnetic flux diffusion and motional induction. Such a spectrum should accomodate the two different, degree dependent, sets of time-scales assigned to these two processes. For each degree the spectrum $P_n(\omega)$ must approach a constant as ω approches the lowest frequency; it must also fall off faster than ω^{-1} at high frequencies to ensure finite total power. These properties should apply to individual factors in $P_n(\omega)$ representing the two main processes. It further seems desirable for $P_n(\omega)$ to be derivable from the difference between two time series so as to represent, in at least a qualitiative way, competition between motional induction and magnetic diffusion. The temporal power spectrum advanced by McLeod [1994; 1996, equation (16)] meets all these requirements.

Finally, McLeod's derivation of (6) uses the high degree approximations $(n + 1/2) \approx [n(n + 1)]^{-1/2} \approx n$, so our empirically rediscovered form (5) is not in severe disagreement with the theoretically motivated (6). Indeed, empirical tests of such slightly different spectral forms might lead to new insights about the lower degree, broader scale geomagnetic field.

When multiplied by 2π, the longest time constant appearing in McLeod's temporal power spectrum (the dipole diffusive T_{a1} in equation (22a) of McLeod [1994, 1996]) might suggest a period of about 30 kyr. This is near the dominant period of between 30 and 40 kyr in relative paleointensity suggested by the spectral analysis of Tauxe & Shackleton [1994]. We have no further comment on periodicity, quasi-periodicity, and aperiodicity in either the geomagnetic field or records thereof; we do suggest that McLeod's Rule has described the field for almost 10^8 years and perhaps much longer.

2.2 Variations

In and above a source-free mantle (for $r \geq c$), McLeod's Rule becomes

$$\{R_{nc}{}^{M}(r)\} = K_M (a/c)^4 (n + 1/2)^{-1} (c/r)^{2n+4} . \tag{7}$$

Here we use (7) for degrees $3 \leq n \leq 12 < N_E$ and occasionally for degrees $1 \leq n \leq 12 \leq N_E$. We extend McLeod's Rule to the core geomagnetic dipole on the hypothesis that, over geologic time, this dipole is exceptional in regards to its axiality but not its energy. Those who cannot agree with this hypothesis are invited to entertain it as a null-hypothesis against which widely held views favoring exceptionally large dipole power might be more firmly established.

It is thought that McLeod's Rule holds under more general conditions than those from which it was originally derived; indeed, the simplified derivation above suggests that it does. Combined with the high degree approximations used by McLeod [1994, 1996], this leads us to keep a factor of $[n(n + 1)]^{-1/2}$ without approximation by $(n + 1/2)^{-1}$. So we also work with

$$\{R_{nc}{}^{M'}(r)\} = K_{M'} [n(n + 1)]^{-1/2} (c/r)^{2n+4} \qquad \text{for } 1 \leq n \leq N_E . \tag{8}$$

$K_{M'}$ is about eleven times K_M due mainly to the factor $(a/c)^4$.

Particular spectral forms (5), (7), and (8) are of the general form

$$\{R_{nc}(r)\} = K\, q(n)\, (c/r)^{2n+4} \qquad \text{for } 1 \leq n \leq N_E . \tag{9}$$

In (5), constant K is K_P and polynomial q(n) is n^{-1}; in (7), K is $K_M(a/c)^4$ and q(n) is $(n + 1/2)^{-1}$; and in (8), K is $K_{M'}$ and q(n) is $[n(n + 1)]^{-1/2}$. The various q(n) are all approximately 2/(2n+1), so these forms are but minor variations on McLeod's Rule when compared with our extending it to the first degree and to geologic time intervals.

2.3 Core Field Hypothesis

The statistical core field hypothesis advanced here is that suitably normalized R_{nc} is distributed as χ^2 with 2n+1 degrees of freedom and the expectation value given by McLeod's Rule (7) or variation (8). This will be so if the Gauss coefficients of degree n are random samples of a population with a zero mean Gaussian probability distribution function of variance $\{R_{nc}(a)\}/[(n+1)(2n+1)]$; however, it may well be that some R_{nc} (notably R_{1c}) are very nearly so distributed, even though the Gauss coefficients of that degree are not normally distributed. Indeed, there are an infinite number of distributions for the individual coefficients of any particular degree n that yield the distribution χ^2 with 2n+1 degrees of freedom for the normalized multipole power $(2n+1)R_{nc}/\{R_{nc}\}$; examples are given for the dipole in Appendix C. This situation may arise when the energy flow through the core geodynamo typically produces a mean magnetic energy density on the core surface of about $\{R_{nc}(c)\}/2\mu$ per degree n, but geometric effects distribute the field abnormally among various orders m within that degree. Earth rotation is the most obvious source of such anisotropy.

The Coriolis pseudo-force does no work on the fluid motion sustaining the core field, so rotational polarization of (geologically) turbulent core flow might produce such anisotropy without much change to the core surface magnetic energy per harmonic degree as given by McLeod's Rule. To the extent that it results by inductive effects ($-\nabla\times(\mathbf{B}\times\mathbf{v})$ for fluid velocity $\mathbf{v}$) that match the geometry of the Coriolis vorticity effect ($-\nabla\times(2\Omega\times\mathbf{v})$ for bulk angular velocity Ω), such anisotropy would mainly distinguish the axial dipole from the equatorial dipoles. In particular, if the radial component of the curl of the Coriolis pseudo-force density at the top of the core is negligible (so, for mass density ρ, $\mathbf{r}\bullet\nabla\times(2\rho\Omega\times\mathbf{v}) \approx 0$ by the CMB), then the core fluid velocity is surficially geostrophic and fluid downwelling implies poleward flow. Fluid downwelling can sweep in or attract magnetic field line footpoints to form regions of strong radial field known as core spots; the accompanying poleward flow implies poleward drift of such core spots. This provides a flux partioning mechanism for axial dipole formation, growth, and fluctuation (as noted and described in some detail elsewhere [Voorhies, 1991; 1992]). This particular mechanism for inducing planetary scale magnetic anisotropy relies on the kinematic boundary condition appropriate to the simplified terrestrial case of a rigid insulator mantling a convecting fluid conductor; it might also operate in cases where a stably stratified fluid bounds a convecting conductor. It remains to be seen whether or not this particular mechanism indicates any deviations from McLeod's Rule (e.g., a shift in magnetic energy from the quadrupolar to the dipolar configuration). There may of course be other mechanisms, perhaps involving the electrically conducting solid inner core or the mantle, that systematically partition core surface magnetic energy unequally among the harmonic orders within a particular harmonic degree for intervals of approaching 10^8 years.

The specific probability distribution function advanced for normalized core magnetic multipole power $(2n+1)R_{nc}/\{R_{nc}\}$ at radii $r \geq c$ is chi-squared with $2n+1$ degrees of freedom,

$$\wp[(2n+1)R_{nc}/\{R_{nc}\}] = \wp_{2n+1}(\chi^2) = [2^{n+1/2}\Gamma(n+1/2)]^{-1}(\chi^2)^{n-1/2}\exp[-\chi^2/2]\,, \tag{10}$$

where Γ is the gamma function and $\{R_{nc}\}$ is given by (7) or (8). Standard tables giving the probability of obtaining a value of χ^2 less than or equal X^2 also give the probability of obtaining a value of $R_{nc}/\{R_{nc}\}$ less than or equal to $X^2/(2n+1)$. It is emphasized that (10) does not imply equipartioning of multipole power amongst the various orders within harmonic degree n.

Because Earth's squared dipole moment $\mathbf{m}\bullet\mathbf{m}$ is mainly from the core and is $(4\pi a^3/\mu_o)^2[R_1(a)/2]$, equation (10) at the first degree predicts that $3\mathbf{m}\bullet\mathbf{m}/\{\mathbf{m}\bullet\mathbf{m}\}$, which equals $3R_{1c}(a)/\{R_{1c}(a)\}$, will be distributed as χ^2 with three degrees of freedom. Because the probability $\wp(|\chi|)d|\chi|$ of finding $|\chi|$ in the interval $[|\chi|, |\chi|+d|\chi|]$ is $\wp(\chi^2)[d\chi^2/d|\chi|]d|\chi|$, the distribution of absolute dipole moments $|\mathbf{m}|$ is predicted to be Maxwellian. Then the root mean square dipole moment is

$$\text{RMSDM} \equiv \{\mathbf{m}\bullet\mathbf{m}\}^{1/2} = 4\pi a^3\{R_{1c}(a)/2\}^{1/2}/\mu_o\,;$$

the mean absolute dipole moment is

$$|\text{DM}| \equiv \{|\mathbf{m}|\} = 4\pi a^3\{[R_{1c}(a)/2]^{1/2}\}/\mu_o$$

$$= 4\pi a^3[4\{R_{1c}(a)\}/3\pi]^{1/2}/\mu_o = (8/3\pi)^{1/2}\text{RMSDM}\,;$$

and the most probable absolute dipole moment is $(2/3)^{1/2}$RMSDM. Again, the probability distribution function for normalized absolute geomagnetic dipole moment is predicted to be equivalent to the Maxwellian distribution for particle speed.

In following sections, one or two parameters of various spectral forms $\{R_n\}$ are estimated by least squares fits to elementary functions of values of R_n. The R_n values are calculated from geomagnetic field models derived using truncation as the sole regularization. By (10), it seems that normally distributed residuals ought not be expected; moreover, the magnitude of the residuals will be large compared with expected errors in R_n indicated by geomagnetic field model covariance due to the natural variability of the core field over geologic time. Here we use plain least squares for simplicity and can but invite more rigorous inversions offering more sensitive tests of McLeod's Rule. Our main goal is to investigate the absolute accuracy of predictions based on simple, preliminary calibrations of McLeod's Rule (and (10)) rather than engage in statistical analyses of uncertainty.

3. Magneto-Location of Earth's Core

If McLeod's Rule is correct, then the two parameter fit of $\{R_{nc}(a)\}$ to values of $R_n(a)$ calculated from geomagnetic field models should give accurate magnetic estimates of the core radius, denoted c_m, as well as the amplitude K. We computed such estimates by fitting $\ln[\{R_{nc}(a)\}/q(n)]$ from (9) to $R_n(a)$ data calculated from the degree 13 model GSFC 12/83 [Langel and Estes, 1985] and, to reduce aliasing of residual crustal fields, the degree 60 model M102189 [Cain et al., 1990].

The top three rows of Table 1 list our estimates of c_m for the q(n) shown in the first column. The values in the second and third columns were fitted only to $R_n(a)$ for degrees 3 through 12; those in the fourth and fifth columns were fitted to degrees 1 through 12. The fourth row shows radii c* obtained from the plain exponential form $A^*(a/c^*)^{2n+4}$; however, most studies presuming q(n) = 1 exclude the dipole, so values obtained by fitting only degrees 2 through 12 are included in parentheses. Such omission of the dipole raises c* by about 102 km.

The top row of Table 1 shows that core radii determined with McLeod's Rule, c_M, agree very well with the independently determined seismologic core radius of 3480 km. The closest agreement comes from the fit of McLeod's Rule (6-7) to the $R_n(a)$ values from M102189 for degree 3 through 12. This is illustrated in Figure 1, which graphs $R_n(a)$ (diamonds connected by dashed line segments), the fit (solid curve with c_M = 3484.5 km and $K_M(a/c_M)^4 = 5.3649\times10^{10}$ nT^2), and the extrapolation to degrees 2 and 1 (fine dashed curve). The 'errorbars' show the 80% likelihood range deduced from χ^2 per 2n+1 degrees of freedom and are thus attached to the theory (solid curve), not the data; as expected, two of the 10 points fitted lie just outside the 80% confidence interval. Although R_1 is greater, and R_2 less, than expected, both dipole and quadrupole terms are within the 80% likelihood range attached to the low degree extrapolation.

Table 1 shows McLeod's Rule (6-7) as most accurate (top row) and variation (8) as almost as accurate (second row); even form (5) gives core radii at most 1.8% high (third row). The average of the first two values in the top row of Table 1 is 3488.2 $\pm$ 10.3 km (2σ). The four values in the upper left quadrant average to 3489.9 $\pm$ 8.6 km (2σ). The last two columns show that inclusion of dipole and quadrupole terms raises c_m by at most 22.3 km. This fact is considered particularly important when combined with the ongoing relaxation of the dipole and quadrupole powers towards their expected values.

The fourth row of Table 1 shows that plain exponential fits, which violate McLeod's Rule, systematically underestimate the core radius by about 200 km. We doubt errors this large result from crustal contributions to the $R_n(a)$ fitted; indeed, coestimation of parameters describing recently advanced forms for R_{nx} would very slightly decrease c_M rather than increase it. Perhaps a failure of the plain exponential form for $R_n(a)$ has been overlooked by insufficiently well-founded omission of the dipole term from previous fits.

