CORRELATED LIGHT AND SCHRÖDINGER CATS

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Abstract

The Schrödinger cat male and female states are discussed. The Wigner and Q–functions of generalized correlated light are given. Linear transformator of photon statistics is reviewed.

1 Introduction

The integral of motion which is quadratic in position and momentum was found for classical oscillator with time-dependent frequency by Ermakov [1]. Two time-dependent integrals of motion which are linear forms in position and momentum for the classical and quantum oscillator with time-dependent frequency were found in [2]; for a charge moving in varying in time uniform magnetic field, this was done in [3]. For the multimode nonstationary oscillatory systems, such new integrals of motion, both of Ermakov's type (quadratic in positions and momenta) and linear in position and momenta, generalizing the results of [2] were constructed in [4]. We will consider below the parametric oscillator using the integrals of motion.The Wigner function of multimode squeezed light is studied using such special functions as multivariable Hermite polynomials. The theory of parametric os cillator is appropriate to consider the problem of creation of photons from vacuum in a resonator with moving walls (with moving mirrors) which is the phenomenon based on the existence of Casimir forces (so-called nonstationary Casimir effect). The resonator with moving boundaries (moving mirrors, media with time-dependent refractive index) produces also effect of squeezing in the light quadratures. In the high energy physics very fast particle collisions may produce new types of states of boson fields (pions, for example) which are squeezed and correlated states studied in quantum optics but almost unknown in particle physics, both theoretically and experimentally.

2 Multimode Quadratic Systems

The generic nonstationary linear system has the Hamiltonian

$$H = \frac{1}{2}\mathbf{Q}B(t)\mathbf{Q} + \mathbf{C}(t)\mathbf{Q}, \tag{1}$$

where we use 2N–vectors $\mathbf{Q} = (p_1, p_2, \ldots, p_N, q_1, q_2, \ldots, q_N)$ and $\mathbf{C}(t)$, as well as 2N×2N–matrix $B(t)$, the Planck constant $\hbar = 1$. This system has 2N linear integrals of motion [5], [6] which may be written in vector form

$$\mathbf{Q}_0(t) = \Lambda(t)\mathbf{Q} + \boldsymbol{\Delta}(t). \tag{2}$$

The real symplectic matrix $\Lambda(t)$ is the solution to the system of equations

$$\begin{aligned} \dot{\Lambda}(t) &= \Lambda(t)\Sigma B(t), \\ \Lambda(0) &= 1, \end{aligned} \tag{3}$$

where the real antisymmetrical matrix Σ is 2N–dimensional analog of the Pauli matrix $i\sigma_y$, and the vector $\mathbf{\Delta}(t)$ is the solution to the system of equations

$$\begin{aligned} \dot{\mathbf{\Delta}}(t) &= \Lambda(t)\Sigma\mathbf{C}(t), \\ \mathbf{\Delta}(0) &= 0. \end{aligned} \tag{4}$$

If for time $t=$ 0, one has the initial Wigner function of the system in the form

$$W(\mathbf{p},\mathbf{q},t=0)=W_0(\mathbf{Q}), \tag{5}$$

the Wigner function of the system at time t is (duc to the density operator is the integral of motion)

$$W(\mathbf{p},\mathbf{q},t)=W_0[\Lambda(t)\mathbf{Q}+\mathbf{\Delta}(t)]. \tag{6}$$

This formula may be interpreted as transformation of input Wigner function into output Wigner function due to symplectic quadrature transform (2). An optical linear transformator of photon distribution function using this output Wigner function i s suggested in [7].

