QUANTUM MECHANICAL NOISE IN A MICHELSON INTERFEROMETER WITH NONCLASSICAL INPUTS — NONPERTURBATIVE TREATMENT

Sun-Kun King
Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300, ROC

Abstract

The variances of the quantum-mechanical noise in a two-input-port Michelson interferometer within the framework of the Loudon-Ni model were solved exactly in two general cases: (i) one coherent state input and one squeezed state input, and (ii) two photon number states inputs. Low intensity limit, exponential decaying signal and the noise due to mixing were discussed briefly.

1 Introduction

In 1981 the effects of intensity fluctuations in the two light beams and radiation pressure on the mirrors in a Michelson interferometer were modeled in a unified way by Loudon [1]. In 1987, Ni extended the model to include the intrinsic uncertainties of the mirrors and obtained an exact solution of the variance of the quantum mechanical noise for a coherent light source with arbitrary intensity [2]. These results were used recently by Ni in proposing an experimental scheme for controlling a macroscopic quantum state of a mirror by light shining [3].

Quantum-mechanical noise of a Michelson interferometer is an important noise source in gravitational waves detection. Experiments had reached the shot noise limit already. Photon shot noise decreases as the intensity goes up. Roughly speaking, it is propotional to the inverse square root of intensity. But, it was argued [4] that the fluctuation of radiation pressure on the mirrors would increase as the intesity becomes higher so that a minimum would be reached, called the "standard quantum limit" [4]. On the other hand, various works [5] show that such a measurement (without loss) implies no limit while squeezed states were used.

With the knowledge of squeezed states and that semi-classical model of interaction between a macroscopic object and photons, we got chance to probe this problem in detail. Within the framework of the Loudon-Ni model, the method of Ref. [2] was extended to obtain an exact solution for the variance of a two-input-port Michelson interferometer where a squeezed-state light source and a coherent light source (both with arbitrary intensity) were applied on each port respectively. Faithful matrix representations [6] were used in this calculation. The final result is more complicated. Nevertheless, it can be organized in a similar form as in Ref. [2]. Photon number state inputs can be treated in a similar way.

Due to the complicated results we've got, the physical implication is yet under study. However, some features and observation were discussed in the final section.

2 The Model

The usually input port was identified as "a-mode", with an annihilation operator "a" ('^' was omitted for simplicity). The usually unused port was called "b-mode", with an annihilation operator "b" ('^' was omitted, also). Both satisfy the canonical commutation relations, say, $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$. Beam splitter played the role of a mixer here. It combined both inputs from a-mode and b-mode then the mixture was sent into two arms as a_1-mode and a_2-mode. Therefore,

$$a = (a_1 + a_2)/\sqrt{2}, \quad b = (a_1 - a_2)/\sqrt{2}. \tag{1}$$

A phase was chosen. Nevertheless, it losts no generality. The reflection coefficient of arm one is

$$\hat{\mathrm{R}}_1 = \exp(i \arg r + 2ik\hat{z}_1) \tag{2}$$

where "$\arg r$" is a constant phase (real) and $2\hat{z}_1$ is the optical path length of the first arm. k is wave vector as usual. Since $\hat{z}$ is a hermitian operator, $\hat{\mathrm{R}}$ is a unitary operator. The annihilation operator on one of the output ports, which was named "d-mode" as in Ref. [1][2], would be a linear combination of those from both arms. The other output port was called "c-mode". Therefore,

$$d = (\hat{\mathrm{R}}_1 a_1 + \hat{\mathrm{R}}_2 a_2)/\sqrt{2} \quad \text{and} \quad c = e^{i\psi_c}(\hat{\mathrm{R}}_1 a_1 - \hat{\mathrm{R}}_2 a_2)/\sqrt{2} \tag{3}$$

where the phase term $e^{i\psi_c}$ was kept for generality. Energy conservation was fulfilled.