The accuracy of the present core magneto-locations is largely superior to those obtained using Hide's method [Hide and Malin, 1981; Voorhies and Benton, 1982]. This is not surprising because the latter method relies heavily upon the frozen-flux core approximation, uncertain phase information in harmonic orders m and, more importantly, upon uncertain secular change information from either secular variation models or main field models at different epochs. The present method, being based upon the more complete physics underlying McLeod's Rule, allows for magnetic flux diffusion as well as motional induction, depends only upon the power spectrum R_n, and relies only upon R_n at a single epoch. Indeed, the average of 44 core magneto-locations using diverse field models and variations of Hide's method is 3506.2 $\pm$ 300.9 km [Voorhies 1984, equation (3.21)], while the twelve estimates of c_m in Table 1 average to 3504.8 $\pm$ 37.0 km (2σ).

4. Preliminary Calibration, Prediction of Mean Paleomagnetic Intensity, and Comparisons with Archeo-Paleointensity Data

McLeod's Rule enabled accurate estimation of the radius of Earth's core from models of satellite geomagnetic data; however, the seismologically inferred value is considered more accurate. So we fixed c to 3480 km when estimating the calibration coefficients K appearing in spectral forms (9). To do so, we fitted the single parameter ln[K] to values of $\ln[R_{nc}(c)/q(n)]$ computed from the field models for degrees 3 through 12 only. This restriction enables us to predict expected geomagnetic dipole and quadrupole powers by extrapolation to degrees 1 and 2. The results are shown in Table 2; K in the top row is $K_M(a/c)^4$ and confirms that McLeod's constant is about 5×10^9 nT^2 (4.915 $\pm$ 0.006 (2σ) $\times$ 10^9 nT^2).

Figure 2 shows the single parameter fit (solid curve) of variation (8) of McLeod's Rule (so q(n) is $[n(n+1)]^{-1/2}$) to $R_n(c)$ from model GSFC 12/83 (diamonds) for degrees 3 through 12 at epoch 1980. The curve is not quite a straight line on this log-log plot. As expected, 80% of the ten multipoles fitted fall within the 80% likelihood range from χ^2 per 2n+1 degrees of freedom shown as "errorbars" attached to the solid curve (not to the data diamonds). The fit is quite close and the trend is clear enough; indeed, it seems difficult to consider a degree-independent flat line drawn through these points (but see Constable and Parker [1988]).

Figure 2 shows that $R_8(c)$ is slightly lower, and $R_9(c)$ is slightly higher, than expected 80% of the geologic time. This uncommon spectral feature is partly associated with the fine structure of the South Atlantic Anomaly. Contour plots of the radial field at the top of the core as a function of the truncation level of expansion (1) [Voorhies, 1984, Figures 3.13-17] show substantial changes, particularly in the field topology under this region, when the trunction level is increased from degree 8 to 9, but not from 7 to 8. This may well be due to emergence of one or more reversed flux patches in the southern hemisphere, perhaps at a mean rate of about -3MWb/yr since the 1600s.

The prediction for degrees 1 and 2 (fine dashed curve in Figure 2) shows that the epoch 1980 dipole has more power, and the quadrupole less power, than expected on the basis of our variation of McLeod's Rule and the higher degree multipolar powers fitted. However, both dipole and quadrupole powers are clearly within their 80% likelihood ranges. The predicted value of $\{R_{1c}(c)\}$ is 3.908×10^{10} nT^2, or 55.51% of the 1980 value calculated from model GSFC 12/83; yet the predicted expectation value of Earth's root mean square magnetic dipole moment

$$\text{RMSDM} = \{\mathbf{m}\bullet\mathbf{m}\}^{1/2} = 4\pi a^3\{R_{1c}(a)/2\}^{1/2}/\mu_0 = 5.891\times10^{22}\ \text{Am}^2,$$

is 74.50% of the 1980 absolute dipole moment (7.907×10^{22} Am^2 from model GSFC 12/83). Moreover, the expected rms magnetic intensity at Earth's surface, the square root of the sum from degrees 1 through 12 of $\{R_{nc}(a)\}$, is predicted to be $\{\mathbf{B}\bullet\mathbf{B}^2\}^{1/2} = 35{,}578$ nT. Are these predicted expectation values for Earth's rms dipole moment and rms intensity accurate over the geologic time intervals for which they are intended?

Table 1: Estimates of Earth's Core Radius in km from McLeod's Rule and Other Forms

q(n)	GSFC 12/83	M102189	GSFC 12/83	M102189
	$3 \leq n \leq 12$	$3 \leq n \leq 12$	$1 \leq n \leq 12$	$1 \leq n \leq 12$
$[n + 1/2]^{-1}$	3491.79	3484.52	3501.24	3496.04
$[n(n+1)]^{-1/2}$	3493.30	3486.03	3506.82	3501.61
n^{-1}	3511.44	3504.13	3543.06	3537.79
1	3261.44	3254.65	3208.56	3203.79
		(3311.8)	(3306.0)	

Table 2: Estimates of the Coefficient K of McLeod's Rule

and other forms (in 10^{10} nT^2)

q(n)	GSFC 12/83	M102189
	$3 \leq n \leq 12$	$3 \leq n \leq 12$
$[n + 1/2]^{-1}$	5.54430	5.49894
$[n(n+1)]^{-1/2}$	5.52664	5.48142
n^{-1}	5.13595	5.09394

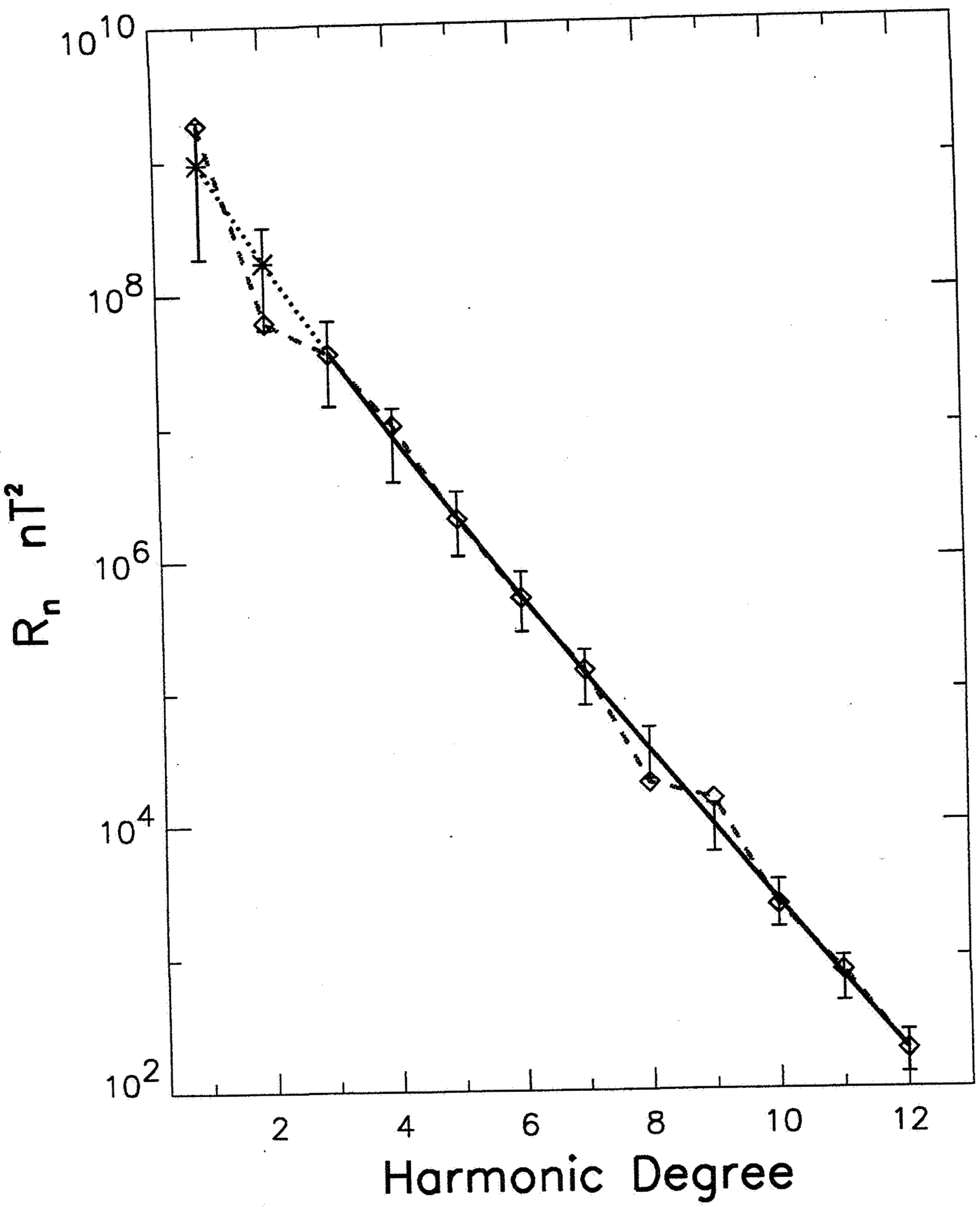

Figure 1 shows the multipole powers $R_n(a)$ from model M102189 [Cain et al., 1990] for degrees $1 \leq n \leq 12$ (diamonds connected by dashed line segments), the two parameter fit of McLeod's Rule (solid curve with $c_M = 3484.5$ km and $K_M(a/c_M)^4 = 5.3649\times10^{10}$ nT^2) to degrees 3 through 12, and the extrapolation to degrees 2 and 1 (fine dashed curve). Errorbars show the 80% likelihood range from χ^2 with 2n+1 degrees of freedom and are attached to the theory (solid curve), not the data. As expected, two of ten powers fitted lie outside this range. R_1 is greater, and R_2 less, than predicted, but both are within their 80% likelihood ranges.

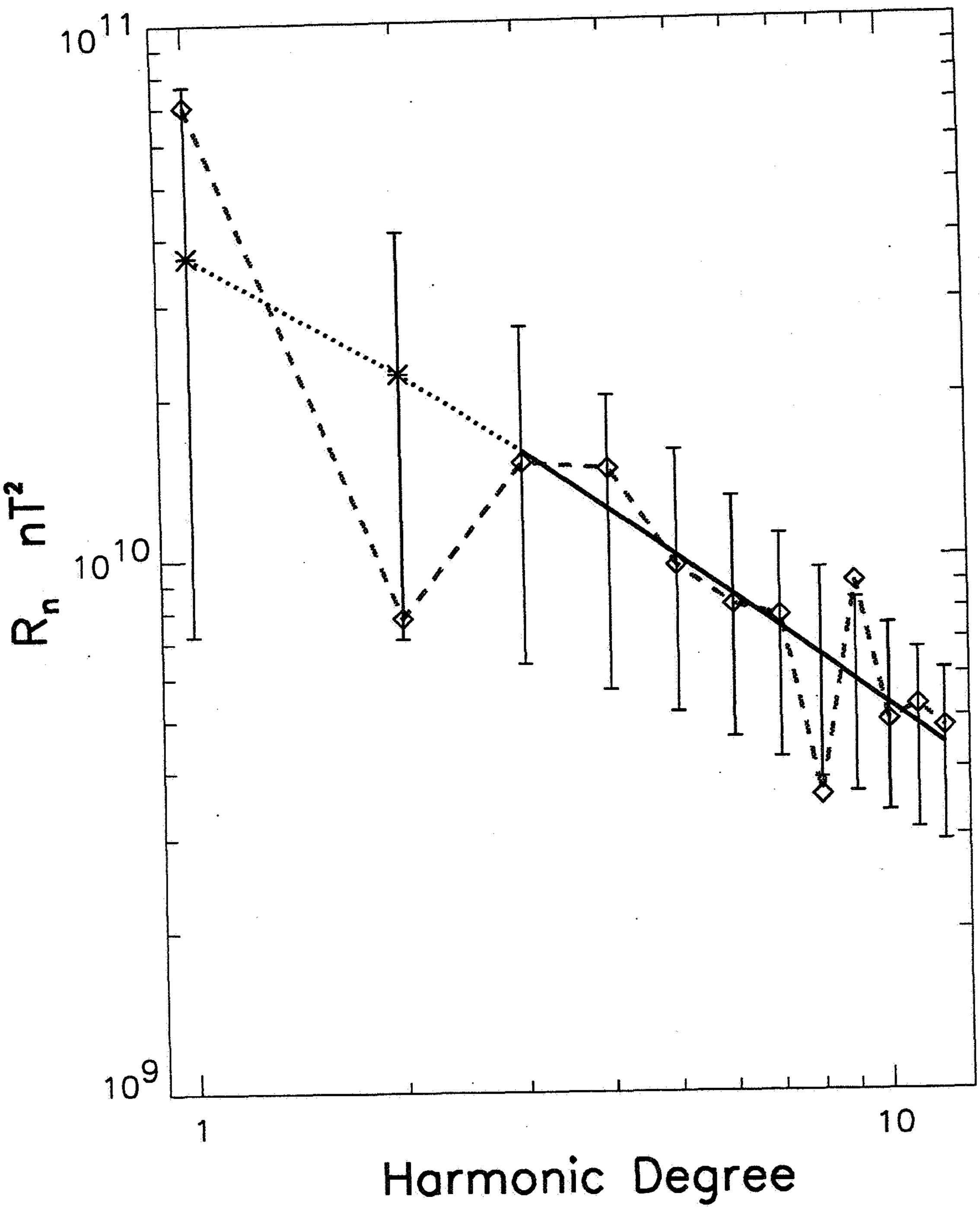

Figure 2 shows the single parameter fit (solid curve) of equation (8) to core multipole powers $R_n(c)$ of degrees 3 through 12 from model GSFC 12/83 [Langel et al., 1985] (diamonds) and the extrapolation to degrees 2 and 1 (fine dashed curve). The curve is not quite a straight line on this log-log plot. As expected, eight of the ten multipoles fitted fall within the 80% likelihood range from χ^2 with 2n+1 degrees of freedom (errorbars attached to solid curve, not diamonds). R_1 is greater, and R_2 less, than predicted, but both are within their 80% likelihood ranges.

