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# Study of Nonclassical Fields in Phase-Sensitive Reservoirs

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## Abstract

We show that the reservoir influence can be modeled by an infinite array of beam splitters. The superposition of the input fields in the beam splitter is discussed with the convolution laws for their quasiprobabilities. We derive the Fokker-Planck equation for the cavity field coupled with a phase-sensitive reservoir using the convolution law. We also analyse the amplification in the phase-sensitive reservoir with use of the modified beam splitter model. We show the similarities and differences between the dissipation and amplification models. We show that a super-Poissonian input field cannot become sub-Poissonian by the phase-sensitive amplification.

## 1 Introduction

A cavity with imperfect mirrors not only lets the cavity field out but also allows the field outside to leak into the cavity so that the reservoir surrounding the cavity gradually influences the cavity field. The mirrors in the cavity can be considered as beam splitters. An array consisting of an infinite number of beam splitters can model a reservoir coupled to the cavity field [1].

It is convenient to utilise quasiprobability distributions such as the  $Q$  function to describe states of quantum-mechanical systems in phase space. We briefly show that the fields at the output ports of the beam splitter can be expressed by the convolution of the quasiprobabilities of the input fields. We then use the convolution relation to derive the Fokker-Planck equation for attenuation of the cavity field coupled with the phase-sensitive reservoir.

It is known in quantum mechanics that an amplification process is inevitably accompanied by the increase of the quantum noise in the system. In other words the amplification degrades an optical signal and rapidly destroys quantum features that may have been associated with the signal. The nature of the amplifier affects the physical properties of the amplified states of light. The phase-sensitive amplifier is conceptually based on the establishment of squeezed light and enables a squeezed input to keep the property for a gain larger than the cloning limit [2].

The amplification process can also be modeled by an array of beam splitters, where the beam splitters are somewhat modified from the usual sense. We find the convolution relation for this modified beam splitters and the Fokker-Planck equation is derived from it. We then formally solve

the Fokker-Planck equation for the phase-sensitive amplifier for an arbitrary input field and study the photon statistics of the amplified field.

## 2 Phase-Sensitive Attenuation

Consider that two fields at the two input ports of a lossless beam splitter are superposed. For convenience we call one input field the signal and the other the noise (Fig. 1). The input signal mode  $b$  with its annihilation operator  $\hat{b}$  is superposed on the noise mode  $a$  with its annihilation operator  $\hat{a}$  by the beam splitter whose amplitude reflectivity is  $r = \sin \theta$  and transmittivity  $t = \cos \theta$ . The two output field annihilation operators  $\hat{c}$  and  $\hat{d}$  are related to the beam splitter input fields by the transformation using the beam splitter operator  $\hat{B}$  [3].

$$\begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = \hat{B} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \hat{B}^\dagger = \begin{pmatrix} t\hat{a} + r\hat{b} \\ t\hat{b} - r\hat{a} \end{pmatrix}; \quad \hat{B} = \exp \left[ \theta \left( \hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b} \right) \right]. \quad (1)$$

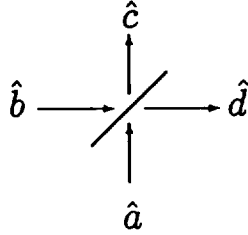


Fig. 1 Beam splitter with the signal input in mode  $b$  and the noise field input in mode  $a$ .

A field state can be represented in phase space by quasiprobabilities. Let us choose to use the positive  $P$ -representation to describe the fields. The density operators  $\hat{\rho}_a$  and  $\hat{\rho}_b$  for the noise and signal fields are then written with the positive  $P$ -representation  $P_{a,b}(\alpha, \gamma)$  as

$$\hat{\rho}_i = \int d^2\alpha d^2\gamma \frac{P_i(\alpha, \gamma)}{\langle \gamma^* | \alpha \rangle} |\alpha\rangle \langle \gamma^*|, \quad i = a, b \quad (2)$$

