COHERENT STATES FOR KRONECKER PRODUCTS OF NON COMPACT GROUPS: FORMULATION AND APPLICATIONS

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Abstract

We introduce and study the properties of a class of coherent states for the group SU(1,1) X SU(1,1) and derive explicit expressions for these using the Clebsch-Gordan algebra for the SU(1,1) group. We restrict ourselves to the discrete series representations of SU(1,1). These are the generalization of the 'Barut Girardello' coherent states to the Kronecker Product of two non-compact groups. The resolution of the identity and the analytic phase space representation of these states is presented. This phase space representation is based on the basis of products of 'pair coherent states' rather than the standard number state canonical basis. We discuss the utility of the resulting 'bi-pair coherent states' in the context of four-mode interactions in quantum optics.

1 FORMULATION

1.1 Coupling of Pair coherent states in the fock state basis

For two mode systems the traditional SU(1,1) coherent states which have been extensively studied in the context of squeezing have been the Caves-Schumaker states [1], defined by the relation

$$|\zeta >= \exp(\zeta a^\dagger b^\dagger - \zeta^* ab)|0,0>, \tag{1}$$

In addition to these states many authours [2] [3] have considered the SU(1,1) coherent states of two mode systems or the 'pair coherent states' which were simultaneous eigenstates of ab and $a^\dagger a - b^\dagger b$

$$\begin{aligned} ab|\zeta, q> &= \zeta|\zeta, q>, \\ Q_1|\zeta, q> &= q_1|\zeta, q>. \end{aligned} \tag{2}$$

These can be mapped onto the SU(1,1) group by means of the two Boson realisation:

$$K_1^+ = a^\dagger b^\dagger \ , \ K_1^- = ab \ , \ K_1^z = \frac{1}{2}(a^\dagger a + b^\dagger b + 1) \ , \tag{3}$$

which form an SU(1,1) algebra with the commutation relations

$$\left[K_1^+, K_1^-\right] = -2K_1^z \,, \left[K_1^z, K_1^\pm\right] = \pm K_1^\pm. \tag{4}$$

The conservation law for Q_1 is related to the Casimir operator C for the SU(1,1) group; which can be written as

$$C = \frac{1}{4}(1-(a^\dagger a - b^\dagger b)^2) = \frac{1}{4}(1-Q_1^2). \tag{5}$$

Thus the eigenstate of Q_1 is also an eigenstate of C and the pair coherent state is related to the eigenstate of K_1^- by Barut and Girardello.

These generate a representation D^{q_1} that correspond to the positive discrete series representation of SU(1,1) [4]. In the number state basis, this corresponds to the basis states $|n_1 + q_1, n_1>$, where

$$|n_1+q_1, n_1>= \frac{(a^\dagger)^{n_1+q_1}(b^\dagger)^{n_1}}{((n_1!)(n_1+q_1)!)^{\frac{1}{2}}}|0,0>, \tag{6}$$

The pair coherent state in the number state basis labelled as $|\zeta_1, q_i>$ is

$$|\zeta_1, q_1>= N_{q_1}\sum_{n_1=0}^{\infty}\frac{\zeta_1^{n_1}}{\sqrt{n_1!(n_1+q_1)!}}|n_1+q_1, n_1>, \tag{7}$$

with

$$N_{q_1} = \left[(|\zeta_1|)^{-q_1} I_{q_1}(2|\zeta_1|)\right]^{-1/2}. \tag{8}$$

These states constitute a complete set in each sector q_i and the completeness relation is given by

$$\int d^2\zeta_1 \frac{2}{\pi} I_{q_1}(2|\zeta_1|) K_{q_1}(2|\zeta_1|)|\zeta_1, q_1><\zeta_1, q_1| \tag{9}$$

for the normalized states .

We now consider the group obtained by the addition of two SU(1,1) generators defined for four modes a,b,c,d.

$$\begin{aligned} K^+ &= a^\dagger b^\dagger + c^\dagger d^\dagger = K_1^+ + K_2^+ \,, \\ K^- &= ab + cd = K_1^- + K_2^- \,, \\ K^z &= \frac{1}{2}(a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d + 2) = K_1^z + K_2^z. \\ C &= \frac{(K^+K^- + K^-K^+)}{2} - K_z^2. \end{aligned} \tag{10}$$

The 'bi- pair coherent states' or the coherent states for the Kronecker Product are now the eigenstates of K^- ,C_1, C_2 and C . If we restrict ourselves to the positive discrete series representations of SU(1,1) then the Kronecker Product $D^{q_1}XD^{q_2}$ i.e the Clebsch Gordan series for SU(1,1) given by

$$D^{q_1}XD^{q_2} = \sum_{q=q_1+q_2+1}^{\infty} D^q. \tag{11}$$

Thus a given representation in the Kronecker product is fixed by q, q_1, q_2

The eigenvalue problem that we wish to solve is

$$K^{-}|\zeta,q>=\zeta|\zeta,q> \quad ;C|\zeta,q>=(1/4-q^2/4)|\zeta,q> . \tag{12}$$