4.1 A Comparison with Archeointensity Data

Tests of our predictions against archeo- and paleomagnetically inferred intensities are complicated not only by the variety of experimental methods used to infer past intensity $F = |\mathbf{B}(r{\approx}a,\dot{\theta},\phi;t{<<}1980)|$ from diverse materials, but from the common conversion of archeo- and paleointensities to Virtual Axial Dipole Moment (VADM $\equiv 4\pi a^3 F/[\mu_o(1 + 3\cos^2\theta)^{1/2}]$) or Virtual Dipole Moment (VDM $\equiv 4\pi a^3[1 + 3\cos^2 I]^{1/2}F/[2\mu_o]$) for magnetic inclination I) [Merrill and McElhinny, 1983]. VADMs and VDMs must be converted back to F before testing the prediction that the root spatio-temporal mean square of F is about 35.6 μT, but not all compilations tabulate requisite vaules of θ (or I) with VADM (or VDM). Moreover, the distribution of samples of F should be both fairly complete and uniform in both geographic position and geologic time; these conditions are not easily met! Furthermore, VADM and VDM selection criteria and spatial and/or temporal averaging can introduce systematic effects. Indeed, one paleomagnetist expressed a conviction that paleointensity data are too imprecise to be predicted accurately. We remain more optimistic because predicted expectation values can only be compared with averages over many individual values and the averaging can reduce random, if not systematic, errors.

Consider some results highlighted by McElhinny and Senanake [1982]. For the interval 0-10 kyr BP, ten, 1 kyr mean values of archeomagnetic VDMs (VADMs when necessary) average to $8.75 \pm 1.58 \times 10^{22}$ Am^2; for the interval 15-50 kyr BP, four VDMs and ten VADMs average to $4.44 \pm 0.64 \times 10^{22}$ Am^2. The former value is greater, and the latter somewhat less, than our predicted RMSDM, but by (10) dipole moments as large or larger than the former are expected about 9% of the geologic time. The standard deviations are small enough to indicate that the dipole moment was, on average, different during these different intervals; therefore, an estimate of the single spatio-temporal root mean square dipole moment should weight the squared average values by the duration of the intervals (10 kyr and 35 kyr, respectively). The square root of the duration weighted mean square of these two values is $5.69{\times}10^{22}$ Am^2 or 96.5% of our predicted RMSDM. Although not highlighted by McElhinny and Senanake [1982], the rms of their twelve tabulated 1 kyr means from 0-12 kyr BP is $8.676{\times}10^{22}$ Am^2; the rms of their 14 selected values from 17-50 kyr BP is $4.578{\times}10^{22}$ Am^2. The root duration weighted mean square of these two values, $5.953{\times}10^{22}$ Am^2, is only 1% greater than our predicted RMSDM.

The foregoing comparisions suggest we have a very accurate prediction of rms archeointensity; however, precise tests against archeo- and paleomagnetically inferred intensities are further complicated by (i) the likelihood that expectation values change over geologic time with evolving core geodynamo boundary conditions and (ii) the difficulty in distinguishing dipole from non-dipole contributions to intensity. The former are outside the main focus of this paper; as to the latter, the predicted rms intensity of 35.58 μT would be produced by a purely dipolar field of absolute moment $6.506{\times}10^{22}$ Am^2. This is 10% greater than RMSDM, so more care is needed to account for the difference between virtual and absolute dipole moments.

To make closer contact with reduced archeo- and paleointensity data, separate $\mathbf{B}$ into dipole and non-dipole parts and suppose that (i) the dipole is usually mainly axial (so $\mathbf{B}_D{}^2 \approx R_{1c}(a)(1 + 3\cos^2\theta)/2$), (ii) the broad-scale, non-dipole core field is fairly independent of (θ,φ) when averaged over geologic time, and (iii) the dipole and non-dipole field vectors eventually decorrelate (so very long term averages of $2\mathbf{B}_D\bullet\mathbf{B}_{ND}$ are much less than those of $\mathbf{B}_{ND}{}^2$); then

$$\{F\} \equiv \{|\mathbf{B}|\} = \{[\mathbf{B}_D{}^2 + 2\mathbf{B}_D\bullet\mathbf{B}_{ND} + \mathbf{B}_{ND}{}^2]^{1/2}\} \tag{11a}$$

$$\approx \{(R_D/2)(1 + 3\cos^2\theta) + R_{ND}]^{1/2}\} \tag{11b}$$

$$\approx \{(3R_D/8)^{1/2}\int_{-1}^{+1}(x^2 + k^2)^{1/2}dx\} \tag{11c}$$

where R_D is $R_{1c}(a)$, R_{ND} is the sum of $R_{nc}(a)$ for $n \geq 2$, $x = \cos\theta$, and k^2 is $(1 + 2R_{ND}/R_D)/3$. We predict $\{R_D\} = (32{,}214\ nT)^2$ and, summing over degrees 2 through 12 only, $\{R_{ND}\} = (15{,}100\ nT)^2$; however, in (11c) we replace $R_D^{1/2}$ with the smaller $\{R_D^{1/2}\} = 29{,}680$ nT and, without fear of much error, approximate $2R_{ND}/R_D$ by $\{2R_{ND}\}/\{R_D^{1/2}\}^2$. The resulting integral is easily evaluated by the substitution $x = k\sinh(z)$. We obtain $\{F\} \approx 32.8\ \mu T$ – less than $\{F^2\}^{1/2} = 35.6\ \mu T$ in accord with Schwartz's inequality. Similar treatments yield $\{VADM^2\}^{1/2} \approx 6.43\times10^{22}$ Am^2, which is 9.1% more than RMSDM, and

$$\{VADM\} \equiv 4\pi a^3\{[F/(1+3\cos^2\theta)]^{1/2}\}/\mu_o \qquad (12a)$$

$$\approx \frac{4\pi a^3}{\mu_o}\{[\frac{R_D(1+2R_{ND}/R_D+3\cos^2\theta)}{2(1+3\cos^2\theta)}]^{1/2}\} \qquad (12b)$$

$$\approx \frac{4\pi a^3}{\mu_o}\{R_D/8\}^{1/2}\int_{-1}^{+1}[\frac{x^2+k^2}{x^2+1/3}]^{1/2} dx \qquad (12c)$$

which, after consulting elliptic integral tables, gives $\{VADM\} \approx 6.21\times10^{22}$ Am^2. This is 11.4% greater than the our predicted mean absolute dipole moment $|DM|$ of 5.43×10^{22} Am^2; it is also 5.4% greater than RMSDM. More accurate approximations to {F} and {VADM} require more 'R&D on $\wp(R_{ND})$'.

With the foregoing adjustments for the expected non-dipole paleomagnetic fields, the predictions calibrated using the 1980 non-dipole field can be directly compared with reduced archeointensity data. The root duration weighted mean square dipole moment from the 10 kyr average VDM and the 35 kyr average VADM highlighted by McElhinny and Senanake [1982] is now seen to be 88.5% of our predicted value $\{VADM^2\}^{1/2}$. The root duration weighted mean square of the rms values for the intervals 1-12 kyr BP and 17-50 kyr BP is 92.6% of this prediction. Allowance for the expected non-dipole field shows our prediction overestimates archeointensity by about 10%. This is close enough to sustain a claim of a reasonably accurate prediction of archeointensity.

4.2 Comparisons with Paleointensity Data

McFadden and McElhinny [1982] analysed 166 non-transitional VDMs for the past 5 Myr. They note reasons to reject a Gaussian, but perhaps not a lognormal distribution of VDMs. They find support for a model in which the non-dipole intensity is proportional to a "True Dipole Moment" that has a truncated Gaussian distribution with a standard deviation of about 3.6×10^{22} Am^2 and a peak at the $8.65 \pm 0.65 \times 10^{22}$ Am^2 "Paleomagnetic Dipole Moment" (PDM). Their lognormal distribution, however, peaks near 6.5×10^{22} Am^2. These distributions do not seem to follow from (10). Indeed, (10) predicts that the absolute dipole moment will have a Maxwellian distribution that requires no truncation at weak moment, falls off much like a Gaussian at very strong moment, and, using our calibration of form (8), has a peak at 4.81×10^{22} Am^2. Of course, VDM and VADM distributions are expected to differ from a Maxwellian, and to peak at larger values, because higher degree multipoles contribute to such virtual moments. Our predicted {VADM} of 6.21×10^{22} Am^2 is nontheless about 0.68 standard deviations less than the PDM.

The 8.65×10^{22} Am^2 PDM for the past 5 Myr reported by McFadden and McElhinny [1982] is about 139% of our predicted {VADM}. The $3.9 \pm 1.9 \times 10^{22}$ Am^2 mean VADM for the past 4 Myr reported by Valet and Meynadier [1993] is about 63% of our prediction; they also cite values of $5 \pm 2 \times 10^{22}$ Am^2 for the past 140 kyr and $4.3 \pm 1.5 \times 10^{22}$ Am^2 for the interval 15-50 kyr BP. The 4.8×10^{22} Am^2 difference between the 5 Myr PDM and the 4 Myr mean VADM is somewhat larger than the root sum square of their standard deviations (4.1×10^{22} Am^2); this is discussed below. The 4 Myr mean VADM and the 5 Myr PDM average to 6.28×10^{22} Am^2. This is only 1% more than our predicted {VADM} of about 6.21×10^{22} Am^2. Averaging the superior temporal distribution of VADMs inferred from sediments with the reliability of VDMs obtained from volcanics shows our prediction to be accurate.

There appears to be some controversy among paleomagnetists about the Valet and Meynadier [1993] data. The accuracy and stability of the absolute calibration of relative paleointensity inferred from sediments would be the main concern here. It seems possible that the reported 4 Myr mean VADM could underestimate that obtained via an improved calibration. There also seems to be some controversy among rock magneticists about paleointensities inferred from volcanics. Systematic overestimation of paleointensity resulting from neglect of small but important curvature in NRM-pTRM curves casued by trace concentrations of multidomain grains (as described by Dunlop [Xu and Dunlop, 1995]) would be the main concern here. Whether or not this effect seriously biases the VDM tabulation of McFadden and McElhinny [1982] towards large values is not known to us. Their intentional exclusion of all transitional moments is, however, expected to make their PDM overestimate the value obtained from a temporally uniform sample. We suggest that averaging their 5 Myr PDM with the 4 Myr mean VADM of Valet and Meynadier [1993] has led to some cancellation of undesirable systematic effects.

