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# HIGHER-ORDER SQUEEZING OF QUANTUM FIELD AND THE GENERALIZED UNCERTAINTY RELATIONS IN NON-DEGENERATE FOUR-WAVE MIXING

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**Abstract**

It is found that the field of the combined mode of the probe wave and the phase-conjugate wave in the process of non-degenerate four-wave mixing exhibits higher-order squeezing to all even orders. And the generalized uncertainty relations in this process are also presented.

With the development of techniques for making higher-order correlation measurement in quantum optics, the new concept of higher-order squeezing of the single-mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985[1,2]. Lately Xi-zeng Li and Ying Shan have calculated the higher-order squeezing in the process of degenerate four-wave mixing[3] and presented the higher-order uncertainty relations of the fields in single-mode squeezed states[4]. As a natural generalization of Hong and Mandel's work, we introduced the theory of higher-order squeezing of the quantum fields in two-mode squeezed states in 1993. In this paper we study for the first time the higher-order squeezing of the quantum field and the generalized uncertainty relations in non-degenerate four-wave mixing (NDFWM) by means of the above theory.

# 1 Definition of higher-order squeezing of two mode quantum fields

The real two mode output field $\hat{E}$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E} = \hat{E}_1 cos(\Omega t - \phi) + \hat{E}_2 sin(\Omega t - \phi), \tag{1}$$

$$[\hat{E}_1, \hat{E}_2] = 2iC_0. \tag{2}$$

Then the field is squeezed to the $N$th-order in $\hat{E}_1 (N = 1, 2, 3, \cdots)$ if there exists a phase angle $\phi$ such that $< (\Delta \hat{E}_1)^N >$ is smaller than its value in a completely two-mode coherent state of the field, viz.,

$$< (\Delta \hat{E}_1)^N > \quad < \quad < (\Delta \hat{E}_1)^N >_{two-mode \quad coh.s.} \tag{3}$$

This is the definition of higher-order squeezing of two mode quantum fields.

## 2 Scheme for generation of higher–order squeezing via NDFWM

The scheme is shown in the following figure:

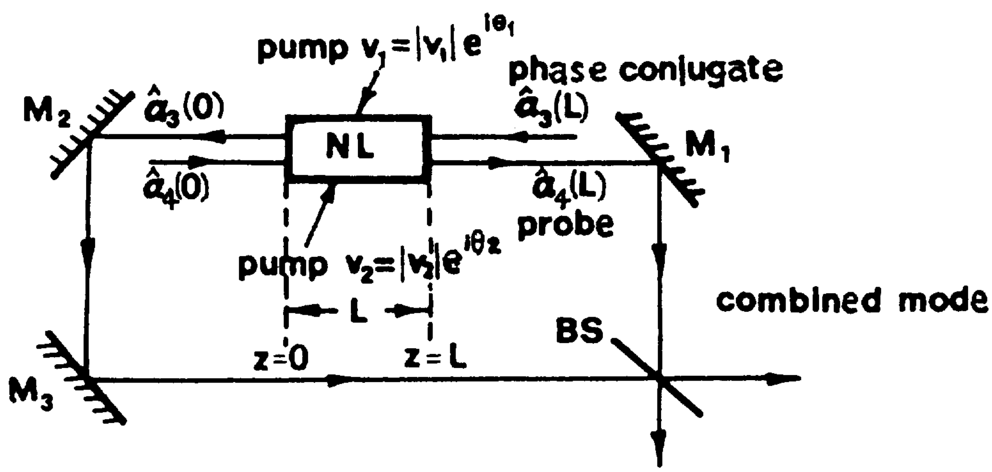


FIG. 1. Schematic for generation of higher–order squeezing via NDFWM. $M_1, M_2, M_3$ are mirrors, BS is the 50%–50% beam splitter

Where two strong, classical pump waves of complex amplitude ($v_1 = |v_1|e^{i\theta_1}$ and $v_2 = |v_2|e^{i\theta_2}$) with the same frequency $\Omega$ are incident on a nonlinear crystal possessing a third–order ($\chi^{(3)}$) nonlinearity. The length of the medium is L. $\hat{a}_4$ is the annihilation operator of the transmitted –probe wave with frequency $\omega_4$, $\hat{a}_3$ is the annihilation operator of the phase–conjugate wave with frequency $\omega_3$, and

$$\Omega = \frac{\omega_3 + \omega_4}{2} \tag{4}$$

The effective Hamiltonian of this interaction system has the form of

$$\hat{H} = \hbar\omega_3\hat{a}_3^+\hat{a}_3 + \hbar\omega_4\hat{a}_4^+\hat{a}_4 + \hbar g_0(v_1 v_2 \hat{a}_3^+\hat{a}_4^+ e^{-2i\Omega t} + H.C) \tag{5}$$

where $g_0$ is the coupling const, t is the time propagation of light in NL crystal.