The fields  $\hat{\rho}_c$  and  $\hat{\rho}_d$  at the output ports are calculated by the beam splitter transformation:  $\hat{B}\hat{\rho}_a\hat{\rho}_b \rightarrow \hat{\rho}_c\hat{\rho}_d$ . With use of the transformation matrix Eq.(1) we find that the positive  $P$ -representation for the field at the port  $d$  is found in the form of the convolution relation:

$$P_d(\phi, \psi) = \frac{1}{t^4} \int d^2\alpha d^2\gamma P_a(\alpha, \gamma) P_b \left( \frac{\phi - r\alpha}{t}, \frac{\psi - r\gamma}{t} \right), \quad (3)$$

The  $Q$  function is another quasiprobability function and is well-defined even for the nonclassical state. The positive  $P$ -representation is defined in four-dimensional space. The  $Q$  function, which is defined in two-dimensional space, is therefore sometimes easier to treat, so we will extend the

convolution law for the uses of the  $Q$  function. The positive  $P$ -representation may be defined as the Fourier transform of the characteristic function. The characteristic function  $C^{(p)}(\xi)$  is related to the characteristic function  $C^{(q)}(\xi)$  for the  $Q$  function as  $C^{(p)}(\xi) = C^{(q)}(\xi) \exp(|\xi|^2)$ . By using the convolution theorem, we can factorise the inverse Fourier transform of the convolution law (3) as

$$C_d^{(p)}(\xi) = C_a^{(p)}(r\xi)C_b^{(p)}(t\xi). \quad (4)$$

Using the relation between the characteristic functions, the convolution relation for the  $Q$  function is found as

$$Q_d(\phi) = \frac{1}{t^2} \int d^2\alpha Q_a(\alpha)Q_b\left(\frac{\phi - r\alpha}{t}\right). \quad (5)$$

We now derive the Fokker-Planck equation for the phase-sensitive reservoir using the model of an infinite array of beam splitters (Fig.2) [4]. The total duration of time when the field is coupled with the lossy channel is denoted by  $T$ , the total number of the beam splitters by  $\aleph$ , and the interval between the adjacent beam splitters by  $\Delta\tau$ . The beam splitters are first taken to be discrete components, but their number,  $\aleph = T/\Delta\tau$  is later taken to infinity in order to model a continuous attenuating reservoir. Under the assumption that the reflectivity is very small for the beam splitter, Eq. (5) is written as

$$Q(\alpha) \approx (1 + R) \int d^2\alpha Q_a(\beta) Q_b\left(\frac{\alpha - r\beta}{t}\right). \quad (6)$$

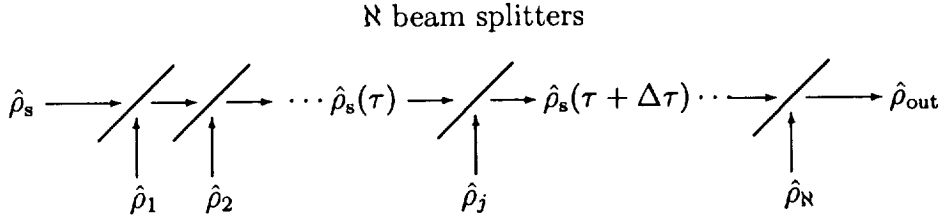


Fig.2 The phase-sensitive reservoir modeled as an array containing an infinite number of beam splitters. The signal is injected from left and the independent squeezed fields (all with the same properties) are injected into the other ports. The transmittivity is considered to be nearly unity.

To calculate the effects of attenuation, we need an expression for the output signal operator in terms of the input operators. To simulate an attenuator, we consider the beam splitters forming a continuous array by taking the limits  $\aleph \rightarrow \infty$ ,  $\Delta\tau \rightarrow 0$ , and  $R \rightarrow 0$ . These limits cannot be taken independently:  $\aleph\Delta\tau$  should be kept constant. Also, the total energy loss within  $T$  is described by  $1 - \exp(-\kappa T)$ , where  $\kappa$  is the attenuation coefficient, and this loss should be equivalent to the beam splitter loss so that  $R \approx \kappa\Delta\tau$ .