Tanaka et al. [1995] constructed a global paleointensity data base of 1123 volcanic flow means. Their mean of 427 VDMs inferred using either Thellier or Shaw methods is $7.4 \pm 4.9 \times 10^{22}$ Am^2; this is 119% of our reasonably accurate {VADM}. We questioned the accuracy of our prediction based on the $\pm0.24 \times 10^{22}$ Am^2 standard error of the mean, but again found more fundamental questions of sampling bias. For example, Tanaka et al. [1995] judge 87 (20.4%) of these VDMs to be transitional. The mean of the non-transitional values is $8.3 \pm 4.9 \times 10^{22}$ Am^2 (about 4{VADM}/3), so the mean of the transitional values is about 3.9×10^{22} Am^2. Exclusion of some transitional moments is appropriate if transitions are indeed oversampled (in accord with our prediction of excursion and reversal frequencies in section 5). Exclusion of some non-transitional moments is appropriate if weak paleointensities from non-transitional flows are either undersampled or occasionally not reported, perhaps because they are of less widespread interest than transitional flows or are more difficult to determine. Still, the model fitted to (non-transitional) VDMs by Tanaka et al. [1995] yields $F \approx 31.3\ \mu T(1 + 3\cos^2 p)^{1/2}$. Averaging their model over the sphere gives $\langle F\rangle = 43.2\ \mu T$, some 132% of our predicted $\{F\} \approx 32.8\ \mu T$, and $\langle F^2\rangle^{1/2} = 44.3\ \mu T$, some 124% of our predicted $\{F^2\}^{1/2}$.

4.3 Further Paleo-Intensity Comparisons

To see how paleointensity data might be used to better test precise predictions like $\{F^2\}^{1/2} = 35.6$ μT, we worked with the McFadden and McElhinny [1982] tabulation of 141 non-transitional VDMs and related parameters. For $1 \le k \le 141$ we recalculated the intensity $B_k = [2\mu_0/4\pi a^3]\ VDM_k/[1 + 3\cos^2 I_k]^{1/2}$. The geographically weighted mean square

$$\langle B^2\rangle_{141} = \left[\sum_{k=1}^{141} B_k^2 \sin\theta_k\right] / \left[\sum_{k=1}^{141} \sin\theta_k\right] \qquad (13)$$

is $(51.7\ \mu T)^2$. The distribution is geographically non-uniform (48% of the samples are from Czechoslovakia or Japan) and temporally non-uniform (over 48% of the samples are post-Pleiocene). The distribution is geographically biased in that the unweighted mean of $\sin\theta_k$ is 4% less than the expected $\pi/4$; to the extent that the dipole is mainly axial, this bias towards high latitudes implies a bias towards stronger field. The distribution is also biased by the omission of transitional moments. Moreover, the root mean square is more strongly influenced by large outlying samples than is the arithmetic mean. To compensate we tried omitting the two strongest VDMs ($<B^2>_{139} = (49.9\ \mu T)^2$) and then the remaining seven VDMs from the 0.5 Myr Hakone Nagao-Toga lavas; this gives $<B^2>_{132} = (47.2\ \mu T)^2$. The 132 value unweighted mean VDM of 8.44×10^{22} Am^2 is 6% less than the 141 value mean; therefore, it is much closer to the Tanaka et al. [1995] 340 non-transitional mean VDM of 8.3×10^{22} Am^2. This fact lends credence to our selection, as does the fact that $<|B|>_{132} = 43.9\ \mu T$ is near the 43.2 μT value of $<F>$ calculated from the Tanaka et al. [1995] model of non-transitional F.

Although the $<B^2>_{132}$ statistic has the basic geographic weighting needed to estimate mean square intensity from the selected lava flow means, we need to estimate the rms intensity averaged over geologic time as well as position. When the sampling distribution is neither random nor uniform, the unweighted sample mean does not necessarily provide the best estimate of the population mean. We think duration weighting is needed to prevent many samples from a sequence of lava flows formed during a geologically short interval from overwhelming a few samples from a sequence of lava flows formed over a long interval. A naive attempt at such weighting gave $(45.8\ \mu T)^2$; however, the sampling distribution is sparse because lava cools and is magnetized by the ambient field quickly when compared with geologic time intervals. Moreover, the liklihood that the sampling is at least partly random and the central limit theorem indicate that having more samples from a particular geologic interval gives a more reliable mean value for that interval.

The 132 samples were therefore divided into 22 groups according to position and time. The group mean square intensity was calculated for each group, as was the group mean $\sin\theta$. The weighted average of the group mean values of $\sin\theta$, with weights equal to the square root of the number of sample VDMs in the group times the apparent duration spanned by the group, is within 0.2% of $\pi/4$. The grouping and weighting thus reduces the effect of geographic bias. The weighted average of spatio-temporal group mean square intensities, with weights equal to the square root of the number of sample VDMs in the group multiplied by the apparent duration spanned by the group and by the group mean $\sin\theta$, is $(44.95\ \mu T)^2$. We reduce this mean square intensity by 2.05% (2.5% of geologic time multiplied by $1 - [15.10/35.58]^2$) to compensate for the omission of transitional intensities and, finally, obtain the experimental rms paleointensity estimate $<B^2>^{1/2} = 44.49\ \mu T$. This is 14% less than the uncompensated value, but remains 25% over our predicted 35.58 μT. That it is close to the value of $<F^2>^{1/2} = 44.3\ \mu T$ calculated from the Tanaka et al. [1995] model of F indicates that efforts to compensate for non-uniformity and bias in small data sets can be worthwhile.

In summary, there is appreciable *prima facie* paleointensity evidence to support the hypothesis that Earth's magnetic field is, when averaged over geologic time scales, about 25% stronger than we predict. We stress that our predictions are based on (i) a low degree extrapolation of a simple variation of McLeod's Rule and (ii) further extrapolating one preliminary calibration of this rule with less than one year of geomagnetic data to an interval of many millions of years. Nonetheless, our predictions are reasonably accurate when compared with archeo- and paleointensity data. This is particularly true when the superior temporal distribution of VADMs inferred from sediments is combined with the reliability of VDMs obtained from volcanics. As a final example of this, the average of the sedimentary mean VADM (3.9×10^{22} Am^2 from Valet and Meynadier [1993]) and the non-transitional volcanic mean VDM (8.3×10^{22} Am^2 from Tanaka et al [1995]) is 6.10×10^{22} Am^2; this is but 1.8% less than our predicted {VADM} of 6.21×10^{22} Am^2.

4.4 Mesozoic Dipole Low

M. Prèvot has called our attention to the Mesozoic dipole low [Prèvot et al., 1990], provided a very helpful review of our 13 November 1995 manuscript, and extended an invitation to comment on non-stationarity.

We have reservations about extending our predictions, which already extapolate McLeod's Rule to the dipole, into the Mesozoic. These stem from the intervening normal polarity Cretaceous Superchron, a 34 Myr interval (84 - 118 Myr) with few if any axial dipole reversals - particularly in its terminal 21 Myr [Jacobs, 1994]. Based on our prediction for the mean rate of axial dipole reversals (see section 5), it seems impossible that so long an interval should pass without one by pure chance. Perhaps the superchron marks a time of different geodynamo boundary conditions for which the modern day geomagnetic field is simply not a very useful guide.

We suggest that a superchron indicates an interval when a stably stratified layer in the upper reaches of the outer core prevents convective motion near the top of the core (see Appendix A). Such a stable conducting layer may stabilize an axial dipole, but might yield a low degree power spectrum $R_n(c)$ of dissipation form K'/n^3 instead of the energetic, McLeodian form K/n. As is well known, such a layer might form from an excess concentration of buoyant slag liberated during an era of relatively rapid inner core solidification from the multi-component outer core melt. Latent heat released by solidification of slag at the base of the mantle may further stabilize the layer while reducing or eliminating the stability of the deepest mantle (D"). If the stabilizing buoyant components solidify onto, or are included into, the lower mantle, then their concentration in the layer (and/or the layer thickness) is reduced. Then the stability of the layer is reduced until the layer is broken up and mixed by ongoing thermo-compositional convection deeper in the core. Both formation and break-up of this (still hypothetical) intermittent layer are considered continuous, non-monotonic processes; however, it seems easier to view the layer as if it is either present or absent. Reversing (ordinary) or non-reversing (superchron) axial dipole states of the core geodynamo are thus viewed as marking the end-members of outer core solidification at its inner or outer boundaries, respectively.

The remarkable tabulation of Thellier-Thellier Cenozoic and Mesozoic paleointensities by Prèvot et al. [1990, Table 1] consists of 12 group mean VDMs and contains no data from the Cretaceous Superchron [Prèvot, 1996 personal communication]. So we suspended our reservations and found the unweighted mean of all 12 values to be 6.27×10^{22} Am^2, or 101% of our predicted {VADM} of 6.21×10^{22} Am^2. The weighted average of group mean VDMs, with weights equal to the square root of the number of sample VDMs in the group times the duration spanned by the group, is 6.31×10^{22} Am^2; this is 102% of our predicted {VADM}. The groups labeled Coniacian-Santonian, Hettangian-Sinemurian, and Early Triassic have few samples and uncertain ages (rather than definite durations) and may thus have been over-weighted; eliminating these gives a 9 value weighted mean VDM of 5.88×10^{22} Am^2, or 95% of our predicted {VADM}. All these mean values show our prediction to be quite accurate. Perhaps the Mesozoic dipole low is more notable for its contrast with the Cenozoic dipole high.

5. Prediction of Mean Geomagnetic Dipole Excursion and Reversal Frequency

Equation (10) predicts that there is a small but non-zero probability of finding Earth with very little power in its magnetic dipole ($R_1 << \{R_{1c}\}$) and thus with a weak absolute dipole moment. To get a better idea of the geomagnetic field during such a time, consider the 1980 field after removal of the dominant, axial part of the dipole. From model GSFC 12/83, we find $([(g_1^1)^2 + (h_1^1)^2]/[(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2])^{1/2}$ to be 19.4%. So here we define "geomagnetic dipole excursion" as an interval when Earth's absolute dipole moment $|\mathbf{m}|$ is less than or equal to 20% of its 1980 value $|\mathbf{m}_{80}|$. During such an excursion, the geomagnetic field could resemble the present day field with the axial, but perhaps not the equatorial, dipole removed.

5.1 Dipole Excursions Occupy 2.5% of Geologic Time

We must predict that the probability of finding Earth during such an excursion is 1/40. To see why, recall that downward continuatuation of model GSFC 12/83 gives $R_1(c) = 7.0401\times10^{10}$ nT^2; yet when extrapolated to the first degree, our calibration of (8) from degrees 3 through 12 of this model predicts expected dipole power $\{R_{1c}(c)\}$ equal to 3.9079×10^{10} nT^2. It follows that $[\{R_{1c}\}/R_1(1980)]^{1/2}$ = 74.50% = RMSDM/$|\mathbf{m}_{80}|$. By definition, excursional $|\mathbf{m}| \leq 0.200|\mathbf{m}_{80}| = 0.2684 \times$ RMSDM, so excursional $R_{1e} \leq 0.072\{R_{1c}\} = 0.216\{R_{1c}\}/3$. By (10) with n = 1, $3R_{1c}/\{R_{1c}\}$ is distributed as χ^2 with three degrees of freedom. From standard tables of χ^2, the probability of obtaining $R_{1c} \leq 0.216\{R_{1c}\}/3$ is thus 2.5%, as is the probability of obtaining $|\mathbf{m}| \leq 0.268$ RMSDM = 0.200 $|\mathbf{m}_{80}|$. So the probability of finding the Earth in such a geomagnetic dipole excursion is 2.5% or 1/40.

5.2 An Uncertain but Stable Dipole Time-Scale

We predict geomagnetic dipole excursions down to 20% or less of the 1980 absolute moment occur during 1/40 of geologic time. To convert this probability into a mean temporal frequency of dipole excursions, we need both a time-scale for changes in dipole power and a model of dipole behavior during excursions. From main field and secular variation model GSFC 12/83, $\partial_t|\mathbf{m}|/|\mathbf{m}|$ is (-1030 yr)$^{-1}$ and $[\partial_t R_1/R_1]$ is (-515 yr)$^{-1}$ at epoch 1980. The dipole power time-scale more representative of the present era obtained from the average and the difference of models GSFC 12/83 and 1945 DGRF [Langel et al., 1988] is $|\langle R_1\rangle/\langle\partial_t R_1\rangle| = |T_1| = 830$ years. Changes in annual values of $R_1/\partial_t R_1$ are due to second and higher time derivatives of the first degree Gauss coefficients; these have short time-scales (see, e.g., McLeod [1996]), so their effect is largely averaged out over the 35 year interval. Still, such variations suggest uncertainties up to a factor of 1.4 might accompany our use of the present era dipole power time-scale T_1. The recent decline of dipole power has been monotonic, so there is no compelling need to adjust T_1 for vacillations which eventually cause $\partial_t R_1$ to change sign; this is so even though such changes may occur many times during an interval as long as T_1.

