TWO DIFFERENT SQUEEZE TRANSFORMATIONS

D. Han
National Aeronautics and Space Administration, Goddard Space Flight Center, Code 910.1, Greenbelt, Maryland 20771, U.S.A.

Y. S. Kim
Department of Physics, University of Maryland, College Park, Maryland 20742, U.S.A.

Abstract

Lorentz boosts are squeeze transformations. While these transformations are similar to those in squeezed states of light, they are fundamentally different from both physical and mathematical points of view. The difference is illustrated in terms of two coupled harmonic oscillators, and in terms of the covariant harmonic oscillator formalism.

The word "squeezed state" is relatively new and was developed in quantum optics, and was invented to describe a set of two photon coherent states [1]. However, the geometrical concept of squeeze or squeeze transformations has been with us for many years. As far as the present authors can see, the earliest paper on squeeze transformations was published by Dirac in 1949 [2], in which he showed that Lorentz boosts are squeeze transformations. In this report, we show that Dirac's Lorentz squeeze is different from the squeeze transformations in the squeezed state of light. The question then is how different they are. In order to answer this question, we shall use a system of two coupled harmonic oscillators.

Let us look at a phase-space description of one simple harmonic oscillator. Its orbit in phase space is an ellipse. This ellipse can be canonically transformed into a circle. The ellipse can also be rotated in phase space by canonical transformation. This combined operation is dictated by a three-parameter group $Sp(2)$ or the two- dimensional symplectic group. The group $Sp(2)$ is locally isomorphic to $SU(1,1), O(2,1)$, and $SL(2,r)$, and is applicable to many branches of physics. Its most recent application was to single- mode squeezed states of light [1, 3].

Let us next consider a system of two coupled oscillators. For this system, our prejudice is that the system can be decoupled by a coordinate rotation. This is not true, and the diagonalization requires a squeeze transformation in addition to the rotation applicable to two coordinate variables [3, 4]. This is also a transformation of the symplectic group $Sp(2)$.

If we combine the $Sp(2)$ symmetry of mode coupling and the $Sp(2)$ symmetry in phase space, the resulting symmetry is that of the (3 + 2)-dimensional Lorentz group [5]. Indeed, it has been shown that this is the symmetry of two-mode squeezed states [6, 7]. It is known that the (3 + 2)-dimensional Lorentz group is locally isomorphic to Sp(4) which is the group of linear canonical transformations in the four-dimensional phase space for two coupled oscillators. These canonical transformations can be translated into unitary transformations in quantum mechanics [7].

In addition, for the two-mode problem, there is another Sp(2) transformation resulting from the relative size of the two phase spaces. In classical mechanics, there are no restrictions on the area of phase space within the elliptic orbit in phase space of a single harmonic oscillator. In quantum mechanics, however, the minimum phase-space size is dictated by the uncertainty relation. For this reason, we have to adjust the size of phase space before making a transition to quantum mechanics. This adds another $Sp(2)$ symmetry to the coupled oscillator system [8]. However, the transformations of this $Sp(2)$ group are not necessarily canonical, and there does not appear to be a straightforward way to translate this symmetry group into the present formulation of quantum mechanics. We shall return to this problem later in this report.

If we combine this additional $Sp(2)$ group with the above- mentioned $O(3,2)$, the total symmetry of the two-oscillator system becomes that of the group $O(3,3)$, which is the Lorentz group with three spatial and three time coordinates. This was a rather unexpected result and its mathematical details have been published recently by the present authors [8]. This $O(3,3)$ group has fifteen parameters and is isomorphic to $SL(4,r)$. It has six $Sp(4)$-like subgroups and many $Sp(2)$ like subgroups.

Let us consider a system of two coupled harmonic oscillators. The Lagrangian for this system is

$$L = \frac{1}{2}\left\{m_1\dot{x}_1^2 + m_2\dot{x}_2^2 - A'x_1^2 + B'x_2^2 + C'x_1x_2\right\}, \tag{1}$$

with

$$A' > 0, \qquad B' > 0, \qquad 4A'B' - C'^2 > 0. \tag{2}$$

Then the traditional wisdom from textbooks on classical mechanics is to diagonalize the system by solving the eigenvalue equation

$$\begin{vmatrix} A' - m_1\omega^2 & C' \\ C' & B' - m_2\omega^2 \end{vmatrix} = 0. \tag{3}$$

There are two solutions for ω^2, and these solutions indeed give correct frequencies for the two normal modes. Unfortunately, this computation does not lead to a complete solution to the diagonalization problem. The above eigenvalue equation seems similar to that for the rotation, but it is not.