Let us define  $Q(\tau; \alpha)$  as the  $Q$  function of the signal field incident on the beam splitter at time  $\tau$ ,  $Q_{\text{sq}}(\beta)$  as the  $Q$  function for the noise added to the signal at the beam splitter, and  $Q(\tau + \Delta\tau; \alpha)$  as the  $Q$  function for the signal leaving from the beam splitter. The squeezed thermal fields produced by the independent stationary sources act as noise in our model.

From Eq. (6), we obtain the relation

$$Q(\tau + \Delta\tau; \alpha) = (1 + R) \int d^2\beta Q\left(\tau; \frac{\alpha - r\beta}{t}\right) Q_{\text{sq}}(\beta), \quad (7)$$

where  $\frac{\alpha - r\beta}{t} \approx \alpha + \frac{R}{2} \alpha - \sqrt{R} \beta$ . With the Taylor expansion for a function having a complex argument:

$$Q\left(\tau; \frac{\alpha - r\beta}{t}\right) = Q + \frac{\partial Q}{\partial \alpha_1} \frac{\alpha_1}{2} + R \left( \frac{\partial Q}{\partial \alpha_2} \frac{\alpha_2}{2} + \frac{1}{2} \frac{\partial^2 Q}{\partial \alpha_1^2} \beta_1^2 + \frac{1}{2} \frac{\partial^2 Q}{\partial \alpha_2^2} \beta_2^2 + \frac{\partial^2 Q}{\partial \alpha_1 \partial \alpha_2} \beta_1 \beta_2 \right), \quad (8)$$

where the real and imaginary parts of  $\alpha$  and  $\beta$  are respectively denoted by  $\alpha_1$ ,  $\alpha_2$  and  $\beta_1$ ,  $\beta_2$ . The function  $Q$  is the simplified notation of the function  $Q(\tau; \alpha)$ . Substituting Eq. (8) into Eq. (7) we obtain

$$\begin{aligned} \frac{dQ(\tau; \alpha)}{d\tau} &= \frac{\kappa}{2} \left[ \frac{\partial}{\partial \alpha_1} (\alpha_1 Q) + \frac{\partial}{\partial \alpha_2} (\alpha_2 Q) \right] \int d^2\beta Q_{\text{sq}}(\beta) \\ &\quad + \frac{\kappa}{2} \left[ \frac{\partial^2 Q}{\partial \alpha_1^2} \int d^2\beta \beta_1^2 Q_{\text{sq}}(\beta) + \frac{\partial^2 Q}{\partial \alpha_2^2} \int d^2\beta \beta_2^2 Q_{\text{sq}}(\beta) \right]. \end{aligned} \quad (9)$$

Taking the squeezed thermal state as noise we substitute the simple Gaussian integration of  $Q_{\text{sq}}(\beta)$  into Eq. (9) and obtain the Fokker-Planck equation for the field coupled to a phase-sensitive attenuation reservoir:

$$\frac{dQ(\tau, \alpha)}{d\tau} = \frac{\kappa}{2} \left[ \frac{\partial}{\partial \alpha_1} \alpha_1 + \frac{\partial}{\partial \alpha_2} \alpha_2 + \frac{1}{2} (1 + N + M) \frac{\partial^2}{\partial \alpha_1^2} + \frac{1}{2} (1 + N - M) \frac{\partial^2}{\partial \alpha_2^2} \right] Q(\tau, \alpha), \quad (10)$$

where  $N$  is the mean photon number for the squeezed thermal field and the phase-dependent term  $M$  is zero when the field is not squeezed [4]. The Fokker-Planck equation is relatively simple and observables can be calculated as correlations of the quasiprobability function.