It is important to realize that time-scales such as T_1 need not, ought not, and arguably cannot, be vastly different during excursions than they are in modern times. To see this, recall that the well known magnetic induction equation for a fluid of uniform permeability μ and conductivity σ,

$$\partial_t \mathbf{B} = \nabla\times(v\times\mathbf{B}) + (\mu\sigma)^{-1}\nabla^2\mathbf{B} , \tag{14a}$$

is linear in the field vector. Following Elsasser [1946a], when **B** is represented as a sum of elementary vector modes with amplitude coefficients $\Gamma_i(t)$, then the induction equation becomes

$$\partial_t \Gamma_i = \Sigma_j \, Z_{ij} \Gamma_j \tag{14b}$$

where the generalized coupling matrix Z(t) depends upon the fluid velocity as well as magnetic diffusivity $(\mu\sigma)^{-1}$. (Voorhies [1995] offers a simple example confirming that select Γ_i correspond to the Gauss coefficients). The solution to (14b) is of propagator form

$$\Gamma(t) = \exp[\int_{t_o}^{t} Z(\tau)d\tau]\,\Gamma(t_o) \tag{14c}$$

but is discussed in general terms of typical (e.g., rms) field strength B and flow speed v. Time-scales for the field, $B/|\partial_t B|$, depend upon flow speed v, magnetic diffusivity $(\mu\sigma)^{-1}$, and detailed geometric couplings that are coarsely represented via a single length scale L for field and flow. Neither the inductive time-scale $B/|\partial_t B| \approx \tau_B \approx L/v$ nor the diffusive time-scale $\tau_\sigma \approx \mu\sigma L^2$ depend explicitly upon field strength. We have no compelling reason to presume that either L or v are dramatically different during an excursion than during other times, so we expect excursional values of both τ_B and τ_σ, and thus the dipole power time-scale, to be similar to their non-excursional values. Even if such differences were presumed, they might largely cancel out so as to leave τ_B largely unchanged. The twentieth century dipole power seems somewhat large, but its relaxation towards the expected value seems efficient. Seeing no anomaly, we thus expect the dipole power time-scale during a dipole power excursion to be similar to its contemporary value T_1.

The position that T_1 can be applied to dipole power excursions may, of course, be viewed as a uniformitarian hypothesis to be tested. Because we support this position with a kinematic view of the induction equation, it may also be viewed as a subject for core geodynamical debate. Such a debate is outlined in Appendix A; it leads to the conclusion that τ_B, and thus arguably T_1, remains independent of the field strength in both magnetogeostrophic and geostrophic dynamical regiemes. Appendix A also notes why we associate a stable layer in the uppermost core with a superchron and circumstances under which stable T_1 need not conflict with the view that changes in the axial dipole field are largely decoupled from the magnitude of the axial dipole itself.

5.3 Fast, Mean, Multiple and Reversing Dipole Excursions

Despite the foregoing demonstrations, the fastest excursion takes not hundreds of years but next to no time at all. To meet our defintion of an excursion, dipole power need only reach the threshold value $0.072\{R_{1c}\}$ for an instant before rebounding to larger values. When normalized dipole power $f(t) \equiv 3R_{1c}(t)/\{R_{1c}\}$ reaches the threshold value $f_{th} = 0.216$, equal probabilities of 0.5 are assigned to terminating and to continuing the excursion. Then half of the excursions are indeed quite short. The other half dwell at $f(t) < f_{th}$ for a time which, on average, is the mean excursion duration τ_{ex}. As such an excursion ends, however, there remains the 50% chance that another begins; if so, we have a double excursion and perhaps a triple or multiple excursion. Accounting for both zero duration and multiple excursions shows that the mean duration for an excursion remains

$$\overline{\tau_{ex}} = 0.5\tau_{ex} \sum_{k=1}^{\infty} k\, 2^{-k} = 0.25\tau_{ex}(1 - 0.5)^{-2} = \tau_{ex}\,. \qquad (15)$$

One may think of τ_{ex} as the half-life of geomagnetic dipole excursions.

We stress that PDF (10) neither predict nor preclude dominant axiality of Earth's vector magnetic dipole moment (see Appendix C). It merely predicts that Earth's surface field is mainly dipolar except during rare intervals. During such intervals the three components of the dipole moment are each of very small absolute value compared with $\{|\mathbf{m}|\}$. Provided an excursion is not too brief, there is a fair chance that one or more of these components, including the axial dipole, will reach zero and change sign – albeit not all at the same time. Indeed, given enough time, they may change sign many times (e.g., during an unusually long or a multiple excursion). On average, however, brief excursions with durations less than τ_{ex} are expected to be "shallow" in that $f(t) \le f_{th}$ but there is not enough time for f(t) to descend to values much less than f_{th}; therefore, in accord with the simple thinking underlying (15) whereby half of all excursions are quite brief, we predict that one half of all excursions are long enough for such sign changes.

(The excursional waveform derived below shows why brief excursions should typically be shallow and thus why durable polarity changes typically require excursion durations of τ_{ex} or more).

If the axial dipole moment vanishes, then it will eventually return to either the initial polarity or to the reversed polarity. We have no excuse to assign other than equal 0.5 probabilities to the post-excursion polarities. Of the half of the dipole power excursions that are typically long enough for the axial dipole component to reach zero, half will thus result in a durable reversal of the axial dipole moment. So one quarter of the excursions are predicted to be durable geomagnetic axial dipole reversals. Such reversing excursions are, on average, necessarily predicted to take twice the mean duration of all excursions. The remainder are intervals of weak, mainly non-dipolar, field. In particular, with excursional $R_{1e} \leq 7.2\%\{R_{1c}\} = (8.64\ \mu T)^2$ and $R_{ND} \approx (15.1\ \mu T)^2$ at Earth's surface, the rms intensity during excursions is expected to be less than or equal to 17.4 μT. Samples of a field with rms intensity 17.4 μT might be mistaken for samples of an equivalent rms intensity dipole field with moment 3.2×10^{22} Am^2. This is close to, indeed 82% of, the 3.9×10^{22} Am^2 mean of transitional VDMs calculated from the selection of Tanaka et al. [1995], suggesting that many transitional fields meet our definition of a geomagnetic dipole power excursion.

5.4 Simple Excursion Models

We think the approach to a dipole power excursion reflects stochastic driving away from the expected dipole power $\{R_{1c}\}$ by chaotic core flow and is thus more akin to a pseudo-random walk than, say, simple exponential decay of R_{1c}. In this view, the approach to an excursion takes a long time; however, the excursion itself is necessarily brief in comparison. The excursion itself might therefore be modeled as the approximately free response of a primed, but otherwise ordinary, dynamical system to a single equivalent perturbation.

One simple model for an absolute dipole moment excursion from threshold amplitude ($|\mathbf{m}_{th}| = 4\pi a^3(\{0.072R_{1c}(a)/2\}^{1/2}/\mu_0 = 1.58\times10^{22}$ Am^2) to zero and back during the interval $t_i \leq t \leq t_f$ is the cosinusoidal form $|\mathbf{m}_{th}|(1 + \cos\omega t)/2$ for $|\mathbf{m}(t)|$, where $\omega = 2\pi/(t_f - t_i)$ is the free oscillation frequency assigned to the absolute dipole moment. With core flow during the excursion treated as a single equivalent initial perturbation, the duration of the excursion is $2\pi/\omega$. With ω taken to be $|(\partial_t|\mathbf{m}|)/|\mathbf{m}|| = (1030$ yr$)^{-1}$ from model GSFC 12/83, one obtains 6472 yr for the approximate duration of excursions that comprise 2.5% of geologic time. This would suggest geomagnetic excursions occur at the rate of once per 258,900 years, or 3.86 per million years, when averaged over geologic time.

An equally simple model, but for a dipole power excursion from threshold $0.072\{R_{1c}\}$ to zero and back during the interval $t_i \leq t \leq t_f$, is the form $A(1 + \cos\omega' t)/2$ for $R_{1c}(t)$, where $\omega' = 2\pi/(t_f - t_i)$ is now the free oscillation frequency of dipole power. With ω' taken to be $|\partial_t R_1/R_1| = (515$ yr$)^{-1}$ from model GSFC 12/83, the duration of the excursion is $2\pi/\omega'$ or 3236 yr. If excursions of this duration occupied 2.5% of geologic time, then geomagnetic excursions would occur at the rate of once per 129,400 years, or 7.73 per million years, when averaged over geologic time.

The second example gives an excursion frequency twice that of the first; moreover, when the mean 1945-1980 rates replace the single 1980 rates, these frequencies are reduced by a factor of 0.62. So these examples merely suggest a plausible range of 2-to-8 excursions per Myr. If a simple functional form for $R_{1c}(t)$ or $|\mathbf{m}(t)|$ is presumed, and the time-scale identified with recent secular variation, then an excursion frequency can be calculated. Such calculations are not satisfactory because the functional forms presumed for $R_{1c}(t)$ were arbitrary, if not wholly unreasonable, so the resulting excursion frequencies are also arbitrary. So let us return to (10) and derive both the mean excursion duration and a more satisfactory waveform for excursional $R_{1c}(t)$.

5.5 Statistical Model of Geomagnetic Dipole Power Excursions

During excursions the positive normalized dipole power $f(t) = 3R_{1c}(t)/\{R_{1c}\}$ is less than the threshold value, so $f(t) \leq f_{th} = 0.216 << 1$ and (10) for the first degree becomes

$$\wp(3R_{1c}/\{R_{1c}\}) = (2\pi)^{-1/2} f^{1/2} \exp[-f/2] \tag{16a}$$

$$\approx (2\pi)^{-1/2} f^{1/2} [1 - f/2] \tag{16b}$$

$$\approx \kappa(2\pi)^{-1/2} f^{1/2} \tag{16c}$$

where the factor $\kappa = 0.9364$ maintains the 2.5% probability of finding $R_{1c} \leq 0.072\{R_{1c}\}$ via

$$0.025 = \kappa(2\pi)^{-1/2} \int_0^{0.216} f^{1/2}\, df = \frac{2\kappa(0.216)^{3/2}}{3(2\pi)^{1/2}}$$

Note κ is close to unity because (16c) is a good approximation to (10) at very low dipole power.

During an excursion, dipole power declines from the threshold value to some lesser value and then recovers. The variation need not be monotonic; however, a bitonic symmetric excursion consistent with (16c) is derived in Appendix B to provide a prototypical, if not archetypal, excursional waveform. An alternate excursion model might involve a pseudo-random walk in normalized dipole power f(t) from the initial, threshold value $f(t_i) = f_{th}$ down to a value near zero $f(t_o) \approx 0$, whereupon f(t) returns towards $f(t_f) = f_{th}$ by a similar process. For any particular excursion, the dwell time at sub-threshold dipole power is then a sample of a distribution of excursion durations with mean value τ_{ex} to be predicted.

Statistical properties of excursional waveforms $f(t) \leq f_{th}$ follow by renormalizing (16c) into the conditional, excursion-specific distribution function

$$\wp_{ex}(f) = (3/2)f_{th}^{-3/2} f^{1/2} \tag{17}$$

with mean $\{f\}_{ex} = 3f_{th}/5$ and variance $(12/175)f_{th}^2$. During decay from f_{th} to near zero values of f, the mean value of $\partial_t f$ is $f_{th}/(t_o - t_i)$; during regrowth to f_{th}, the mean value of $\partial_t f$ is $f_{th}/(t_f - t_o)$. The single value of $|\partial_t f|$ characterizing the excursion is thus $2f_{th}/|t_f - t_i|$. This is also the expected value of $|\partial_t f|$ for the excursional waveform derived in Appendix B.

5.6 Prediction of 9.04 Dipole Power Excursions and 2.26 Reversals per Million Years

To predict the mean duration of dipole power excursions, recall that (i) the expected value of f during an excursion is $3f_{th}/5$ and (ii) the characteristic absolute rate of change of f during an excursion is just $2f_{th}/|t_f - t_i|$. Because $t_f - t_i$ is τ_{ex}, the dipole power time-scale during the excursion is

$$\{f\}_{ex}/|\partial_t f|_{ex} = (3f_{th}/5)/(2f_{th}/\tau_{ex}) = 3\tau_{ex}/10\,. \tag{18}$$

For the reasons stated above, we equate this with the time-scale $T_1 = 830$ years for the present era dipole power, thereby obtaining the predicted mean duration of geomagnetic dipole excursions

$$\tau_{ex} = 10T_1/3 = 2767 \text{ years}\,. \tag{19}$$

Because we predict that, on average, excursions occupy 2.5% of geologic time, excursions are, on average, predicted to occur every 110,667 years (40×2767 years). So the predicted frequency of geomagnetic dipole excursions is

$$\nu_{ex} = 9.04 \text{ per million years} \tag{20}$$

One excursion in four is expected to be accompanied by a durable change in the sign of the axial dipole moment, so we predict the mean frequency of geomagnetic axial dipole reversals to be

$$\nu_{rev} = 2.26 \text{ per million years.} \tag{21}$$

On average, such reversals are expected to take $2\tau_{ex} = 5534$ years. Are these predictions consistent with paleomagnetic data?