Let us go back to Eq.(1). This quadratic form cannot be diagonalized by rotation alone. Indeed, the potential energy portion of the Lagrangian can be diagonalized by one rotation, but this rotation will lead to a non-diagonal form for the kinetic energy. For this reason, we first have to replace x_1 and x_2 by y_1 and y_2 with the transformation matrix

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (m_2/m_1)^{1/4} & 0 \\ 0 & (m_1/m_2)^{1/4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4}$$

In terms of these new variables, the Lagrangian can be written as

$$L = \frac{\sqrt{m_1m_2}}{2}\left\{\dot{y}_1^2 + \dot{y}_2^2\right\} - \frac{1}{2}\left\{Ay_1^2 + By_2^2 + Cy_1y_2\right\}, \tag{5}$$

with

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \sqrt{m_2/m_1} & 0 & 0 \\ 0 & \sqrt{m_1/m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}.$$

The Lagrangian of Eq.(5) can now be diagonalized by a simple coordinate rotation:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{6}$$

with

$$\tan(2\alpha) = \frac{C}{A-B}. \tag{7}$$

In this Lagrangian formalism, momenta are not independent variables. They are strictly proportional to their respective coordinate variables. When the coordinates are rotated by the matrix of Eq.(6), the momentum variables are transformed according to the same matrix. When the coordinates undergo the scale transformation of Eq.(4), the momentum variables are transformed by the same matrix. Thus, the phase-space volume is not preserved for each coordinate.

Let us approach the same problem using the Hamiltonian

$$H = \frac{1}{2}\left\{\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + A'x_1^2 + B'x_2^2 + C'x_1x_2\right\}. \tag{8}$$

Here again, we have to rescale the coordinate variables. In this formalism, the central issue is the canonical transformation, and the phase-space volume should be preserved for each mode. If the coordinate variables are to be transformed according to Eq.(4), the transformation matrix for the momenta should be the inverse of the matrix given in Eq.(4). Indeed, if we adopt this transformation matrix, the new Hamiltonian becomes

$$H = \frac{1}{2\sqrt{m_1m_2}}\left\{p_1^2 + p_2^2\right\} + \frac{1}{2}\left\{Ax_1^2 + Bx_2^2 + Cx_1x_2\right\}. \tag{9}$$

As for the rotation, the rules of canonical transformations dictate that both the coordinate and momentum variables have the same rotation matrix. The above Hamiltonian can be diagonalized by the rotation matrix given in Eq.(6).

We can now consider the four-dimensional phase space consisting of variables in the following order.

$$(\chi_1, \chi_2, \chi_3, \chi_4) = (x_1, x_2, p_1, p_2). \tag{10}$$

For both the non-canonical Lagrangian system and the canonical Hamiltonian system, the mode-coupling rotation matrix is

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{11}$$

On the other hand, they have different matrices for the scale transformation. For the canonical Hamiltonian system, the matrix takes the form

$$S_-(\eta) = \begin{pmatrix} e^{\eta} & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & e^{-\eta} & 0 \\ 0 & 0 & 0 & e^{\eta} \end{pmatrix}. \tag{12}$$

Here, the position and momentum variables undergo anti-parallel squeeze transformations. On the other hand, for non-canonical Lagrangian system, the squeeze matrix is written as

$$S_{+}(\eta) = \begin{pmatrix} e^{\eta} & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & e^{\eta} & 0 \\ 0 & 0 & 0 & e^{-\eta} \end{pmatrix}. \tag{13}$$

We use the notation S_{+} and S_{-} for the parallel and anti- parallel squeeze transformation respectively.

If we rotate the above squeeze matrices by 45^{o} using the rotation matrix of Eq.(11), the anti-parallel squeeze matrix become

$$S_{-}(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & \cosh\eta & -\sinh\eta \\ 0 & 0 & -\sinh\eta & \cosh\eta \end{pmatrix}, \tag{14}$$

and the parallel squeeze matrix takes the form

$$S_{+}(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & \cosh\eta & \sinh\eta \\ 0 & 0 & \sinh\eta & \cosh\eta \end{pmatrix}. \tag{15}$$

Now the difference between these two matrices is quite clear. The squeeze matrix of Eq.(14) is applicable to two-mode squeezed states of light [7, 9, 10].

As for the squeeze matrix of Eq.(15), let us consider the Lorentz transformation of a particle along the z direction:

$$z' = (\cosh\eta)z + (\sinh\eta)t, \qquad t' = (\sinh\eta)z + (\cosh\eta)t. \tag{16}$$

Then the momentum and energy are transformed according to

$$P' = (\cosh\eta)P + (\sinh\eta)E, \qquad E' = (\sinh\eta)P + (\cosh\eta)E. \tag{17}$$

If we regard z and t as the two coordinate variables, the four- component vector of Eq.(10) takes the form

$$(\chi_1, \chi_2, \chi_3, \chi_4) = (z, t, P, E). \tag{18}$$

Thus, the parallel squeeze matrix performs a Lorentz boost. According to classical mechanics of coupled harmonic oscillators, this transformation appears like a non-canonical transformation. Then, is the Lorentz boost a non-canonical transformation? The answer is NO.