In terms of the product number state basis $|n_1+q_1,n_1>|n_2+q_2,n_2>$ we get:

$$\begin{aligned} |\zeta,n,q_1,q_2> &= N_n \sum_{k=0}^{\infty} \frac{(\zeta)^k}{[(k)!(k+2n+q_1+q_2+1)!]^{\frac{1}{2}}} \\ &\times \sum_{n_1,n_2} C^{q_1,q_2,n}_{n_1,n_2,n+k}\delta_{(n_1+n_2,n+k)}|n_1+q_1,n_1>|n_2+q_2,n_2> . \end{aligned} \tag{13}$$

we get an expression for the Kronecker Product states in terms of the CG coefficients in the photon number basis.

1.2 Clebsh Gordan Problem in the pair coherent state basis

Consider the four mode bases of the Hilbert space characterised by the product of two pair (SU(1,1) coherent states $|\zeta_1,q_1>|\zeta_2,q_2>$. Since these coherent states form an overcomplete set any vector in the four mode Hilbert space can be expanded in terms of these states. In particular the coherent state of the product SU(1,1) X SU(1,1) $|\zeta,q>$ can be expanded directly in terms of the unnormalized states

$$\begin{aligned} |\zeta_1,q_1>> &= \sum_{n=0}^{\infty} \frac{\zeta_1^n}{\sqrt{n!(n+q_1)!}}|n+q_1,n> , \\ |\zeta_2,q_2>> &= \sum_{m=0}^{\infty} \frac{\zeta_2^m}{\sqrt{m!(m+q_2)!}}|m+q_2,m> . \end{aligned} \tag{14}$$

The completeness relation for the unnormalised states $|\zeta_i,q_i>>$ can be deduced from (2.18) to be

$$\int d^2\zeta_i \frac{2}{\pi}|\zeta_i|^{q_i}K_{q_i}(2|\zeta_i|)|\zeta_i,q_i>><<\zeta_i,q_i|=1. \tag{15}$$

The unnormalised states have the advantage that the operators $K_i^{\pm}$ and K_i^z can be expressed as differential operators. The completeness relation and resolution of the identity ensures that the product states $|\zeta_1,q_1>>|\zeta_2,q_2>>$ form the basis states for $D^{q_1}XD^{q_2}$ and any four mode state $|\psi>$ can be expanded as

$$|\psi>=\int <<\zeta_1,q_1<<\zeta_2,q_2|\psi>|\zeta_1q_1>>|\zeta_2,q_2>> d^2\sigma(\zeta_1)d^2\sigma(\zeta_2). \tag{16}$$

In this representation the quantity $<<\zeta_1,q_1,zeta_2,q_2|\psi>$ is an analytic function $\psi(\zeta_1^*,\zeta_2^*,q_1,q_2)$ and the operators K_1 and K_2 act as ifferential operators on this function. In particular the coherent state vector $|\zeta,q>$ in this four mode hilbert space can be written as:

$$|\zeta,q,q_1,q_2>=\int <<\zeta_1,q_1<<\zeta_2,q_2|\zeta,q>|\zeta_1q_1>>|\zeta_2,q_2>> d^2\sigma(\zeta_1)d^2\sigma(\zeta_2). \tag{17}$$

This becomes the equivalent of the Clebsch Gordon equation in the pair coherent state basis and the quantity The overlap function $<<\zeta_1,q_1|\zeta_2q_2|\zeta,q>=f(\zeta_1^*,\zeta_2^*,\zeta q_1,q_2)$ is the equivalent of the Clebsch Godon coefiient for the SU(1,1) COHERENT STATE BASIS. The action of the generators of SU(1,1) X SU(1,1) on f is given by

$$
\begin{aligned}
(K_1^+ + K_2^+)f &= (\zeta_1^* + \zeta_2^*)f \\
(K_1^- + K_2^-)f &= [(\frac{\partial}{\partial\zeta_1^*}(q_1 + \zeta_1^*\frac{\partial}{\partial\zeta_1^*} + (\frac{\partial}{\partial\zeta_2^*}(q_2 + \zeta_2^*\frac{\partial}{\partial\zeta_2^*}]f
\end{aligned}
\tag{18}
$$