6. Comparison with Paleomagnetic Excursion and Reversal Data

We predict geomagnetic excursions down to 20% of the 1980 absolute dipole moment at an average rate of 9.04/Myr. Jacobs [1994, Figure 4.9 after Champion and others] shows 12 named paleomagnetic events in the past 1 Myr. The paleointensity record of Valet and Meynadier [1993, Figure 3] shows the same 12 named events, but occuring over 1.1 myr. Nine events (Laschamp, Blake, Jamaica, Levantine, Biwa III, Emperor, Big Lost, Delta, and Kamikatsura) appear to be excursions; three events (Brunhes/Matuyama, the end of the Jaramillo, and the beginning of the Jaramillo) appear to be durable axial dipole reversals. If our interpretation of these events is correct, then the geologically recent rate of dipole excursions is 12/Myr, or 33% faster than predicted; however, one quarter of the excursions are reversals as predicted. This agreement over 10^6 years is close enough to sustain a claim of reasonable accuracy when it is recalled that our prediction used but the single time-scale from the mid-twentieth century.

Valet and Meynadier [1993, Figure 3], identify 25 named events in 3.8 Myr, 11 of which appear to be durable polarity changes. This gives 6.6 named events per million years and 2.9 durable axial dipole reversals per million years. The former figure is 27% less than our predicted excursion frequency; the latter is 28% more than our predicted axial dipole reversal frequency.

Our definition of excursions implies expected VADMs of about 3.2×10^{22} A/m^2 or less during an excursion. By laying a straightedge across the paleointensity record of Valet and Meynadier [1993, Figure 3] at a VADM of 3.0×10^{22} A/m^2, we counted 17 intervals below this value in the first Myr and about 64 in 3.8 Myr. The event rates of 17/Myr and 16.8/Myr, respectively, seemed large – as did the amount of time spent below this moment. Purely subjective counts gave 53 ± 6 events in 3.8 Myr, or about 14/Myr. Recall, however, that the mean VADM of this record, 3.9×10^{22}, is less than our predicted {VADM} of 6.21×10^{22} Am^2. The scale factor of (3.9/6.21) is needed to fairly test our excursion frequency prediction, as distinct from our paleointensity prediction, against this record. (We stress that these predictions are distinct: the mean paleointensity prediction depends upon extrapolation of McLeod's Rule for multipole power to the lowest degree, but not upon the PDF assigned to normalized multipole power $(2n+1)R_{nc}/\{R_{nc}\}$; the mean excursion (and reversal) frequency prediction depends upon the PDF assigned to normalized dipole power, but only uses a particular value of $\{R_{1c}\}$ to scale our definition of threshold dipole power to $0.072\{R_{1c}\}$). Because 3.2(3.9/6.21) = 2.01, only intervals Valet and Meynadier [1993, Figure 3] show with VADMs of 2.0×10^{22} or less are counted as excursions. We then count about 40 excursions in the past 3.8 million years. The average observed rate of such excursions, about 10.5/Myr, remains 16% more than the predicted rate of 9.04 excursions per Myr, but demonstrates that the prediction is indeed fairly accurate.

Because only one quarter of geomagnetic dipole power excursions are expected to be accompanied by durable reversals of the axial dipole, the frequency of durable axial dipole reversals is predicted to be 2.26/Myr. To test this prediction, we used the polarity time scale of Cande and Kent [1992]. Their Figure 29 shows 186 reversals in 84 million years, so the observed mean rate of reversals is 2.21/Myr. Our prediction of 2.26/Myr is therefore quite accurate. The observed rate is, however, not constant; the eight, 10 million year intervals give rates of 4.8, 3.6, 4.0, 2.0, 0.9, 1.4, 1.0, and 0.8 per Myr (see Appendix A). The total 185/80 Myr average rate of 2.31/Myr nontheless confirms the absolute accuracy of our prediction.

M. Prevot (1996, personal communication) calls attention to the work of Kristjansson [1985], who found the average transition time for durable geomagnetic polarity changes recorded in Icelandic lavas over the past 10 Myr to be 5-6000 years. His statistically estimated mean transition time of 5500 years shows our predicted mean duration of 5533 years for durable axial dipole reversals to be quite accurate. The agreement seems particularly remarkable because the polarity transitions are defined by detailed paleodirection data, while our prediction is derived from a PDF for normalized dipole power and the 1945-1980 dipole moment.

7. Summary and Conclusions

In geomagnetically source-free regions, Earth's magnetic field can be represented mathematically by spherical harmonic expansions of scalar magnetic potentials. The mean square value of the magnetic induction represented by potential harmonics of degree n averaged over a sphere gives the spatial magnetic power spectrum at degree n on the sphere. McLeod's Rule for the magnetic field generated by Earth's core says that the internal spatial geomagnetic power spectrum of the core field at the core surface, $R_{nc}(c)$, is expected to be inversely proportional to (2n + 1) for finite degrees $1 < n \leq N_E$ [McLeod, 1985; 1994; 1996]

The distribution χ^2 with 2n+1 degrees of freedom is assigned to $(2n+1)R_{nc}/\{R_{nc}\}$. We extend this even to the first degree on the hypothesis that the small tilt of Earth's magnetic dipole moment relative to its rotation axis is mainly a geometric, rather than an energetic, effect of the Coriolis pseudo-force on outer core field and flow. Such anisotropy is consistent with the χ^2 distribution in that there are an infinte number of triads of probability distribution functions for the three dipole components that give the distribution χ^2 with three degrees of freedom for normalized dipole power.

McLeod's Rule (equations (6) and (7)) was verified by using it to locate the core-mantle boundary with single epoch main field models of satellite geomagnetic data. This method is more accurate than other core magneto-location methods, with the estimated core radius of 3485 km being very close to, and within 0.2% of, the seismologic value of 3480 km. With three minor variations of McLeod's Rule, two main field models of Magsat data, and using both degrees 3 through 12 and degrees 1 through 12, we obtained 12 estimates of the core radius with mean 3504.8 $\pm$ 37.0 km (2σ). All the estimates are within 1.8% of the seismologic value, even those including Earth's magnetic dipole and quadrupole powers. McLeod's Rule thus enabled accurate magneto-location of Earth's core.

With the core radius fixed at 3480 km, preliminary calibrations of McLeod's Rule and similar spectral forms were performed using main field model values of R_n for degrees 3 through 12. By extrapolation to the lower degrees, we predict the expectation value of Earth's dipole moment to be 5.89×10^{22} Am^2 rms (74.5% of its 1980 value) and the expectation value of geomagnetic intensity to be 35.6 μT rms. Archeo- and paleomagnetic intensity estimates show this and related predictions to be reasonably accurate, though our non-expert analysis of a volcanic paleointensity data subset suggests it is 20% low. By treating the dipole as axial, the expected multipolar powers were converted to a prediction of Earth's expected Virtual Axial Dipole Moment {VADM} = 6.21×10^{22} Am^2. When the mean virtual dipole moment from a temporally well-distributed 4 Myr sedimentary record [Valet & Meynadier, 1993] is averaged with that from reliable, geographically better-distributed volcanic records [McFadden & McElhinny, 1982; Tanaka et al., 1995], the result is within 1.8% of our prediction. Our prediction is also within 5% of various mean virtual dipole moments calculated from the Prèvot et al. [1990] tabulation of group mean Thellier-Thellier paleointensities from the Mesozoic and the Cenozoic. (This tabulation appropriately excludes intensities from the Cretaceous superchron, which we suggest was a time of very stable stratification in the upper reaches of the outer core when McLeod's Rule for the low degree multipole powers may have given way to a dissipation limited n^{-3} spectrum and a more axisymmetric, perhaps more Saturnian, field). Simple extension of McLeod's Rule to the first degree thus enabled accurate prediction of geomagnetic intensity averaged over geologic time intervals.

Our extension of McLeod's Rule and multipole power PDFs forced us to predict that exceptionally weak absolute dipole moments (<20% of the 1980 value) will occur during 2.5% of geologic time. We predicted the mean duration of such major geomagnetic dipole excursions, one quarter of which feature durable axial dipole reversal, using the dipole time-scale from modern geomagnetic field models [Langel et al. 1985; 1988] and a statistical model of dipole power excursions. Use of the modern dipole power time-scale for other epochs is justified and is indicated by the convective core geodynamo hypothesis; an excursional dipole waveform was derived from the statistical model. We predict a mean excursion duration of 2767 years, 9.04 excursions per million years, and thus about 2.26 axial dipole reversals per million years with a mean duration of 5533 years. Paleomagnetic data show this purely geomagnetic prediction to be quite accurate. Our non-expert analysis of the Valet & Meynadier [1993] record gives a mean rate of 10.5 power excursions per million years over the past 4 million years; this is 16% more than predicted. The polarity time scale of Cande and Kent [1992] shows the mean rate of durable dipole reversals for the past 84 Myr to be 2.21 per million years; this is within 2.3% of our prediction. The average polarity transition time determined by Kristjansson [1985] is 5500 years; this is within 1% of our prediction. Simple extension of McLeod's Rule to the first degree thus enabled accurate prediction of the frequency of geomagnetic excursions and reversals averaged over geologic time intervals.

McLeod's Rule for the core field led to (1) very accurate magneto-location of the core-mantle boundary; (2) reasonably, and it seems very, accurate prediction of paleomagnetic field intensity; and (3) fairly accurate prediction of the mean frequency of major absolute geomagnetic dipole excursions, including very accurate prediction of the mean frequency of durable axial dipole reversals. It also enabled remarkably accurate prediction of the mean duration of durable axial dipole reversals. The accuracy of these predictions serves to unify geomagnetism and paleomagnetism and indicates that McLeod's [1994, 1996] model correctly relates the inductive basis of core geodynamo theory to geomagnetic observation. We conclude that McLeod's Rule provides *bona fide a priori* information required for stochastic inversion of paleo-, archeo-, and/or historical magnetic measurements. Some caution in applying it to the first and second harmonic degrees is nontheless advised, particularly for historical times; moreover, it might not hold during polarity superchrons. We hope that McLeod's rule will not only be better used, but better tested, by geomagnetists and paleomagnetists in the future.

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Appendix A

Core Field Time Scales and Superchrons

We see no reason to presume that geomagnetic dipole power excursions represent extraordinary core thermal, compositional, or stress boundary conditions, and thus do not presume exceptional thermo-compositional buoyancy forces during excursions. These forces are thought to drive the core geodynamo, but pressure ($-\nabla p$), Coriolis ($2\rho\Omega\times v$), and Lorentz ($\mathbf{J}\times\mathbf{B}$) forces may be of similar magnitude. If so, and with a strong field of curl $\mu\mathbf{J}$, rough magnetogeostrophic balance indicates $|\mathbf{J}\times\mathbf{B}| \approx B^2/\mu L \approx 2\rho\Omega v \approx |2\rho\Omega\times v|$. The inductive time-scale becomes $\tau_B \approx L/v \approx 2\rho\mu\Omega L^2/B^2$, which might seem to increase as the field decreases; however, uncompensated buoyancy due to non-hydrostatic density perturbations ρ' is the driver, so (except near the poles) $|\rho' g| \approx |2\rho\Omega\times v| \approx |\mathbf{J}\times\mathbf{B}|$, $B^2 \approx \mu L\rho' g$, and $\tau_B \approx 2\rho\Omega L/\rho' g$ remains independent of the field strength.

In contrast with the qualitative excursion scenario outlined by Voorhies [1992], Lorentz forces might be uncommonly weak during an excursion. This depends upon the importance of the poloidal dipole compared with other poloidal and the toroidal field multipoles within the core. Yet values of τ_B (and T_1) need not be altered if deep core flow is more nearly geostrophic than magnetostrophic during an excursion. Even such a profound change in the dynamical balance might alter flow geometry rather more than typical flow speed. Moreover, with $|\rho' g| \approx |2\rho\Omega\times v|$ (except near the poles), $v \approx \rho' g/2\rho\Omega$, and $\tau_B \approx 2\rho\Omega L/\rho' g$ remains independent of the field strength.

The foregoing order-of-magnitude reckonings show that if the core geodynamo is driven by uncompensated (e.g., thermo-compositional) buoyancy, then the inductive time-scale τ_B ought not, indeed cannot, vary greatly with field strength. Our predicted excursion and reversal frequencies use the modern, emipirical dipole power time-scale T_1 during geologically remote times. The accuracy of our predictions for geomagnetic dipole power excursion (and reversal) frequency might thus be viewed as lending support to the convective core geodynamo hypothesis. In fact, the accuracy of our predictions merely provides no evidence against this hypothesis; however, the accuracy does seem higher than might reasonably have been anticipated by advocates for this hypothesis.