The Hamiltonian (1) may be rewritten in terms of creation and annihilation operators

$$H=\frac{1}{2}\mathbf{A}D(t)\mathbf{A}+\mathbf{E}(t)\mathbf{A}, \tag{7}$$

where we use 2N–vectors $\mathbf{A}=(a_1, a_2,\ldots, a_N, a_1\dagger, a_2\dagger,\ldots, a_N\dagger)$ and $\mathbf{E}(t)$, as well as 2N×2N–matrix $D(t)$. This system has 2N linear integrals of motion [5], [6] which are written in vector form

$$\mathbf{A}_0(t)=M(t)\mathbf{A}+\mathbf{N}(t). \tag{8}$$

The complex matrix $M(t)$ is the solution to the system of equations

$$\begin{aligned} \dot{M}(t) &= M(t)\sigma D(t), \\ M(0) &= 1, \end{aligned} \tag{9}$$

where the imaginary antisymmetric matrix σ is 2N×2N–analog of the Pauli matrix $-\sigma_y$, and the vector $\mathbf{N}(t)$ is the solution to the system of equations

$$\begin{aligned} \dot{\mathbf{N}}(t) &= M(t)\sigma\mathbf{E}(t), \\ \mathbf{N}(0) &= 0. \end{aligned} \tag{10}$$

Analogously to the Wigner function evolution, if for time $t=$ 0, one has the initial Q–function of the system in the form

$$Q(\alpha,\alpha^*,t=0)=Q_0(\mathcal{A}), \qquad \mathcal{A}=(\alpha,\alpha^*), \tag{11}$$

the Q–function of the system at time t is

$$Q(\alpha, \alpha^*, t) = Q_0[M(t)\mathcal{A} + \mathbf{N}(t)]. \tag{12}$$

Here $\alpha = (\mathrm{q} + i\mathrm{p})/\sqrt{2}$.

For time-independent Hamiltonian (1), the matrix $\Lambda(t)$ is

$$\Lambda(t) = \exp(\Sigma B t), \tag{13}$$

and the vector $\mathbf{\Delta}(t)$ is

$$\mathbf{\Delta}(t) = \int_0^t \exp(\Sigma B \tau)\ \Sigma\ \mathbf{C}(\tau)\ d\tau. \tag{14}$$

For time-independent Hamiltonian (7), the matrix $M(t)$ is

$$M(t) = \exp(\sigma D t), \tag{15}$$

and the vector $\mathbf{N}(t)$ is

$$\mathbf{N}(t) = \int_0^t \exp(\sigma D \tau)\ \sigma\ \mathbf{E}(\tau)\ d\tau. \tag{16}$$

For time-dependent linear systems, the Wigner function of generic squeezed and correlated state (generalized correlated state [8]) has Gaussian form and it was calculated in [5].

Thus the evolution of the Wigner function and Q–function for systems with quadratic Hamiltonians for any state is given by the following prescription. Given the Wigner function $W(\mathbf{p}, \mathbf{q}, t = 0)$ for the initial moment of time $t = 0$. Then the Wigner function for time t is obtained by the replacement

$$W(\mathbf{p}, \mathbf{q}, t) = W(\mathbf{p}(t),\ \mathbf{q}(t),\ t = 0),$$

where time-dependent arguments are the linear integrals of motion of the quadratic system found in [5], [4], and [9]. This formula was given as integral with δ–function kernel in [10]. The linear integrals of motion describe initial values of classical trajectories in the phase space of the system. The same ansatz is used for the Q–function. Namely, given the Q–function of the quadratic system $Q(\mathbf{B},\ t = 0)$ for the initial moment of time $t = 0$. Then the Q–function for time t is given by the replacement

$$Q(\mathbf{B},\ t) = Q(\mathbf{B}(t),\ t = 0),$$

where the 2N–vector $\mathbf{B}(t)$ is the integral of motion linear in the annihilation and creation operators. This ansatz follows from the statement that the density operator of the Hamiltonian system is the integral of motion, and its matrix elements in any basis must depend on appropriate integrals of motion.