To treat photon shot noise and the fluctuation or radiation pressure separately was criticized by Marx [7] on the ground that it seems to assume some knowledge of the routes through the interferometer followed by individual photons, which is contrary to our understanding of quantum mechanics today. Loudon proposed a unified calculation [1] by introducing a coupling constant C. Ni pointed out that the position of the mirror itself should be a quantum-mechanical operator [2]. It'll contribute its intrinsic uncertainty to the total quantum uncertainty of the position of mirrors. In the low intensity limit, it was shown that the total uncertainty can be expressed as the sum of all three noise sources. Situation gets complicated at high intensity. The independence and correlation at different intensities among these noise moments provide ways to monitor and control a macroscopic quantum-mechanical object [3].

The Loudon-Ni model can be rephrased as the following:

$$k\hat{z}_1 = k\hat{z}_{1i} - C_1' {a_1}^\dagger a_1 \qquad \text{and} \qquad k\hat{z}_2 = k\hat{z}_{2i} - C_2' {a_2}^\dagger a_2 \tag{4}$$

where C_1' is the coupling constant which might be different from each mirror. The prime was to differ our notation from the previous one. There might be a factor of 2 difference. $\hat{z}_{1i}$ corresponds to the position operator of the first mirror without including the coupling effect. $\hat{z}$ is the mirror position operator we measured finally.

The coupling constant C' can be estimated as below. Suppose the mirror was hanged as a simple pendulum with mass M and length l. Its restoring force would be $\Delta x \cdot \mathrm{Mg}/l$ where Δx is a small displacement and g is the gravitational acceleration. Each photon suffers a momentum change $2\hbar k$ after been reflected back from the mirror. On balance we got $C' = 2\hbar k^2 l/(\mathrm{Mg})$.

Photon detector usually has its own quantum efficiency, denoted by ξ, which was assumed identical for both c-mode and d-mode. The measured photon intensity would, therefore, differ from $\langle d^\dagger d\rangle$ and $\langle c^\dagger c\rangle$ by a factor ξ. We've considered two detection schemes as in Ref. [8]. The first is direct detection:

$$\langle m\rangle_{\text{dir}} = \langle m_d\rangle = \xi\langle d^\dagger d\rangle. \tag{5}$$

The other is difference detection, which is the difference between the two output ports:

$$\langle m\rangle_{\text{diff}} = \langle m_c\rangle - \langle m_d\rangle = \xi(\langle c^\dagger c\rangle - \langle d^\dagger d\rangle). \tag{6}$$

Only the variances of difference detection were presented in this article:

$$\begin{aligned}(\Delta m)^2_{\text{diff}} &= \xi^2(\langle c^\dagger c^\dagger c\,c\rangle + \langle d^\dagger d^\dagger d\,d\rangle - 2\langle c^\dagger d^\dagger c\,d\rangle - \langle c^\dagger c\rangle^2 - \langle d^\dagger d\rangle^2 + 2\langle c^\dagger c\rangle\langle d^\dagger d\rangle)\\ &\quad + \xi(\langle c^\dagger c\rangle + \langle d^\dagger d\rangle)\end{aligned} \tag{7}$$

or, equivalently, in its expansion form:

$$\begin{aligned}(\Delta m)^2_{\text{diff}} &= \xi^2\big[2\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle - (\langle a_1^\dagger \hat{\mathrm{R}}_1^\dagger \hat{\mathrm{R}}_2 a_2\rangle + \langle a_2^\dagger \hat{\mathrm{R}}_2^\dagger \hat{\mathrm{R}}_1 a_1\rangle)^2\\ &\quad + \langle a_1^\dagger \hat{\mathrm{R}}_1^\dagger a_1^\dagger \hat{\mathrm{R}}_1^\dagger \hat{\mathrm{R}}_2 a_2 \hat{\mathrm{R}}_2 a_2\rangle + \langle a_2^\dagger \hat{\mathrm{R}}_2^\dagger a_2^\dagger \hat{\mathrm{R}}_2^\dagger \hat{\mathrm{R}}_1 a_1 \hat{\mathrm{R}}_1 a_1\rangle\big] + \xi(\langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle).\end{aligned} \tag{8}$$

These are what we want to calculate with our various inputs. It would be more complicated and model-dependent when considering photons with different frequencies.