Curiously, our position regarding the stability of T_1 need not conflict with the view that changes in the dipole field are largely decoupled from the magnitude of the dipole itself because near surface core flow is largely tangentially geostrophic [LeMouël, 1984]. Analysis of the present day field shows that the mean square radial core surface field is about equally divided between dipole and resolvable non-dipole fields (through degree 12) [Voorhies, 1984], so $T_1 = \langle R_1(c)\rangle/\langle\partial_t R_1(c)\rangle \approx R_{ND}(c)/[\partial_t R_1(c)] = T^*$. By (10) we do not expect the non-dipole field to be unusually weak during excursions; however, if this were so, then time-scale T^* could remain on the order of many hundreds of years.

Although we do not presume that geomagnetic dipole power excursions represent extraordinary core thermal and compositional boundary conditions, these boundary conditions likely evolve with Earth's mantle and inner core. If boundary condition evolution increases the amplitude of uncompensated density perturbations ρ', one anticipates increased convective speeds, a reduced inductive time-scale, reduced T_1, reduced mean excursion duration by (19), and thus an increased frequency of geomagnetic dipole power excursions and axial dipole reversals. In contrast, buoyancy braking in a stably stratified uppermost outer core reduces convective speeds there, increases the inductive time-scale, increases T_1, increases a mean excursion duration indicated by relations akin to (19), and thus decreases the frequency of dipole power excursions and axial dipole reversals. If thick enough and sufficiently stable, such a layer might eliminate convective motion and drive the excursion and reversal frequency to zero. This indicates a polarity superchron.

Whether or not either McLeod's Rule or (10) holds during such an era of superficial stability remains to be seen. More nearly zonal toroidal flow in such a layer may suggest different inductive time constants than used by McLeod, while more complete suppresion of motion suggests increased importance of possibly lengthened diffusive time constants. It is conceivable that dynamo action might fail in the stable layer; if so, the R_n spectrum for the field outside the core might be fairly described by extending the diffusive range down to the first degree (so $\{R_n\} \approx Kn^{-3}$). Turning to the PDFs consistent with (10) derived in Appendix C, it may seem unlikely that a form like (C5) would succeed; however, as σ_y approaches zero, (C5) approaches (C1). The limiting axial dipole PDF (C1a) actually vanishes at zero axial moment, as does its slope; this may describe inhibition, or perhaps prohibition, of axial dipole reversals in the special case of small, or perhaps zero, equatorial dipole. So it is conjectured that, by decreasing reversal frequency and hastening diffusion of higher order multipoles, the hypothetical stable layer renders superchrons a time of almost purely axial dipole field.

As to the nonstationarity suggested by a non-constant rate of reversals inferred from the Cande and Kent [1992] polarity time scale, it may well be that the formation and removal of the layer should be viewed as a long and complex geologic process rather than a sequence of simple catastrophies.

If, eventually, the density perturbations become too small to drive convection, then the relevant time-scale for the field would become the diffusive, rather than the inductive, time scale and the core field would decay away.

Appendix B

Excursional Waveform

Consider a special ensemble average of many particular dipole power excursions $f_j(t)$ whereby idiosyncratic features of individual excursions can cancel out, yet a single mean excursional waveform consistent with (16c) can remain. This archtypal excursion must initiate at the threshold value $f(t_i) = f_{th}$, decrease to the minimum at $f(t_o)$, and finally return to $f(t_f) = f_{th}$; therefore, the ensemble averaging is a variation of superimposed epoch analysis keyed to minimum f. A symmetric waveform can be ensured by averaging each particular excursion record with its reflection about its minimum. The waveform must account for the fact that $\wp$ (f=0) = 0, but is non-zero for other f; therefore, it has a cusp at $f(t_o) = 0$. To construct this cusp, individual records that do not descend to exceptionally small values of f must be cut at their minimum f and the two parts separated in time; in effect, such records receive zero weight in constructing the portion of the waveform near the centered cusp. We can but suppose that the resulting symmetrized waveform will decline monotonically from f_{th} to the cusp at zero f and then grow monotonically back to f_{th}.

During the interval $[t_o, t_f]$, the probability that a monotonically increasing function M(t) is in the range $[\alpha, \alpha+\delta]$, denoted $P[\alpha = M(t_\alpha) \le M(t) \le M(t_{\alpha+\delta}) = \alpha+\delta]$, is proportional to $t_{\alpha+\delta} - t_\alpha$; indeed,

$$\frac{P[\alpha = M(t_\alpha) \le M(t) \le M(t_{\alpha+\delta}) = \alpha+\delta]}{P[M(t_o) \le M(t) \le M(t_f)]} = (t_f - t_o)^{-1} \int_{t_\alpha}^{t_{\alpha+\delta}} dt \qquad \text{(B1)}$$

A sufficient condition for M(t) to satisfy (B1) throughout the interval $t_o \le t \le t_f$ is $\int \wp(M)dM = V(t - C)$, or $\wp(M) = V dt/dM$, where V is the probability of finding M(t) in range $[M(t_o), M(t_f)]$ divided by the duration of the interval and C is a constant (e.g., t_o).

The probability for finding $f(t) \leq f_{th}$ and increasing is half that given by (16c), but by symmetry such epochs occupy but 1.25% of geologic time. For monotonic growth from an $f(t_o)$ of zero to the threshold value $f(t_f) = f_{th} = 0.216$, (16c) and the sufficient condition for (B1) give

$$0.5\int \wp(f)df \approx 0.5(2\pi)^{-1/2}(2\kappa/3)[f(t)]^{3/2} = V(t - t_o) \tag{B2}$$

where V is $0.0125/(t_f - t_o)$. It follows that

$$f(t) = [3(\pi/2)^{1/2}/(40\kappa)]^{2/3} \left|\left[\frac{t - t_o}{t_f - t_o}\right]\right|^{2/3} = f_{th}|\tau|^{2/3} \tag{B3}$$

where $\tau = (t - t_o)/(t_f - t_o)$. A similar analysis applied from the onset of an excursion at $t_i = 2t_o - t_f$ to its minimum at t_o shows that (B3) specifies the symmetric bitonic dipole power excursion model during the entire interval $[t_i, t_f]$. The cusp in (B3) at $\tau = 0$ is singular in that $\partial_t f$ approaches infinity as t approaches t_o. In fact $R_{1c}(t)$ must change at a finite rate; this unphysical property of the statistical dipole power excursion model (B3) merely reflects the zero probability (10) assigns to having all three dipole components vanish simultaneously. If not wholly impossible, a complete dipole power outage thus last no time at all.

Because the cusp in (B3) at t_o is singular, the duration of an excursion waveform reconstructed from (B3) is lessened but slightly if f(t) does not quite reach zero. For example, model (B3) spends but 3.2% the excursion duration below 10% of the f_{th}. The typical form of very brief excursions is reconstructed by cutting out a very large portion of form (B3) centered on t_o and pasting the wings together. This form for brief excursions is minimum at values of f that are not much less than f_{th}. Brief excursions are thus expected to be shallow and, as also mentioned in the text, are therefore less likely to include a change in sign of one or more of the vector dipole components. Again, brief excursions are expected to be shallow, axial dipole reversals typically require excursion durations of τ_{ex} or more, and only half of all excursions have a 50% chance of becoming a durable reversal.

Excursional waveforms reconstructed from (B3) have a finite jump in $\partial_t f(t)$ where the wings of are joined. This is thought to be an artifact, but may suggest an alternate view wherein the jump indicates a sign change in a dipole component. The reverse is not always true, for spinning a tilted dipole need not change f at all.

In excursion model (B3) $|f(t)/\partial_t f(t)| = 3|t - t_o|/2$, dipole power varies as $|t - t_o|^{2/3}$, and absolute dipole moment varies as $|t - t_o|^{1/3}$. Indeed, the accompanying model of an absolute dipole moment excursion at Earth's surface from the threshold down and back is

$$|\mathbf{m}(t_i \leq t \leq t_f)| = 1.58\times10^{22}\ \mathrm{Am}^2 \left[\frac{|t - t_o|}{t_f - t_o}\right]^{1/3} \tag{B4}$$

It might be possible to test excursion model (B3-B4) against ensemble averages of data recorded during many geomagnetic dipole power excursions, provided effects of the expected 15.1 μT rms non-dipole field could be reduced.

Appendix C

Chi-squared from Abnormal Distributions

If 2n+1 independent random variables x_i are drawn from identical, zero mean Gaussian probability distribution functions of unit variance, then the probability distribution function (PDF) for the sum of their squares, $\Sigma_i(x_i)^2$, is well known to be chi-squared with 2n+1 degrees of freedom. It is apparently less well known that the reverse is not always true. For example, consider three independent real variables (X, Y, Z) on the open interval $(-\infty,\infty)$ with PDFs

$$\wp_M(X) = (2\pi)^{-1/2}\ X^2 \exp(-X^2/2) \tag{C1a}$$

$$\wp_D(Y) = \delta(Y) \tag{C1b}$$

$$\wp_D(Z) = \delta(Z) \tag{C1c}$$

where $\wp_M$ is the bi-Maxwellian and δ is the Dirac delta function. There is no chance of Y and Z being anything but zero and X can be either positive or negative, so $X^2 + Y^2 + Z^2 = \chi^2$ is distributed as

$$\wp(\chi^2)d(\chi^2) = \wp(X^2)d(X^2) = 2\wp_M(X)\left|\frac{dX}{d(X^2)}\right| d(X^2)$$

$$= (2\pi)^{-1/2}(\chi^2)^{1/2}\exp(-\chi^2/2)$$

$$= [2^{3/2}\Gamma(3/2)]^{-1}(\chi^2)^{1/2}\exp(-\chi^2/2) \tag{C1d}$$

where Γ is the gamma function. PDF (C1d) is chi-squared with three degrees of freedom.

The difference between example (C1) and the usual case of three normal distributions with equal variance is important. If one replaces (X, Y, Z) with suitably normalized dipole coefficients ($X \rightarrow g_1^0/D$, $Y \rightarrow 0$, $Z \rightarrow 0$), then the example would describe a dipole with a zero mean bi-modally distributed axial component, variance $\{(g_1^0)^2\} = 3D^2$, and negligible tilt; this approximates a planetary magnetic field dominated by an axial dipole. The usual case (with $(X, Y, Z) \rightarrow (g_1^0/\sigma, g_1^1/\sigma, h_1^1/\sigma)$) would describe a dipole with no preferred direction at all and seems even less relevant to Earth than example (C1). That this isotropic case with typical dipole tilts of order $\tan^{-1}([2]^{1/2}) = 54.7°$ degrees may be of some interest for Uranus and Neptune, while case (C1) may be of some interest for Saturn, helps motivate derivation of potentially more Earth-like, intermediate distributions.