### 3 Phase-Sensitive Amplification

The amplification process can also be modeled by an array of beam splitters similar to that for dissipation. In their experiment with phase-sensitive amplification, Ou et al. have a nondegenerate parametric amplifier where the signal field is amplified and the idler mode is coupled with the squeezed vacuum [5]. As in their experiment a two-mode parametric optical amplifier is modeled here by an amplification beam splitter matrix. For a two-mode parametric amplifier the signal input  $\hat{b}$  is transformed into the amplified output  $\hat{d}$  with unavoidable noise  $\hat{a}^\dagger$ :

$$\begin{pmatrix} \hat{d} \\ \hat{c}^\dagger \end{pmatrix} = \begin{pmatrix} \sqrt{g} & i\sqrt{g-1} \\ -i\sqrt{g-1} & \sqrt{g} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix}, \quad (11)$$

where  $g > 1$  is the infinitesimal amplification factor. We are going to build a beam-splitter-like relation for amplification, and consecutive application of an infinite number of Eq. (11) will give the final amplification result. The actual gain  $G$  by the amplifier will, thus, be proportional to  $g$ . The unitary amplification beam splitter operator has been introduced in analogy with the two-mode squeezing operator as  $B_1 = \exp[i\theta_1(\hat{a}^\dagger \hat{b}^\dagger + \hat{a}\hat{b})]$  with  $\cosh \theta_1 = \sqrt{g}$  and  $\sinh \theta_1 = \sqrt{g-1}$ .

To analyze the beam splitter transformation for the amplifier, let us assume that the input fields are expressed as a weighted sum of diagonal coherent components:

$$\hat{\rho}_{\text{in}} = \int d^2\alpha d^2\beta P_a(\alpha)P_b(\beta)|\alpha\rangle_{\text{aa}}\langle\alpha| \otimes |\beta\rangle_{\text{bb}}\langle\beta|, \quad (12)$$

where  $P_a$  and  $P_b$  are respectively the Glauber  $P$ -representations for modes  $a$  and  $b$ . Tracing the output field over mode  $c$ , we find the output density operator for mode  $d$ :

$$\hat{\rho}_d = \int d^2\alpha d^2\beta P_a(\alpha)P_b(\beta)D_d(\delta)\hat{\rho}_{\text{th}}D_d^\dagger(\delta); \quad \delta = i\sqrt{g-1}\alpha^* + \sqrt{g}\beta \quad (13)$$

and  $\hat{\rho}_{\text{th}}$  is the thermal field density operator for the mean photon number  $\bar{n} = g - 1$ . Even when both the signal and the idler fields are in the vacuum state, i. e.,  $\alpha = \beta = 0$ , the amplifier brings noise into the fields. The density operator for the thermal field can be written with its quasiprobability  $P_T(\phi)$  as

$$\hat{\rho}_{\text{th}} = \int d^2\phi P_T(\phi)|\phi\rangle\langle\phi|; \quad P_T(\phi) = \frac{1/\pi}{g-1} \exp\left(-\frac{|\phi|^2}{g-1}\right). \quad (14)$$

By using Eq. (14), we find the Glauber  $P$ -representation for the output field as convolution of the three  $P$ -representations

$$P_d(\zeta) = \frac{1}{g} \int d^2\phi d^2\alpha P_a(\alpha)P_b\left(\frac{\zeta - \phi - i\alpha^*\sqrt{g-1}}{\sqrt{g}}\right) P_T(\phi). \quad (15)$$

The inverse Fourier transform of the Glauber  $P$ -representation gives the characteristic function for the output field in the form of the product of the characteristic functions for the input modes  $a$  and  $b$  and the thermal field. We can simplify this relation using the relation between the characteristic functions for the various quasiprobabilities. The Fourier transformation of this shows that a modified convolution between  $Q_b$  for the signal  $Q$  function and  $P_a$  for the noise  $P$ -representation results in  $Q_d$  for the output field:

$$Q_d(\zeta) = \frac{1}{g} \int d^2\lambda P_a(\lambda)Q_b\left(\frac{\zeta - i\sqrt{g-1}\lambda}{\sqrt{g}}\right). \quad (16)$$

The convolution relation for amplification differs from that for attenuation (5) because of the unavoidable extra noise due to the thermal field (14).

