# OPERATORIAL APPROACH TO GENERALIZED COHERENT STATES

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**Abstract**

Generalized coherent states for general potentials, constructed through a controlling mechanism, can also be obtained applying on a reference state suitable operators. An explicit example is supplied.

## 1 Introduction

After the seminal works of Glauber, Klauder and Sudarshan [1], relevant generalizations and extensions of coherent states have been introduced, both in a group-theoretical framework [2], and in the direction of squeezing phenomena [3].

Moreover, a celebrated approach for general potentials was given in the case of classically integrable systems by Nieto and collaborators [4].

We have tackled the problem of building generalized coherent states from a point of view which can be useful also in a wider context [5]. In fact, if an interesting physical behaviour has been singled out for a quantum system, one can search for a controlling device which allows for its realization; we call this approach Controlled Quantum Mechanics (CQM) [6].

To this aim we use the methods of stochastic mechanics [7] [8]; however, we will show that these states can also be obtained in the standard operatorial approach.

## 2 The coherence constraint

We proceed now to apply the scheme of CQM to the problem of generalized coherent states. We will use in the following the notations $E(.)$ and $< . >$, respectively, for the stochastic mechanics and quantum mechanics mean values, sending back to [6] for the code of correspondence.

We search for states which are constrained to follow classical-like dynamics:

$$\frac{d}{dt} < \hat{p} >= F(< \hat{q} >, \Delta\hat{q}, t) \tag{1}$$

$$\frac{d^2}{dt^2}\Delta\hat{q} = G(<\hat{q}>, \Delta\hat{q}, t),$$

where $F$, $G$ are known functions and $\Delta\hat{q}$ is the dispersion. A particular, but interesting, case is associated to constant dispersion.

If $\Phi$ denotes the potential associated to our states, the Ehrenfest equation

$$\frac{d}{dt} <\hat{p}> = - <\nabla\Phi> \tag{2}$$

always holds. Then we can write the above classical-like constraint in the more transparent form

$$<\nabla\Phi> = \nabla\Phi(x,t)|_{x=<\hat{q}>} + \delta F(<\hat{q}>, \Delta\hat{q}, t)$$

$$\tag{3}$$

$$\frac{d^2}{dt^2}\Delta\hat{q} = G(<\hat{q}>, \Delta\hat{q}, t),$$

where $\delta F$ is a known function. We note that for harmonic potentials this condition is always satisfied in the strict sense, that is with $\delta F = 0$.

However, for non-harmonic potentials the above condition becomes a true constraint. How this constraint can be imposed?

In the framework of stochastic mechanics the mean deterministic motion of the quantum process is ruled by the current velocity $v(x,t)$, while quantum fluctuations are associated to the osmotic velocity $u(x,t)$.

A given choice of $v(x,t)$ singles out a whole class of quantum states, all sharing a common mean motion. In our case, then, the constraint must be imposed through a suitable choice of the current velocity.

The natural choice is given by the following form of $v(x,t)$

$$v(x,t) = \frac{d}{dt}E(q) + \frac{(x - E(q))}{\Delta q}\frac{d}{dt}\Delta q \tag{4}$$

which is associated to the standard harmonic oscillator coherent and squeezed states [9]. We can expect, in fact, that these states are a sub-set of the whole class of states selected by the form (4), and that all these states exhibit mean classical-like motion in the sense of Eq.s (3).

Note that, in conventional quantum-mechanical formalism, choice (4) corresponds, through $mv = \nabla S$, to the well known quantum mechanical coherent phase

$$S = <\hat{p}> x + \frac{(<\{\hat{q},\hat{p}\}>/2) - <\hat{q}><\hat{p}>}{2(\Delta q)^2}(x - <\hat{q}>)^2 + S_0(t). \tag{5}$$

In order to find explicitely the form of the searched states, we must take into account the constitutive couple of equations

$$\partial_t \rho = -\nabla(\rho v),$$

$$\tag{6}$$

$$\partial_t S + \frac{m}{2}v^2 - \frac{m}{2}u^2 - \frac{\hbar}{2}\nabla u = -\Phi,$$

that is the continuity equation and the Hamilton-Jacobi-Madelung (HJM) equation respectively.