There are an infinite number of sets of three PDFs for three independent variables which give the distribution χ^2 with three degrees of freedom. To see this, consider the possibly singular distributions for independent variables (x, y, z) on the interval $(-\infty, +\infty)$

$$\wp_1(x) = A_x|x|^{-p_x} e^{-x^2/2} \tag{C2a}$$

$$\wp_2(x) = A_y|y|^{-p_y} e^{-y^2/2} \tag{C2b}$$

$$\wp_3(z) = A_z|z|^{-p_z} e^{-z^2/2} \tag{C2c}$$

where (A_x, A_y, A_z) are normalization constants and (p_x, p_y, p_z) are power law indicies which are less than one and sum to zero. Clearly

$$\wp_1'(x^2)d(x^2) = 2\wp(x)\left|\frac{dx}{d(x^2)}\right| d(x^2)$$

$$= A_x' |x^2|^{-[(1+p_x)/2]} e^{-x^2/2} , \tag{C2d}$$

while similar PDFs for y^2 or z^2 follow by replacing x in (C2d) with y or z, respectively. The PDF for $\xi^2 = x^2 + y^2 + z^2$ is obtained in the usual way via

$$\wp(\xi^2) = \int_0^\infty \int_0^\infty \int_0^\infty \wp_1'(x^2)\, \wp_2'(y^2)\, \wp_3'(x^2)\, \delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{C3a}$$

and the fact that the offset delta-function is the inverse Lapalace transform (denoted La^{-1}) of the exponential of its offset

$$\delta[\xi^2 - (x^2+y^2+z^2)] = La^{-1}[e^{\xi^2 - (x^2+y^2+z)^2}] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s\xi^2} e^{-s(x^2+y^2+z^2)} ds \tag{C3b}$$

where s is the Laplace transform domain variable. Substituting (C2a) and (C3b) into (C3a), and introducing $u = x^2$, $v = y^2$ and $w = z^2$, yields

$$\wp(\xi^2) = \frac{A_x'A_y'A_z'}{2\pi i}\int_{-i\infty}^{i\infty} e^{s\xi^2} La[u^{-(1+p_x)/2} e^{-u/2}] \\ La[v^{-(1+p_y)/2} e^{-v/2}]\, La[w^{-(1+p_z)/2} e^{-w/2}]ds . \tag{C4a}$$

With $k_x \equiv (1 - p_x)/2$ and

$$La[u^{-(1+p_x)/2} e^{-u/2}] = La(u^{k_x-1} e^{-u/2}) = \Gamma(k_x)[s + 1/2]^{-k_x} \tag{C4b}$$

it follows that

$$\wp(\xi^2) = \frac{A_x'A_y'A_z'}{2\pi i}\int_{-i\infty}^{i\infty} e^{s\xi^2} \frac{\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)}{(s + 1/2)^{k_x+k_y+k_z}} ds . \tag{C4c}$$

Provided k_x, k_y, and k_z are all positive definite and $k_x + k_y + k_z = 3/2$,

$$\begin{aligned}\wp(\xi^2) &= A_x'A_y'A_z'\,\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)\, La^{-1}[(s + 1/2)^{-3/2}] \\ &= A_x'A_y'A_z' \frac{\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)}{\Gamma(3/2)} (\xi^2)^{1/2} e^{-\xi^2/2} \\ &= [2^{3/2}\,\Gamma(3/2)]^{-1} (\xi^2)^{1/2} e^{-\xi^2/2} .\end{aligned} \tag{C4d}$$

Comparison of (C4d) with (C1d) shows that ξ^2 is in fact distributed as chi-squared (χ^2) with three degrees of freedom. The conditions that (k_x, k_y, k_z) are all positive definite and sum to 3/2 is, in (C2a-c), equivalent to the condition that (p_x, p_y, p_z) are all less than 1 and sum to zero. There are an infinite number of triplets for (p_x, p_y, p_z) that satisfy these conditions. The form of PDFs (C2a-c) is $A|\chi|^{-p}\exp(-\chi^2)]$ and is, in retrospect, one half the chi-squared distribution with typically fractional degree of freedom $(1 - p)$ reflected about a zero mean; for $p = (1 - 2k)$, the corresponding A is $1/[2^k\Gamma(k)]$.

It has been demonstrated that supposing some variable ξ^2 to be distributed as chi-squared with 2n+1 degrees of freedom does not necessarily imply that it is the sum of 2n+1 normally distributed variables of equal variance. It particular, there are an infinite number of sets of three PDFs for the three degree 1 Gauss coefficients that imply the distribution of chi-squared with three degrees of freedom for normalized dipole power $3R_{1c}/\{R_{1c}\}$. To see this explicitly, denote the variances of (g_1^0, g_1^1, h_1^1) by $(\sigma_x^2, \sigma_y^2, \sigma_z^2)$ which, in turn, sum to $3D^2$; then $3R_{1c}/\{R_{1c}\}$ is $[(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2]/D^2$. This would be identified with ξ^2 in (C4d) if (x, y, z) in (C2a-c) were identified with $(g_1^0/D, g_1^1/D, h_1^1/D)$. One eligible triplet of PDFs would then be (C2a-c) with $p_x = -3/2$, $p_y = p_z = 3/4$; however, with $p_x = -2 + \varepsilon$ for the axial dipole, $p_y = p_z = 1 - \varepsilon/2$ for the equatorial dipoles, then any one of the infinite values of real ε such that $0 < \varepsilon < 2$ gives an eligible triplet of PDFs. Indeed, as ε approaches zero the PDFs (C2a-c) seem to approach (C1a-c) and the dipole becomes axial, while as ε approaches 2 the PDFs (C2a-c) approach the usual, normal distributions and the dipole tilt becomes random. It is perhaps amusing to consider $\varepsilon/2$ as a 'tilt control parameter' that not only describes the distribution of dipole tilts of some class of astronomical objects, but which might be diagnostic of, and perhaps determined by, the character of convection within a stellar or planetary interior. Alternately, slow evolution of a single astronomical object effecting evolution in the style of its internal convection might yield an evolution (non-stationarity) in the distribution of dipole tilts that could be described via time-evolution of ε.

We see no immediate problem with trying forms like (C2a-c); however, the variance of such forms is restricted by the selection of the power law indicies (and is $2^{1/2}\Gamma(k + 1/2)/\Gamma(k)$). Moreover, we feel uncomfortable with the singularities in PDFs (C2b-c) for $1 > p_y = p_z > 0$. These are not the intermediate cases we had sought! Furthermore, it seems worth following Constable & Parker [1988] in so far as assigning normal distributions to g_1^1 and h_1^1 in the terrestrial case. If this is done, then (C1a-c) makes it intuitively obvious that the PDF for g_1^0 yielding chi-squared with three degrees of freedom for the distribution of normalized dipole power will be a linear combination of Gaussian and bi-Maxwellian distribution. This is in fact the case, as shown by the following derivation.

Denote $[(a/c)^3 g_1^0, (a/c)^3 g_1^1, (a/c)^3 h_1^1]$ by [x, y, z] and the variances of independent (x, y, z) by $(\sigma_x^2, \sigma_y^2, \sigma_z^2)$. The axial dipole hypothesis would have the means of y and z be zero, while isotropy of the equatorial dipole would have σ_y and σ_z be equal. We replace (C2a-c) with distributions

$$\wp_x(x)dx = (3/2\pi V^2)^{1/2}(3/V^2)\left[\frac{\sigma_x^2 - \sigma_y^2}{V^2}\, x^2 + \sigma_y^2\right] \exp(-3x^2/2V^2)\, dx \qquad \text{(C5a)}$$

$$\wp_y(y)dy = (2\pi\sigma_y^2)^{-1/2} \exp(-y^2/2\sigma_y^2)dy \qquad \text{(C5b)}$$

$$\wp_z(z)dz = (2\pi\sigma_z^2)^{-1/2} \exp(-z^2/2\sigma_z^2)dz \qquad \text{(C5c)}$$

where $V^2 \equiv \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. With $\xi^2 \equiv x^2 + y^2 + z^2 = V^2\chi^2/3$, the problem is to derive (C5a) for $\wp_x(x)$ given normal distributions (C5b-c) and

$$\wp(\xi^2)d(\xi^2) = (2^{3/2}\Gamma(3/2))^{-1}[3\xi^2/V^2]^{1/2}\exp[-3\xi^2/2V^2](3/V^2)d(\xi^2) \qquad \text{(C5d)}$$

subject to the constraint that the dipole is mainly axial, so $\sigma_x^2 >> \sigma_y^2 \approx \sigma_z^2$. For independent (x, y, z)

$$\wp(\xi^2) = \int_0^\infty\int_0^\infty\int_0^\infty \wp_x'(x^2)\,\wp_y'(y^2)\,\wp_z'(z^2)\,\delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{C6a}$$

$$= (2\pi\sigma_y\sigma_z)^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \wp_x'(x^2)(y^2z^2)^{-1/2}\exp(-y^2/2\sigma_y^2 - z^2/\sigma_z^2)$$

$$\delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{C6b}$$

or, using (C3b) and resetting $u = x^2$, $v = y^2$, $w = z^2$,

$$\wp(\xi^2) = (4\pi^2\sigma_y\sigma_z i)^{-1}\int_{-i\infty}^{i\infty}\int_0^\infty\int_0^\infty\int_0^\infty \wp_x'(u)\exp(-su)(vw)^{-1/2}\exp(-sv - v/2\sigma_y^2)$$

$$\exp(-su - w/\sigma_z^2)\exp(s\xi^2)\,du\,dv\,dw\,ds\,. \tag{C6c}$$

Recognizing and evaluating the Laplace transforms gives

$$\wp(\xi^2) = (4\pi^2\sigma_y\sigma_z i)^{-1}\int_{-i\infty}^{i\infty} La[\wp_x'(u)]\,La[v^{-1/2}\exp(-v/2\sigma_y^2)]$$

$$La[w^{-1/2}\exp(-w/2\sigma_z^2)]\exp(s\xi^2)ds$$

$$= (2\pi\sigma_y\sigma_z)^{-1}La^{-1}[La[\wp_x'(u)]\,[\Gamma(1/2)]^2[(s + 1/2\sigma_y^2)(s + 1/2\sigma_z^2)]^{-1/2}\,. \tag{C6d}$$

The Laplace transforms of (C6d) and (C5d) imply

$$La[\wp_x'(u)] = \sigma_y\sigma_z(3/2V^2)^{1/2}\left[\frac{(s + 1/2\sigma_y^2)(s + 1/2\sigma_z^2)}{(s + 3/2V^2)^3}\right]^{1/2}(3/V^2) \tag{C7a}$$

In the special case $\sigma_y = \sigma_z$,

$$\wp_x'(x^2)] = \sigma_y^2(3/2V^2)^{1/2}\,La^{-1}\left[\frac{s + 1/2\sigma_y^2}{(s + 3/2V^2)^{3/2}}\right](3/V^2) \tag{C7b}$$

and the inverse transform gives

$$\wp_x'(x^2) = \frac{\sigma_y^2(3/2V^2)^{1/2}}{2\Gamma(3/2)}(3/V^2)\,[(x^2)^{-1/2} + \frac{V^2 - 3\sigma_y^2}{\sigma_y^2V^2}(x^2)^{-1/2}]\exp(-3x^2/2V^2)\,. \tag{C7c}$$

Because x can be of either sign, $\wp_x(x)dx$ is but half $\wp_x'(x^2)2xdx$ and (C7c) in fact reduces to the linear combination of a zero mean bi-Maxwellian and a zero mean Gaussian (C5a).

In (C5a) note that $V^2 - 3\sigma_y^2 = \sigma_x^2 - \sigma_y^2 > 0$; with $\sigma_x^2 >> \sigma_y^2$, this distribution (C5a) enjoys the two peaks on either side of the local minumum at $g_1^0 = 0$ corresponding to normal and reversed axial dipole polarity. A suitable index of anisotropy, or 'tilt control parameter', for PDFs (C5a-c) is now $\varepsilon \equiv [(\sigma_x^2 - \sigma_y^2)/V^2]$, which is 1 for purely axial dipoles, zero for randomly oriented dipoles, and -1/2 for purely equatorial dipoles. We conjecture that the probability distibution functions (C5a-c) describe the terrestrial dipole with fair accuracy; however, they are but one class of PDFs consistent with the distribution chi-squared with three degrees of freedom advanced for normalized dipole power in the text.

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13. ABSTRACT *(Maximum 200 words)*

The geomagnetic spatial power spectrum $R_n(r)$ is the mean square magnetic induction represented by degree n spherical harmonic coefficients of the internal scalar potential averaged over the geocentric sphere of radius r. McLeod's Rule for the magnetic field generated by Earth's core geodynamo says that the expected core surface power spectrum $\{R_{nc}(c)\}$ is inversely proportional to $(2n + 1)$ for $1 < n \leq N_E$. McLeod's Rule is verified by locating Earth's core with main field models of Magsat data; the estimated core radius of 3485 km is close to the seismologic value for c of 3480 km. McLeod's Rule and similar forms are then calibrated with the model values of R_n for $3 \leq n \leq 12$. Extrapolation to the degree 1 dipole predicts the expectation value of Earth's dipole moment to be about 5.89×10^{22} Am^2 rms (74.5% of the 1980 value) and the expected geomagnetic intensity to be about 35.6 μT rms at Earth's surface. Archeo- and paleomagnetic field intensity data show these and related predictions to be reasonably accurate.

The probability distribution χ^2 with 2n+1 degrees of freedom is assigned to $(2n+1)R_{nc}/\{R_{nc}\}$. Extending this to the dipole implies that an exceptionally weak absolute dipole moment (≤20% of the 1980 value) will exist during 2.5% of geologic time. The mean duration for such major geomagnetic dipole power excursions, one quarter of which feature durable axial dipole reversal, is estimated from the modern dipole power time-scale and the statistical model of excursions. The resulting mean excursion duration of 2767 years forces us to predict an average of 9.04 excursions per million years, 2.26 axial dipole reversals per million years, and a mean reversal duration of 5533 years. Paleomagnetic data show these predictions to be quite accurate.

McLeod's Rule led to accurate predictions of Earth's core radius, mean paleomagnetic field intensity, and mean geomagnetic dipole power excursion and axial dipole reversal frequencies. We conclude that McLeod's Rule helps unify geo-paleomagnetism, correctly relates theoretically predictable statistical properties of the core geodynamo to magnetic observation, and provides *a priori* information required for stochastic inversion of paleo-, archeo-, and/or historical geomagnetic measurements.

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