3 The Solutions

First, a solution for coherent state - squeezed state inputs was solved. We'll have to deal with an expectation value, $\langle\exp[A(a^\dagger a + b^\dagger b) + B(b^\dagger a + a^\dagger b)]\rangle$, where the state vector $|\,\rangle_a$ is in coherent state $|\alpha\rangle$, $|\,\rangle_b$ is in general squeezed state $|\beta,\zeta\rangle$. $\zeta = se^{i\theta}$, where s is squeezing factor and θ is squeezing angle. The coherent parameters α and β are complex numbers with their phases ϕ_α and ϕ_β respectively.

Since coherent states $|\alpha\rangle$ is a's eigenstates, it is reasonable to reorder those operators as

$$\exp[A(a^\dagger a + b^\dagger b) + B(b^\dagger a + a^\dagger b)] = \exp(Ua^\dagger b)\exp(Va^\dagger a)\exp(Yb^\dagger b)\exp(Zb^\dagger a). \tag{9}$$

To get the coefficients U, V, Y, Z, it would be much easier to use faithful matrix representations of those four operators. Suppose $X_{11}, X_{22}, X_{12}, X_{21}$ are their corresponding matrices, which satisfy the same commutators as $a^\dagger a, b^\dagger b, a^\dagger b$ and $b^\dagger a$ do. It is not difficult to find a set of faithful matrices (2×2) which have the same relations. The operators equation becomes a matrix equation after this substitution. Solving this matrix equation we got

$$U = Z = \tanh B, \quad V = A - \ln(\cosh B) \quad \text{and} \quad Y = A + \ln(\cosh B). \tag{10}$$

After reordering, the calculation of the expectation value on a-mode (coherent state input) can be carried out. $\langle\alpha|\exp(Va^\dagger a)|\alpha\rangle$ was given in Ref. [2]. What left would be a calculation on b-mode (squeezed state input), which looks like $\langle\beta,\zeta|\exp(U\alpha^* b)\exp(Yb^\dagger b)\exp(U\alpha b^\dagger)|\beta,\zeta\rangle$. A squeezed state can be expressed as a vacuum state operated by a squeezing operator $S(s,\theta)$ and a displacement operator $D(a,\alpha)$ where

$$D(b,\beta) = \exp(\beta b^\dagger - \beta^* b) \quad \text{and} \quad S(s,\theta) = \exp[(s/2)(e^{-2i\theta}b^2 - e^{2i\theta}b^{\dagger 2})]. \tag{11}$$

Substitute this definition of squeezed state into b-mode, with some algebra, we may express its expectation value as the vacuum expectation value of a product of seperating terms

$$\langle\beta,\zeta|e^{U\alpha^* b}e^{Yb^\dagger b}e^{U\alpha b^\dagger}|\beta,\zeta\rangle = \langle 0|e^{A(b^2/2)}e^{Bb^\dagger}e^{Cb}e^{Y(b^\dagger b+1/2)}e^{Db^\dagger}e^{Eb}e^{F(b^{\dagger 2}/2)}|0\rangle \times (\text{a number}). \tag{12}$$

To solve this, we turned those operators into their normal ordering. Those operators form a Lie algebra. With corresponding commutators and their structure constants it is possible to find a faithful matrix representation [6]. Therefore, the operator equation could be reordered as

$$\begin{aligned}
& \langle 0|\exp[A(b^2/2)]\exp[Bb^\dagger]\exp[Cb]\exp[Y(b^\dagger b+1/2)]\exp[Db^\dagger]\exp[Eb]\exp[F(b^{\dagger 2}/2)]|0\rangle \\
= & \langle 0|\exp[J(b^{\dagger 2}/2)+Lb^\dagger]\exp[M(b^\dagger b+1/2)]\exp[N(b^2/2)+Pb]\exp[Q]|0\rangle \\
= & \exp(M/2)\exp(Q)
\end{aligned} \tag{13}$$

and, its corresponding matrix equation can be solved easily. We got

$$e^{-M} = e^{-Y}(1-AFe^{2Y}) \tag{14}$$

$$\begin{aligned}
Q = & (e^M/2)(2CD+2CEF+2ABD+2ABEF+AB^2e^{-Y} \\
& +E^2Fe^{-Y}+C^2Fe^Y+AD^2e^Y+2ABCFe^Y+2ADEFe^Y).
\end{aligned} \tag{15}$$