On the other hand

$$
K_f = \zeta f \quad ; Cf = [(1-q^2)/4]f \tag{19}
$$

Thus we get the following two differential equations for f:

$$
\left[\frac{\partial}{\partial\zeta_1^*}(q_1 + \zeta_1^*\frac{\partial}{\partial\zeta_1^*}) + \frac{\partial}{\partial\zeta_2^*}(q_2 + \zeta_2^*\frac{\partial}{\partial\zeta_2^*})\right] f = \zeta f\ ,
$$

and

$$
\begin{aligned}
&\left[\zeta_1^*\zeta_2^*(\frac{\partial^2}{\partial\zeta_1^{*2}} - 2\frac{\partial}{\partial\zeta_1^*}\frac{\partial}{\partial\zeta_2^*} + \frac{\partial^2}{\partial\zeta_2^2})\right] f \\
+&\left[(q_1+1)\zeta_2^*(\frac{\partial}{\partial\zeta_1^*} - \frac{\partial}{\partial\zeta_2^*}) - (q_2+1)\zeta_1^*(\frac{\partial}{\partial\zeta_1^*} - \frac{\partial}{\partial\zeta_2^*})\right] f \\
&= -\left[\frac{q^2}{4} - \frac{(q_1+q_2+1)^2}{4}\right] f
\end{aligned}
$$

Solving these two equations we get [5]:

$$
\begin{aligned}
f &= <<\zeta_1, q_1, \zeta_2, q_2|\zeta, q>> \\
&= N(\zeta(\zeta_1^* + \zeta_2^*))^{-q/2} I_q(\sqrt{4\zeta(\zeta_1^* + \zeta_2^*})(\zeta_1^* + \zeta_2^*)^n P_n^{q_2,q_1}(\frac{\zeta_1^* - \zeta_2^*}{\zeta_1^* + \zeta_2^*})
\end{aligned}
\tag{20}
$$

N is the normalisation . Thus the state $|\zeta, q>$ can be obtained from the relation:

$$
|\zeta, q> = \frac{4N}{\pi^2}\int d^2\zeta_1 \int d^2\zeta_2 K_{q_1}(2|\zeta_1|)K_{q_2}(2|\zeta_2|) <<\zeta_1, q_1, \zeta_2, q_2|\zeta, q>> |\zeta_1, q_1>> |\zeta_2, q_2>> \tag{21}
$$

This is the Clebsch Gordon form for the product basis of Coherent states of SU(1,1) X SU(1,1).

It is interesting to note that by substituting the values of $|\zeta_1, q_1>>$ and $|\zeta_2, q_2>>$ given in equations (14) and using the expansion for the Jacobi Polynomial as well as the expansion of the Bessel function I_q and carrying out the various integrations we have:

$$
\begin{aligned}
|\zeta, q> &= N'\sum_{k=0}^{\infty}\frac{\zeta^k}{(k!(k+q)!)^{\frac{1}{2}}}\sum_{n_1,n_2}\delta_{(n_1+n_2,n+k)} \\
&\left[\frac{n_1!n_2!,(n_1+q_1)!(n_2+q_2)!k!}{(k+q)!}\right]^{\frac{1}{2}}((n+q_1)!(n+q_2)!) \\
&\sum_l(-1)^l\frac{1}{l!(q_2+l)!(n-l)!(n_2-l)!(n_1-n-l)!(n+q_1-l)!}|n_1+q_1, n_1>|n_2+q_2, n_2>
\end{aligned}
\tag{22}
$$

By comparison with expression [13] in the previous section we have:

$$
\begin{aligned}
C^{q_1,q_2,n}_{n_1,n_2,n+k} &= \left[\frac{n_1!n_2!,(n_1+q_1)!(n_2+q_2)!k!}{(k+2n+q_1+q_2+1)!}\right]^{\frac{1}{2}}((n+q_1)!(n+q_2)!)^{1/2} \\
&\sum_m (-1)^m \frac{1}{(q_2+m)!(n-m)!(n_2-m)!(n_1-n-m)!(n+q_1-m)!} \quad . \qquad (23)
\end{aligned}
$$

Which is the Clebsch Gordon coefficient for the canonical number state basis for SU(1,1)XSU(1,1)