Consider an array of  $\aleph$  beam splitters which satisfy the transformation relation (11). To simulate an amplifier we will take  $\aleph \rightarrow \infty$  and let the infinitesimal amplification factor for each beam splitter be given by  $g = 1 + \epsilon \approx 1$ . After a signal passes through the  $\aleph$  beam splitters, it is amplified by the factor of  $G = e^{\gamma T} = (1 + \epsilon)^\aleph$ , where  $\gamma$  is the amplification coefficient. By using

the Taylor expansion of  $Q$  function (16) to the second-order under the assumption  $\epsilon \approx 0$  (we had  $\tau \approx 0$  for attenuation), we obtain the Fokker-Planck equation for amplification:

$$\frac{dQ(\tau, \alpha)}{d\tau} = \frac{\gamma}{2} \left[ -\frac{\partial}{\partial \alpha_1} \alpha_1 - \frac{\partial}{\partial \alpha_2} \alpha_2 + \frac{1}{2}(N+M) \frac{\partial^2}{\partial \alpha_1^2} + \frac{1}{2}(N-M) \frac{\partial^2}{\partial \alpha_2^2} \right] Q(\tau, \alpha). \quad (17)$$

The Fokker-Planck equation (17) is solved for an arbitrary signal amplified in the phase-sensitive reservoir. The  $Q$  function corresponding to an arbitrary input field can be written as a weighted integral of Gaussian functions:

$$Q_s(\alpha) = \frac{1}{\pi} \int d^2\mu d^2\nu P(\mu, \nu) \exp[-(\alpha_1 - A)^2 - (\alpha_2 - B)^2], \quad (18)$$

where  $A = \frac{1}{2}(\mu + \nu)$ ,  $B = \frac{1}{2i}(\mu - \nu)$  and  $P(\mu, \nu)$  is the positive  $P$ -function for the field. It has been recently shown that if the initial  $Q$  function of the quantum system is (complex) Gaussian, then the solution of the Fokker-Planck equation (17) is also Gaussian with time-dependent parameters. The  $Q$  function (18) is a weighted integral of complex Gaussian functions, so one can obtain the time evolution of the input state

$$Q_{\text{amp}}(\alpha, \tau) = \frac{1/\pi}{\sqrt{(N_1(\tau) + 1)^2 - M_1(\tau)^2}} \int d^2\mu d^2\nu P(\mu, \nu) \times \exp \left\{ -\frac{[\alpha_1 - A(\tau)]^2}{1 + N_1(\tau) - M_1(\tau)} - \frac{[\alpha_2 - B(\tau)]^2}{1 + N_1(\tau) + M_1(\tau)} \right\}, \quad (19)$$

where the time-dependencies of the amplification parameters are

$$A(t) = A\sqrt{G} = Ae^{\gamma t}, \quad B(t) = B\sqrt{G}, \quad N_1(\tau) = (N+1)(G-1) \quad \text{and} \quad M_1(\tau) = M(G-1). \quad (20)$$

The inverse Fourier transformation of the  $Q$  function (19) shows that the characteristic function  $C_{\text{amp}}^{(q)}$  for the  $Q$  function of the amplified field is the product of the characteristic function  $C_{\text{sq}}^{(p)}$  for the  $P$ -representation of the squeezed thermal field and that  $C_s^{(q)}$  for the  $Q$  function of the amplified signal without noise

$$C_{\text{amp}}^{(q)}(\zeta) = C_{\text{sq}}^{(p)}(i\sqrt{G-1}\zeta) C_s^{(q)}(\sqrt{G}\zeta). \quad (21)$$

The convolution relation for this relation is then in a form analogous to Eq. (16) for the amplification beam splitter superposition of two input fields.

The antinormally-ordered moments can be calculated from the characteristic function  $C^{(q)}$ :

$$\langle \hat{a}^n (\hat{a}^\dagger)^m \rangle = \sum_{\ell=0}^m \sum_{k=0}^n \binom{m}{\ell} \binom{n}{k} (i\sqrt{G-1})^\ell (-i\sqrt{G-1})^k \langle (\hat{a}^\dagger)^\ell \hat{a}^k \rangle_{\text{sq}} \times (\sqrt{G})^{m+n-\ell-k} \langle \hat{a}^{n-k} (\hat{a}^\dagger)^{m-\ell} \rangle_s. \quad (22)$$