Inserting expression (4) in the first of the Eq.s (6), the selected states result to be all the states with a (normalizable) probability density of the ("wave-like") form

$$\rho(x,t) = \frac{1}{\Delta q} exp\{2R(\xi)\} \quad , \quad \xi = \frac{x - E(q)}{\Delta q}, \tag{7}$$

with the corresponding form for the associated osmotic velocity

$$u \equiv (\frac{\hbar}{2m})\frac{\nabla \rho}{\rho} = \frac{1}{\Delta q} G(\xi). \tag{8}$$

¿From the expressions (5), (7) for $S$ and $\rho$ we obtain the wave functions of the generalized states

$$\Psi(x,t) = \frac{1}{\sqrt{\Delta q}} exp\{R(\xi)\} exp\{\frac{i}{\hbar} S\}. \tag{9}$$

Now, inserting Eq.s (4), (8) in the HJM equation (6), taking the gradient term by term, and computing the resulting identity in $x =< \hat{q} >$ (or in $x = 0$ if the potential is singular), we can simply verify that the classical-like constraint is fulfilled. Then, our aim is reached.

Finally, the HJM equation, with the inputs of Eq.s (4), (8), gives as output the controlling potential $\Phi$.

It is immediately seen, however, that $\Phi$ must be in general a function $\Phi(x,t| < \hat{q} >, \Delta \hat{q})$ also of $< \hat{q} >$ and $\Delta \hat{q}$; namely, in order to control the coherence of the wave packet, it is needed a feed-back mechanism, which allows for readjusting the system at any time.

Let us now look with greater detail at the problem of spreading. Two choices are possible, that is constant or time-dependent dispersion.

a) *Constant dispersion*

If we require $\Delta \hat{q} = const.$, the general relation

$$\Delta q \frac{d}{dt} \Delta q = m\{E(qv) - E(q)E(v)\} \tag{10}$$

forces the current velocity to assume the "classical" value $v = dE(q)/dt \equiv E(v)$, which is exactly expression (4) when $d\Delta q/dt = 0$. Then our states in this case are the unique solution of the problem. Note that the right member of the last equation is connected to the quantum average of the position-momentum anticommutator [9].

b) *Squeezing*

If a time dependence is allowed for $\Delta \hat{q}$, one can ask the following question: are states (9) the natural generalization of the harmonic oscillator squeezed states? The answer is positive, due the following considerations.

First of all, a "stochastic squeezing condition" $\Delta q \Delta u = K\hbar/2m$ is satisfied, where $K^2 = (4m^2/\hbar^2)E(G^2(\xi))$.

Moreover, if we consider the whole quantum uncertainty product for our states, it is immediately proved [10], using Eq.s (4), (8), (10), that

$$(\Delta \hat{q})^2 (\Delta \hat{p})^2 \equiv m^2 (\Delta q)^2 \{(\Delta u)^2 + (\Delta v)^2\} = K^2 \frac{\hbar^2}{4} + \frac{m}{4} (\Delta \hat{q})^2 (\frac{d}{dt} \Delta \hat{q})^2. \tag{11}$$

We see, then, that the uncertainty structure in this case has the same form as in the harmonic oscillator squeezing states, with the only difference of a rescaled Heisenberg part.

Finally, the dispersion satisfy the equation [10]

$$\frac{d^2}{dt^2}\Delta\hat{q} = \frac{K^2\hbar^2}{4m^2\Delta\hat{q}^3} - < \frac{\hat{q} - <\hat{q}>}{\Delta\hat{q}}\nabla\Phi >, \tag{12}$$

which is the natural generalization of that of the harmonic case [9].

Eq.s (11), (12) assure controlled squeezing.

# 3 Displacement and squeezing operators

One can now asks two questions:
-can we construct states (9) directly in the standard quantum mechanical formalism?
-how we can choice in the whole class of states (9) the physically interesting states?

We can answer both questions in the following way [10].

Consider a reference stationary state $\Psi_0$, for example the ground state of a physically relevant potential $V$.