2 SubPoissonian Properties of SU(1,1)XSU(1,1) coherent states

To give an idea of the Sub-Poissonian nature of these states let us consider a special case which is useful in physical applications. Consider the case $q_1 = q_2 = 0$; q=1 ; $\zeta \neq 0$
In this special case , we start with equal number of photons in the modes a and b and in c and d. Then

$$
|\zeta,1,0,0>= N_1 \sum_k \frac{\zeta^k}{[(k+1)!(k)!]^{1/2}} \sum_{n_1,n_2} \frac{1}{(k+1)^{1/2}} \delta_{n_1+n_2,k}|n_1,n_1>|n_2,n_2> , \qquad (24)
$$

where

$$
N_1 = \frac{(|\zeta|)^{1/2}}{[I_1(2|\zeta|)]^{1/2}} \quad . \qquad (25)
$$

The single mode probability distribution P_{n_1} and the mean number of photons $<n_1>$ are given by

$$
P_{n_1}(\zeta) = N_1^2|\zeta|^{2n_1} \sum_{n_2} \frac{|\zeta|^{2n_2}}{(n_2+n_1+1)!^2} \quad , \qquad (26)
$$

and

$$
<n_1>= \frac{|\zeta|I_2(2|\zeta|)}{2I_1(2|\zeta|)} \qquad (27)
$$

A measure of the non-classical nature of the distribution is given by Mandel's Q parameter , which for the mode a is given by

$$
\begin{aligned}
Q &= \frac{<n_1^2> - (<n_1>)^2 - <n_1>}{<n_1>} \qquad (28) \\
&= \frac{2|\zeta|I_3(2|\zeta|)}{3I_2(2|\zeta|)} - \frac{|\zeta|I_2(2|\zeta|)}{2I_1(2|\zeta|)} . \qquad (29)
\end{aligned}
$$

In fig. 1 we plot Q .vs. $|\zeta|$. For values of $|\zeta| < 2$, Q is negative showing the departure from the Poisonnian. The joint probability distribution $P_{n_1+n_2}$ can be calculated from P_{n_1,n_2} by the relation:

$$
P_k = \sum_{n_1,n_2} \delta_{n_1+n_2,k} P_{n_1,n_2} = \frac{N_1^2|\zeta|^{2k}}{k!(k+1)!} \qquad (30)
$$

The average value $< k >$ is given by:

$$\sum_k kP_k = \frac{|\zeta| I_2(2|\zeta|)}{I_1(2|\zeta|)} \tag{31}$$

In figure 2 we plot $P_k.vs.k$ and compare it to the corresponding Poissonian with mean value $< k >$ and it is clear that the distribution is sub Poissonian.

3 Physical Applications

SU(1,1)XSU(1,1) states are useful states in dealing with physical systems involving four modes of the radiation fields. The physical problem could be the passage of two-beams of light each having two polarisation modes passing through a medium in which there is a competition between the non-linear gain due to an external pumping field and the non-linear absorption[7] [8],[9]. The states generated are precisely the states considered in this paper. Let each beam contain both left and right circularly polarised photons. Let a,b, a ,b denote the creation and annilation operators for RIGHT circularly polarised photons from beam 1 and beam 2 and $c, d, c^\dagger, d^\dagger$denote the creation and annihilation operators for LEFT circularly polarised photons in beam 1 and beam 2. The master equation describing the dynamic behaviour of the fields resulting from the competition between two photon absorption and four wave mixing can be shown to be:

$$d\rho/dt = -K/2(O^\dagger O\rho - 2O\rho O^\dagger + \rho O^\dagger O) - i\left[G(O^\dagger + O), \rho\right] [3] \tag{32}$$

Where G denotes the four wave mixing susceptibility. Where K is related to the cross-section for two photon absorption and O = ab+cd. Defining an operator C=O+2iG/K We have:

$$d\rho/dt = -K/2(C^\dagger C\rho + \rho C^\dagger CC - 2C\rho C^\dagger) \tag{33}$$

Whose steady state solution: $C\rho = 0$ with $\rho = |\psi >< \psi|$ so that: $C|\psi >= 0$ implying that $O|\psi >= -2iG/K|\psi >$ or $(ab + cd)|\psi >= \lambda|\psi >$ Where $\lambda = -2iG/K$ Thus the steady state solutions of the master equations are eigenstates of the operator O. Furthermore, if we now impose the condition that the initial state is one in which the difference in the in the the number of photons in the two polarisation modes of each beam is a constant, with q being the constant for the right circularly polarised photons and q being the constant for the left circularly polarised photon in beam 1 and beam 2, the states $|\psi >$ are just the SU(1,1) X SU(1,1) coherent states.

Another examples of processes where four modes of the radiation field are important involve phase conjugate resonators and the process of down conversion in the field of a standing pump wave[6] .In the latter case, the forward wave will produce the modes a and b and the backward pump will give the modes c and d. The Hamiltonian for such interactions will have the form

$$H = (\epsilon_f^* ab + \epsilon_b^* cd + c.c), \tag{34}$$

where ϵ_f and ϵ_b are the forward and backward fields. Again the relevant coherent states are the eigenstates of the operator

$$K^- = (ab + cd) = K_1^- + K_2^-. \tag{35}$$

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