We can also consider the phase-sensitive amplifier which can be implemented as a stream of three-level atoms in a ladder configuration with equispaced levels injected into the cavity where the initial state of the field has been prepared. We denote the population in the uppermost state

by  $\rho_{aa}$ , the population in the lowest state by  $\rho_{cc}$  and the coherences between them by  $\rho_{ac}$  and  $\rho_{ca}$ . The atomic coherences  $\rho_{ac}$  and  $\rho_{ca}$  bring about the phase-sensitive effect in the two-photon linear amplifier. The parameters  $N$  and  $M$  can then be represented by the atomic variables

$$N_1(\tau) = \frac{\rho_{aa}}{\rho_{aa} - \rho_{cc}}(G - 1), \quad M_1(\tau) = \frac{|\rho_{ac}|}{\rho_{aa} - \rho_{cc}}(G - 1). \quad (23)$$

The normally-ordered photon number variance,  $:(\Delta n)^2:$ , where  $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$ , measures the deviation of the photon number fluctuations from the Poissonian photon statistics. The Poissonian field has the normally-ordered photon number variance zero, while the quantum mechanical sub-Poissonian field has it less than zero. It is larger than zero for the noisy super-Poissonian field. With use of Eq.(22) we find the normally-ordered photon number variance

$$:(\Delta n)_{\text{amp}}^2 := G^2 :(\Delta n)_s^2: + \zeta, \quad (24)$$

where the additive noise is

$$\zeta = \frac{(G - 1)^2}{(\rho_{aa} - \rho_{cc})^2}(\rho_{aa}^2 + |\rho_{ac}|^2) + \frac{G(G - 1)}{\rho_{aa} - \rho_{cc}}[2\rho_{aa}\langle a^\dagger a \rangle_s - |\rho_{ac}|(\langle a^2 \rangle_s + \langle (a^\dagger)^2 \rangle_s)]. \quad (25)$$

If the additive noise is negative, the amplified field has less photon number fluctuation than the input field. It is clearly seen that if the atomic coherence,  $\rho_{ac}$ , is zero the additive noise is always positive. However as the atomic coherence is nonzero we can have the negative noise to enhance the signal to noise ratio.

If each atom injected into the cavity is in atomic coherences we have the relation  $\rho_{aa}\rho_{cc} = |\rho_{ac}|^2$  and the additive noise (25) can be written as

$$\zeta = G(G - 1)\frac{c|\rho_{ac}|}{\rho_{aa} - \rho_{cc}} + 2G(G - 1)\frac{\rho_{aa} - \rho_{cc}}{\rho_{aa} - |\rho_{ac}|}\langle a^\dagger a \rangle_{\text{in}} + (G - 1)^2\frac{\rho_{aa}}{(\rho_{aa} - \rho_{cc})^2}, \quad (26)$$

where

$$c = 2\langle a^\dagger a \rangle_{\text{in}} - (\langle a^2 \rangle_{\text{in}} + \langle (a^\dagger)^2 \rangle_{\text{in}}), \quad (27)$$

which has to be negative to have the additive noise negative. The bosonic operators  $a$  and  $a^\dagger$  have a simple restriction,  $2\langle a^\dagger a \rangle - (\langle a^2 \rangle + \langle (a^\dagger)^2 \rangle) \geq -1$ . It is thus required that

$$-1 \leq c < 0 \quad (28)$$

for the noise reduction in the amplified signal. The noise reduction in the photon number fluctuations seems to be possible if the input field satisfies Eq.(28). However we should not fail to notice that the condition (28) is related to the initial photon number fluctuations. Because the expectation value of an operator times its hermitian conjugate is again positive,

$$c \geq -\frac{:(\Delta n)_{\text{in}}^2:}{\langle a^\dagger a \rangle}. \quad (29)$$

It is easily seen from Eqs.(28) and (29) that the input field should be super-Poissonian to have a possibility to reduce the photon number fluctuations by the amplification. If the input field is Poissonian there is no intersection between the two conditions (28) and (29) so that we can say that the Poissonian field does not become sub-Poissonian during the amplification.

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