Consider moreover the standard displacement and squeezing operators

$$\hat{D}_\alpha = exp\{\alpha a^\dagger - \alpha^\star a\},$$
$$\tag{13}$$
$$\hat{S}_{\Delta\hat{q}} = exp\{\frac{\zeta}{2}(a^2 - a^{\dagger 2})\},$$

which are used to construct the harmonic oscillator coherent and squeezed states, and write them in terms of the position and momentum operators

$$\hat{D}_\alpha = exp\{\frac{i}{\hbar}S_0(t)\}exp\{\frac{i}{\hbar}P\hat{q}\}exp\{-\frac{i}{\hbar}Q\hat{p}\},$$
$$\tag{14}$$
$$\hat{S}_{\Delta\hat{q}} = exp\{i[\frac{f(t)}{\hbar}\{\hat{q},\hat{p}\} + \frac{g(t)}{\Delta\hat{q}_0{}^2}\hat{q}^2]\},$$

with

$$Q = <\hat{q}> - <\hat{q}>_0 \quad , \quad P = <\hat{p}>,$$
$$\tag{15}$$
$$f(t) = -\frac{1}{2}ln\frac{\Delta\hat{q}}{\Delta\hat{q}_0} \quad , \quad g(t) = \frac{m}{\hbar}[1 - 2f(t)]^{-1}\frac{\frac{d}{dt}\Delta\hat{q}}{\Delta\hat{q}}.$$

Then it is simple to verify that the states

$$\Psi_G(x,t) = (\hat{D}_\alpha\Psi_0)(x,t) \tag{16}$$

and

$$(\hat{D}_\alpha \hat{S}_{\Delta\hat{q}} \Psi_0)(x,t) \tag{17}$$

belong to our class (9) respectively in the case of constant dispersion and in the case of squeezing.

This answers not only the first question, obviously, but also the second one. In fact, we now have the following scheme, which will be clarified by the subsequent example.

Given a physical system described by a potential $V$, we can choice as reference state, for example, its ground state $\Psi_0$.

Applying on it operators (14), we obtain generalized coherent packets, whose centers follow the classical dynamics ruled by the potential $V$.

Inserting then the current and osmotic velocity associated to these states in the HJM equation (6), we obtain the controlling potential $\Phi$ which allows, through the feed-back mechanism, to retain states (9).

$V$ and $\Phi$ must not be confused: the first ($V$), in fact, is the original potential, for example a molecular one, for which we want to construct generalized coherent states, while the second ($\Phi$) simply describes the controlling device, that is it supplies the feed-back prescriptions needed to retain coherence.

# 4 Example

We develope now an explicit example.

Putting for simplicity $\hbar = m = 1$ in the following, let us consider the potential

$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{1}{x^2} \tag{18}$$

and choose as reference state its ground state

$$\Psi_0(x) = N_0^{\frac{1}{2}} x^2 exp\{-\frac{1}{2}\omega x^2\} \tag{19}$$

where $N_0$ is a normalization constant.

Applying on (19) the operators (14), we obtain the generalized coherent states

$$\Psi^{(g)}(x,t) \equiv (\hat{D}_\alpha \hat{S}_{\Delta\hat{q}} \Psi_0)(x,t) = \frac{1}{\sqrt{\Delta q}} exp\{R(\xi)\} exp\{iS\} \tag{20}$$

where $S$ is the phase (5) and

$$R(\xi) = -a\xi^2 + 2\,ln(a\xi^2) + ln\, b, \tag{21}$$

with a, b suitable functions of $\omega$.

Inserting in the HJM equation (6) expression (4), and the osmotic velocity associated to (20) through Eq. (8), we obtain for the center the classical equation

$$\frac{d}{dt} < \hat{p} >= -\omega^2(t) < \hat{q} > + \frac{\gamma(t)}{< \hat{q} >^3}, \tag{22}$$

and for the controlling potential the form

$$\Phi(x,t| < \hat{q} >, \Delta\hat{q}) = \frac{1}{2}\omega^2(t)x^2 + h(t)x + \frac{1}{2}\frac{\gamma(t)}{(x- < \hat{q} >)^2} + g(t), \tag{23}$$

where

$$\omega^2(t) = 2[a^2 - \Delta\hat{q}^{-1}\frac{d^2}{dt^2}\Delta\hat{q}] \ , \ \gamma(t) = 6(\Delta\hat{q})^4. \tag{24}$$

We see from Eq. (22) that the center follows just the classical motion associated to the potential $V$, Eq. (18), as previously claimed.

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