It is now straightforward to evaluate the uncertainty of photon measurements. The expectation value of the photon number of d-mode is

$$\langle d^\dagger d\rangle = (1/2)(\langle n\rangle+|\beta|^2+|\rho_s|^2)+(E_0|h_0|/2)\langle\cos 2\hat{\phi}'\rangle \tag{16}$$

where $|h_0|$ is roughly proportional to the input intensity and E_0 is an exponential factor which would be discussed later. ρ_s is squeezing related and $\hat{\phi}'$ is essentially the difference in optical path length with additional terms.

$$\hat{\phi}' = (1/2)[H_0+2k(\hat{z}_{2i}-\hat{z}_{1i})-\mathrm{Im}(Q)] \tag{17}$$

$$h_0 = |h_0|e^{iH_0} = \langle n\rangle + \frac{\alpha g'-\alpha^* f'-|\rho_s|^2e^{Y'}}{\rho_c^2-|\rho_s|^2e^{2Y'}} - \frac{f'g'}{(\rho_c^2-|\rho_s|^2e^{2Y'})^2} \tag{18}$$

$$\begin{aligned}
E_0 = (\rho_c^2-|\rho_s|^2e^{2Y})^{-\frac{1}{2}}\exp\Big\{&\frac{-(1-e^Y)}{1-|\Gamma|^2e^{2Y}}[\langle n\rangle(1+|\Gamma|^2e^Y-|\Gamma|(1+e^Y)\cos(\theta-2\phi_\alpha)) \\
&+|\beta|^2(1-|\Gamma|^2e^Y+|\Gamma|(1-e^Y)\cos(\theta-2\phi_\beta))]\Big\}
\end{aligned} \tag{19}$$

$$\mathrm{Im}(Q) = \frac{-iUe^Y}{1-|\Gamma|^2e^{2Y}}[(\alpha^*\beta+\alpha\beta^*)(1-|\Gamma|^2e^Y)+(\alpha^*\beta^*\Gamma+\alpha\beta\Gamma^*)(1-e^Y)] \tag{20}$$

$$f' = (\rho_c^2-|\rho_s|^2e^{Y'})\beta+\rho_c\rho_s(1-e^{Y'})\beta^*+|\rho_s|^2e^{2Y'}U'\alpha-\rho_c\rho_s e^{Y'}U'\alpha^* \tag{21}$$

$$g' = (\rho_c^2-|\rho_s|^2e^{Y'})\beta^*+\rho_c\rho_s^*(1-e^{Y'})\beta+|\rho_s|^2e^{2Y'}U'\alpha^*-\rho_c\rho_s^* e^{Y'}U'\alpha \tag{22}$$

$$U' = i\tan\Sigma C', \quad Y' = -i\Delta C'+\ln(\cos\Sigma C') \tag{23}$$

$$U = -i\tan(\Sigma C'), \quad Y = i\Delta C'+\ln(\cos\Sigma C') \tag{24}$$

where $\rho_c = \cosh s$, $\rho_s = e^{i\theta}\sinh s$, $\Gamma = \rho_s/\rho_c$, $\Delta C' = C_2' - C_1'$ and $\Sigma C' = C_2' + C_1'$. Similarly,

$$\langle c^\dagger c - d^\dagger d\,\rangle = -E_0|h_0|\langle\cos 2\hat{\phi}'\rangle \tag{25}$$

for difference detection. Basically, the Michelson interferometer is a transducer which turns a change of the arm length into a change of light intensity. The measured intensity $\langle m\rangle$ and its variance can be transformed back to the uncertainty of arm length, or more precisely, the difference of the positions of two mirrors. We may write down this uncertainty as

$$(\Delta Z_{total})_{\text{diff}} = \sqrt{(\Delta m)^2_{\text{diff}}}/(2\xi k E_0|h_0||\langle\sin 2\hat{\phi}'\rangle|) \tag{26}$$