We would like to show that the Lorentz boost is an uncertainty- preserving transformation using the covariant oscillator formalism which has been shown to be effective in explaining the basic hadronic features observed in high energy laboratories [11]. According to this model, the ground-state wave function for the hadron takes the form

$$\psi_0(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(z^2 + t^2\right)\right\}, \tag{19}$$

where the hadron is assumed to be a bound state of two quarks, and z and t are space and time separations between the quarks. If the system is boosted, the wave function becomes [11]

$$\psi_\eta(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2 + e^{2\eta}v^2\right)\right\}, \tag{20}$$

where

$$u = (z+t)/\sqrt{2}, \qquad v = (z-t)/\sqrt{2}.$$

The u and v variables are called the light-cone variables [2]. The wave function of Eq.(19) is distributed within a circular region in the uv plane, and thus in the zt plane. On the other hand, the wave function of Eq.(20) is distributed in an elliptic region. This ellipse is a "squeezed" circle with the same area as the circle. The question then is how the momentum-energy wave function is squeezed.

The momentum wave function is obtained from the Fourier transformation of the expression given in Eq.(20):

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{2\pi}\right) \int \psi_\eta(z,t) \exp\left\{-i(q_z z - q_0 t)\right\} dx dt. \tag{21}$$

If we use the variables:

$$q_u = (q_0 - q_z)/\sqrt{2}, \qquad q_v = (q_0 + q_z)/\sqrt{2}. \tag{22}$$

In terms of these variables, the above Fourier transform can be written as

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{2\pi}\right) \int \psi_\eta(z,t) \exp\left\{-i(q_u u + q_v v)\right\} du dv. \tag{23}$$

The resulting momentum-energy wave function is

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(e^{-2\eta}q_u^2 + e^{2\eta}q_v^2\right)\right\}. \tag{24}$$

Because we are using here the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical with that of the space-time wave function given in Eq.(20). The Lorentz-squeeze properties of these wave functions are also the same. This certainly is consistent with the parallel squeeze matrix given in Eq.(15), and the Lorentz boosts appears like a non-canonical transformation.

However, we still have to examine how conjugate pairs are chosen from the space-time and momentum-energy wave functions. Let us go back to Eq.(21) and Eq.(23). It is quite clear that the light-cone variable u and v are conjugate to q_u and q_v respectively. It is also clear that the distribution along the q_u axis shrinks as the u-axis distribution expands. The exact calculation leads to

$$< u^2 >< q_u^2 >= 1/4, \qquad < v^2 >< q_v^2 >= 1/4. \tag{25}$$

Planck's constant is indeed a Lorentz-invariant quantity, and the Lorentz boost is a canonical transformation.

Because of the Minkowskian metric we used in the Fourier transformation of Eq.(21), the non-canonical squeeze transformation of Eq.(15) becomes a canonical transformation for the Lorentz boost. Otherwise, it remains non-canonical. Then, does this non-canonical transformation play a role in physics? The answer is YES. The best known examples are thermally excited oscillator states [12] and coupled oscillator system where one of the oscillator is not observed [13, 14]. These systems serve as simple models for studying the role of entropy in quantum mechanics [15, 16].

These examples are for the cases where the phase space volume for each mode becomes larger than Planck's constant. In the classical mechanics of two coupled harmonic oscillators, the phase-space volume of each oscillator fluctuates. If one becomes larger, the other shrinks. In quantum mechanics, we do not have a theory of shrinking phase-space volumes. Without this, we cannot have a complete understanding of coupled oscillators in quantum mechanics.

References

[1] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[2] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[3] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[4] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980).

[5] P. A. M. Dirac, J. Math. Phys. **4**, 901 (1963).

[6] R. F. Bishop and A. Vourdas, Z. Phyzik B **71**, 527 (1988).

[7] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. A **41**, 6233 (1990).

[8] D. Han, Y. S. Kim, and M. E. Noz, J. Math. Phys. **36**, 3940 (1995).

[9] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); B. L. Schumaker and C. M. Caves, Phys. Rev. A **31**, 3093 (1985). See also Fan, Hong–Yi and J. Vander Linder, Phys. Rev. A **39**, 2987 (1989).

[10] B. Yurke, S. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[11] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[12] H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).

[13] B. Yurke and M. Potasek, Phys. Rev. A **36**, 3464 (1987).

[14] A. K. Ekert and P. L. Knight, Am. J. Phys. **57**, 692 (1989). For an earlier paper, see S. M. Barnett and P. L. Knight, J. Opt. Soc. Am. B **2** 467 (1985).

[15] J. von Neumann, *Die mathematische Grundlagen der Quanten- mechanik* (Springer, Berlin, 1932). See also J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton University, Princeton, 1955).

[16] R. P. Feynman, *Statistical Mechanics* (Benjamin/Cummings, Reading, MA, 1972).