By solving the Heisenberg Equation of motion we get the output mode

$$\hat{a}_3(t) = [\mu\hat{a}_3(L) + \nu\hat{a}_4^+(0)]e^{-i\omega_3 t}, \quad (z = L - ct \quad \text{for} \quad \hat{a}_3) \tag{6}$$

$$\hat{a}_4(t) = [\mu\hat{a}_4(0) + \nu\hat{a}_3^+(L)]e^{-i\omega_4 t}, \quad (z = ct \quad \text{for} \quad \hat{a}_4) \tag{7}$$

where

$$\left.\begin{array}{rcl} \mu &=& sec|k|L, \\ \nu &=& -ie^{i(\theta_1+\theta_2)}tan|k|L, \\ |k| &=& \frac{g_0|v_1||v_2|}{c}. \end{array}\right\} \tag{8}$$

# 3 Combined mode and its quadrature components

It can be verified that the field of either $\hat{a}_3(0)$ or $\hat{a}_4(L)$ mode does not exhibit higher−order squeezing.

We consider the field of the combined mode of $\hat{a}_3(t)$ and $\hat{a}_4(t)$

$$\begin{aligned} \hat{E}(t) &= \sqrt{\frac{\omega_3}{2}}\hat{a}_3(t) - i\sqrt{\frac{\omega_4}{2}}\hat{a}_4(t) + (H.C) \\ &= \sqrt{\frac{\Omega}{2}}\lambda_3\hat{a}_3(t) - i\sqrt{\frac{\Omega}{2}}\lambda_4\hat{a}_4(t) + (H.C) \end{aligned} \tag{9}$$

where

$$\lambda_3 = \sqrt{\frac{\omega_3}{\Omega}}, \lambda_4 = \sqrt{\frac{\omega_4}{\Omega}} \tag{10}$$

and $-i$ denotes the phase delay. The units are chosen so that $\hbar = c = 1$.

$\hat{E}(t)$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E}(t) = \hat{E}_1 cos(\Omega t - \phi) + \hat{E}_2 sin(\Omega t - \phi), \tag{11}$$

where

$$\Omega = \frac{\omega_3 + \omega_4}{2}, \tag{12}$$

and $\phi$ is an arbitrary phase angle that may be chosen at will.

$\hat{E}_1$ can be expressed in term of initial modes $\hat{a}_3(L)$ and $\hat{a}_4(0)$,

$$\hat{E}_1 = g\hat{a}_3(L) + h\hat{a}_4(0) + g^*\hat{a}_3^+(L) + h^*\hat{a}_4^+(0), \tag{13}$$

where

$$\begin{aligned} g &= \sqrt{\frac{\Omega}{2}}[\lambda_3\mu e^{-i\phi} + \lambda_4\nu^* e^{i(\phi+\pi/2)}]e^{i\epsilon t}, & (14) \\ h &= \sqrt{\frac{\Omega}{2}}[\lambda_4\mu e^{-i(\phi+\pi/2)} + \lambda_3\nu^* e^{i\phi}]e^{-i\epsilon t}, & (15) \\ \epsilon &= \Omega - \omega_3 = \omega_4 - \Omega, & (16) \end{aligned}$$

$\epsilon$ is the modulation frequency.

Now we define

$$\hat{B} = g\hat{a}_3(L) + h\hat{a}_4(0), \tag{17}$$

$$\hat{B}^+ = g^*\hat{a}_3^+(L) + h^*\hat{a}_4^+(0), \tag{18}$$

then

$$\hat{E}_1 = \hat{B} + \hat{B}^+, \tag{19}$$

where $\hat{B}^+$ is the adjoint of $\hat{B}$.

# 4 Higher−order noise moment $< (\Delta\hat{E}_1)^N >$ and higher −order squeezing

By using the Campbell−Baker−Hausdorff formula, we get the Nth−order moment of $\Delta\hat{E}_1$,

$$\begin{aligned} < (\Delta\hat{E}_1)^N > &= <:: (\Delta\hat{E}_1)^N ::> + \frac{N^{(2)}}{1!}(\frac{1}{2}C_0) <:: (\Delta\hat{E}_1)^{N-2} ::> + \frac{N^{(4)}}{2!}(\frac{1}{2}C_0)^2 \\ & <:: (\Delta\hat{E}_1)^{N-4} ::> \\ & + \cdots + (N-1)!!C_0^{N/2}. \qquad \text{(N is even)} \end{aligned} \tag{20}$$

where

$$N^{(r)} = N(N-1)\cdots(N-r+1), \quad C_0 = \frac{1}{2i}[\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^+], \tag{21}$$

and :: :: denotes normal ordering with respect to $\hat{B}$ and $\hat{B}^+$.