where $(\Delta m)^2_{\text{diff}}$ was given by Eq. (7) or, more explicitly, by Eq. (8). This is just the inverse of signal to noise ratio. The final result is

$$\begin{aligned}(\Delta Z_{total})^2_{\text{diff}} &= \frac{(\langle n\rangle + |\beta|^2)^2 + 3|\rho_s|^4 + |\rho_s|^2 - 4\langle n\rangle|\beta|^2\cos^2(\phi_\alpha - \phi_\beta)}{8k^2E_0^2|h_0|^2\langle\sin 2\hat{\phi}'\rangle^2}\\ &+ \frac{|\beta|^2[2|\rho_s|^2 - \rho_c|\rho_s|\cos(\theta - 2\phi_\beta)] + \langle n\rangle\rho_c|\rho_s|\cos(\theta - 2\phi_\alpha)}{4k^2E_0^2|h_0|^2\langle\sin 2\hat{\phi}'\rangle^2}\\ &+ \frac{E''|h_2|\langle\cos[4\hat{\phi}' + H_2 - 2H_0 + \mathrm{Im(Q'')} + 2\mathrm{Im(Q)}]\rangle}{8k^2E_0^2|h_0|^2\langle\sin 2\hat{\phi}'\rangle^2}\\ &- \frac{\langle\cos 2\hat{\phi}'\rangle^2}{4k^2\langle\sin 2\hat{\phi}'\rangle^2} + \frac{\langle n\rangle + |\beta|^2 + |\rho_s|^2}{4\xi k^2E_0^2|h_0|^2\langle\sin 2\hat{\phi}'\rangle^2}\end{aligned} \tag{27}$$

for difference detection. Where $\langle n\rangle = |\alpha|^2$ is the intensity of the input coherent state on a-mode. h_2 is shorthand notations of complicated modification on intensity square, H_2 is its phase. E'' is another exponential factor which decreases the interference terms in ΔZ_{total}. $\mathrm{Im}(Q)$ and $\mathrm{Im}(Q'')$ are imaginary parts of Q and Q'', which came from solving the matrix equation Eq. (13) or its similar version. All of them can be evaluated exactly as follow.

$$\begin{aligned}h_2 &= |h_2|e^{iH_2} = \Big[\langle n\rangle + \frac{\alpha g'' - \alpha^* f'' - 2|\rho_s|^2e^{Y''}}{\rho_c^2 - |\rho_s|^2e^{2Y''}} - \frac{f''g''}{(\rho_c^2 - |\rho_s|^2e^{2Y''})^2}\Big]^2 + \frac{\rho_c^2|\rho_s|^2 - 2|\rho_s|^4e^{2Y''}}{(\rho_c^2 - |\rho_s|^2e^{2Y''})^2}\\ &- \frac{\rho_c\rho_s^*}{\rho_c^2 - |\rho_s|^2e^{2Y''}}(\alpha - \frac{f''}{\rho_c^2 - |\rho_s|^2e^{2Y''}})^2 - \frac{\rho_c\rho_s}{\rho_c^2 - |\rho_s|^2e^{2Y''}}(\alpha^* + \frac{g''}{\rho_c^2 - |\rho_s|^2e^{2Y''}})^2\end{aligned} \tag{28}$$

U'' and Y'' are similar to U' and Y' but with $\Delta C'$ and $\Sigma C'$ replaced by $2\Delta C'$ and $2\Sigma C'$. f'', g'', E'' and $\mathrm{Im}(Q'')$ are similar to f, g, E_0 and $\mathrm{Im}(Q)$ except $Y(Y')$ and $U(U')$ were replaced by Y'' and U''. In our calculation, it was assumed that $C_1' = C_2' \equiv C'$.

We now turn to the photon number states input. Photon number state is a quantum state without classical correspondence. Its second order coherence is minimum such that its number variance vanished.

Suppose the input state is $|n_a\rangle_a \otimes |n_b\rangle_b$ with $|n_a\rangle_a = (a^\dagger)^{n_a}/\sqrt{n_a!}|0\rangle$ and $|n_b\rangle_b = (b^\dagger)^{n_b}/\sqrt{n_b!}|0\rangle$ where n_a and n_b are real numbers. With a different reordering from Eq. (9) and the assumption $\Delta C' = 0$, we have

$$\begin{aligned}\langle d^\dagger d\,\rangle &= \frac{1}{2}(n_a + n_b) + \frac{1}{2}\langle\cos[2k(\hat{z}_{2i} - \hat{z}_{1i})]\rangle(\cos 2C')^{n_b - n_a}\\ &\times\{n_a\cos 2C'{}_2F_1(1 + n_b, 1 - n_a, 1; \sin^2 2C') - \frac{n_b}{\cos 2C'}{}_2F_1(n_b, -n_a, 1; \sin^2 2C')\}\end{aligned} \tag{29}$$

where ${}_2F_1(a,b,c;z)$ is hypergeometric function. In perfect detection, that is, $\xi = 1$, the variance of the photon detection becomes

$$\begin{aligned}
\langle(\Delta m)^2\rangle = &\frac{1}{8}n_a(n_a+1) + \frac{1}{8}n_b(n_b+1) - \frac{1}{4}\langle\cos[2k(\hat{z}_{2i}-\hat{z}_{1i})]\rangle^2(\cos 2C')^{2(n_b-n_a)} \\
&\times[n_a\cos 2C'{}_2F_1(1+n_b,1-n_a,1;\sin^2 2C') - \frac{n_b}{\cos 2C'}{}_2F_1(n_b,-n_a,1;\sin^2 2C')]^2 \\
&+\frac{1}{8}\langle\cos[4k(\hat{z}_{2i}-\hat{z}_{1i})]\rangle(\cos 4C')^{n_b-n_a} \\
&\times\Big[n_a(n_a-1)\cos^2 4C'{}_2F_1(1+n_b,2-n_a,1;\sin^2 4C') \\
&+ n_b(n_b-1)\sec^2 4C'{}_2F_1(n_b-1,-n_a,1;\sin^2 4C') \\
&- 4n_a n_b {}_2F_1(n_b,1-n_a,1;\sin^2 4C') \\
&- n_a n_b(n_a-1)(n_b-1)\sin^2 4C'{}_2F_1(1+n_b,2-n_a,3;\sin^2 4C')\Big].
\end{aligned} \tag{30}$$

There are relations between a hypergeometric function and its contiguous functions. Further simplification is possible.

4 Discussion

A low intensity limit can be obtained easily [1] while b-mode was in vacuum state. In short, $(\Delta Z_{total})^2 \geq C'/k^2$. We have searched in a limited parameter space and found no violation of this inequality (within error). Though, it is *not* a proof of that limit. At very high intensity, the reflectance $\hat{\mathrm{R}}$ has little effect. The noise behavior can be explained by the mixing of two input states. High order moments are needed to characterize such a superposition. On the other hand, that noise can be eliminated by setting two input states at nearly the same intensity. Nevertheless, there is still an exponential factor E_0 in signal (cf. Eq. (25) and Eq. (26)). It doesn't show up in classical solution and it *always* decreases the signal. We left further discussion to another work.

References

[1] R. Loudon, *Phys. Rev. Lett.* **47**, 815 (1981).

[2] W.-T. Ni, *Phys. Rev.* **D35**, 3002 (1987).

[3] W.-T. Ni, "Controlling a Macroscopic Quantum State by Light Shining" in *Quantum Control and Measurement*, H. Ezawa & Y. Murayama (edit.), Elsevier Science Publishers B. V., 1993.

[4] C. M. Caves, *Phys. Rev. Lett.* **45**, 75 (1980).

[5] See A. F. Pace *et al.*, *Phys. Rev.* **A47**, 3173 (1993) and references therein.

[6] W. M. Zhang, D. H. Feng and R. Gilmore, *Rev. Mod. Phys.* **62**, 867 (1990).

[7] B. Marx, *Nature* **287**, 276 (1980).

[8] C. M. Caves, *Phys. Rev.* **D23**, 1693 (1981).