We take the initial quantum state to be $|\alpha>_4|0>_3$, which is a product of the coherent state $|\alpha>_4$ for $\hat{a}_4(0)$ mode and the vacuum state for $\hat{a}_3(L)$ mode. Since $|\alpha>_4|0>_3$ is the eigenstate of $\hat{B}$, we get

$$\begin{aligned} <:: (\Delta\hat{E}_1)^N ::> &= <:: (\Delta\hat{B} + \Delta\hat{B}^+)^N ::> \\ &= \sum_{\gamma=0}^{N} \begin{bmatrix} N \\ \gamma \end{bmatrix} \quad {}_3<0|_4<\alpha| :: (\Delta\hat{B}^+)^\gamma(\Delta\hat{B})^{N-\gamma} :: |\alpha>_4|0>_3 = 0. \end{aligned} \tag{22}$$

Then from (20),

$$< (\Delta\hat{E}_1)^N > = (N-1)!!C_0^{N/2}, \tag{23}$$

$$\begin{aligned} C_0 &= [\hat{B}, \hat{B}^+] = |g|^2 + |h|^2, \\ &= \frac{\Omega}{2}\{(\lambda_3^2 + \lambda_4^2)(|\mu|^2 + |\nu|^2) + 2\lambda_3\lambda_4[\mu^*\nu^* e^{i(2\phi+\frac{\pi}{2})} + \mu\nu e^{-i(2\phi+\frac{\pi}{2})}]\}. \end{aligned} \tag{24}$$

where

$$\lambda_3^2 + \lambda_4^2 = 2, \qquad \lambda_3\lambda_4 = \sqrt{1 - \frac{\epsilon^2}{\Omega^2}}.$$

Substituting eqs. (8), (10), (24) into (23), we get the Nth−order moment of $\Delta\hat{E}_1$,

$$\begin{aligned} < (\Delta\hat{E}_1)^N > &= (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L \\ &- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L cos(2\phi - \theta_1 - \theta_2)]^{N/2}. \end{aligned} \tag{25}$$

If $\phi$ is chosen to satisfy

$$2\phi - \theta_1 - \theta_2 = 0, \quad \text{or} \quad cos(2\phi - \theta_1 - \theta_2) = 1,$$

then the above eq. (25) leads to the result

$$\begin{aligned} < (\Delta\hat{E}_1)^N > &= (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L \\ &- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L]^{N/2}. \end{aligned} \tag{26}$$

When $0 < |k|L < \pi$, the right-hand side is less than $(N-1)!!\Omega^{N/2}$, which is the corresponding Nth-order moment for two-mode coherent states. It follows that the field of the combined mode of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders.

The squeeze parameter $q_N$ for measuring the degree of Nth-order squeezing is

$$q_N = \frac{< (\Delta \hat{E}_1)^N > - < (\Delta \hat{E}_1)^N >_{two-mode \ coh.s}}{< (\Delta \hat{E}_1)^N >_{two-mode \ coh.s}} \tag{27}$$

$$= [sec^2|k|L + tan^2|k|L - 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L]^{N/2} - 1. \tag{28}$$

We find that $q_N$ is negative, and $q_N$ increases with N. This gives out the conclusion that the degree of higher-order squeezing is greater than that of the second order.

# 5 Generalized uncertainty relations in NDFWM

$\hat{E}_2$ can be regarded as a special case of $\hat{E}_1$ if $\phi$ is replaced by $\phi + \pi/2$. Then if $\phi$ is chosen to satisfy $2\phi - \theta_1 - \theta_2 = 0$, from eq. (25) it follows that

$$< (\Delta \hat{E}_2)^N > = (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L + 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L]^{N/2}. \tag{29}$$

when $0 < |k|L < \pi$, the right-hand side is greater than $(N-1)!!\Omega^{N/2}$.

From eqs. (26) and (29), we obtain

$$< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N-1)!!]^2\Omega^N[1 + 4\frac{\epsilon^2}{\Omega^2} sec^2|k|L tan^2|k|L]^{N/2}. \tag{30}$$

Eq. (30) shows that $< (\Delta \hat{E}_1)^N >$ and $< (\Delta \hat{E}_2)^N >$ can not be made arbitrarily small simultaneously. We call eq. (30) the generalized uncertainty relations in NDFWM, and the right-hand side is dependent on $\epsilon, \Omega, N$, and $|k|L$.

In the degenerate case $\omega_4 = \omega_2 = \Omega, \epsilon = 0$ from eqs. (26), (28) and (30) we obtain

$$< (\Delta \hat{E}_1)^N > = (N-1)!!\Omega^{N/2}[sec|k|L - tan|k|L]^N, \tag{31}$$

$$q_N = [sec|k|L - tan|k|L]^N - 1, \tag{32}$$

$$< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N-1)!!]^2 \cdot \Omega^N. \tag{33}$$

When $N = 2$,

$$< (\Delta \hat{E}_1)^2 > = \Omega[sec|k|L - tan|k|L]^2, \tag{34}$$

$$q_2 = [sec|k|L - tan|k|L]^2 - 1, \tag{35}$$

$$< (\Delta\hat{E}_1)^2 > \cdot < (\Delta\hat{E}_2)^2 > = \Omega^2 \quad (36)$$

These results are in agreement with the conclusions in the previous relevant references[3][5].

# 6 Acknowledgements

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