3 Multimode Mixed Correlated Light

The most general mixed squeezed state of the N–mode light with a Gaussian density operator $\hat{\rho}$ is described by the Wigner function $W(\mathrm{p}, \mathrm{q})$ of the generic Gaussian form,

$$W(\mathrm{p}, \mathrm{q}) = \det \mathbf{M} \exp\left[-\frac{1}{2}(\mathrm{Q} - <\mathrm{Q}>)\mathrm{M}^{-1}(\mathrm{Q} - <\mathrm{Q}>)\right], \tag{17}$$

where 2N parameters $<p_i>$ and $<q_i>$, $i=1,2,\ldots,N$, combined into vector $<\mathbf{Q}>$, are average values of quadratures,

$$\begin{aligned} <\mathbf{p}> &= \mathrm{Tr}\,\hat{\rho}\hat{\mathbf{p}}, \\ <\mathbf{q}> &= \mathrm{Tr}\,\hat{\rho}\hat{\mathbf{q}}. \end{aligned} \tag{18}$$

A real symmetric dispersion matrix $\mathbf{M}$ consists of $2N^2+N$ variances

$$\mathcal{M}_{\alpha\beta} = \frac{1}{2} < \hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha > - < \hat{Q}_\alpha >< \hat{Q}_\beta >, \qquad \alpha, \beta = 1,2,\ldots,2N. \tag{19}$$

They obey uncertainty relations constraints [5]. According to previous section the Wigner function of parametric linear system with initial value (17) is

$$W(\mathbf{p},\mathbf{q},t) = \det \mathbf{M} \exp\left[-\frac{1}{2}(\Lambda(t)\mathbf{Q} + \boldsymbol{\Delta}(t) - <\mathbf{Q}>)\mathbf{M}^{-1}(\Lambda(t)\mathbf{Q} + \boldsymbol{\Delta}(t) - <\mathbf{Q}>)\right], \tag{20}$$

The photon distribution function of the state (17)

$$\mathcal{P}_{\mathbf{n}} = \mathrm{Tr}\,\hat{\rho}|\mathbf{n}><\mathbf{n}|, \qquad \mathbf{n} = (n_1, n_2, \ldots, n_N), \tag{21}$$

where the state $|\mathbf{n}>$ is photon number state, which was calculated in [11], [12] and it is

$$\mathcal{P}_{\mathbf{n}} = \mathcal{P}_0 \frac{H_{\mathbf{nn}}^{\{\mathbf{R}\}}(\mathbf{y})}{\mathbf{n}!}. \tag{22}$$

The trace (21) may be calculated using the explicit form of the Wigner function of the operator $|\mathbf{m}><\mathbf{n}|$ (see, [5]) which is the product of Wigner functions of one-dimensional oscillator expressed in terms of Laguerre polynomials of the form

$$W_{mn}(p,q) = 2^{m-n+1}(-1)^n \sqrt{\frac{n!}{m!}} \left(\frac{q-ip}{\sqrt{2}}\right)^{m-n} e^{-(p^2+q^2)} L_n^{m-n}\left(2(q^2+p^2)\right). \tag{23}$$

The function $H_{\mathbf{nn}}^{\{\mathbf{R}\}}(\mathbf{y})$ is multidimensional Hermite polynomial. The probability to have no photons is

$$\mathcal{P}_0 = \left[\det\left(\mathbf{M} + \frac{1}{2}\mathbf{I}_{2N}\right)\right]^{-1/2} \exp\left[- <\mathbf{Q}> (2\mathbf{M} + \mathbf{I}_{2N})^{-1} <\mathbf{Q}>\right], \tag{24}$$

where we introduced the matrix

$$\mathbf{R} = 2\mathbf{U}^\dagger(1 + 2\mathbf{M})^{-1}\mathbf{U}^* - \sigma_{Nx}, \tag{25}$$

and the matrix

$$\sigma_{Nx} = \begin{pmatrix} 0 & \mathbf{I}_N \\ \mathbf{I}_N & 0 \end{pmatrix}. \tag{26}$$

The argument of Hermite polynomial is

$$\mathbf{y} = 2\mathbf{U}^t(\mathbf{I}_{2N} - 2\mathbf{M})^{-1} <\mathbf{Q}>, \tag{27}$$

and the 2N–dimensional unitary matrix

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\mathbf{I}_N & i\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{I}_N \end{pmatrix} \tag{28}$$

is introduced, in which $\mathbf{I}_N$ is the N×N–identity matrix. Also, we use the notation

$$\mathbf{n}! = n_1! n_2! \cdots n_N!.$$

The mean photon number for j–th mode is expressed in terms of photon quadrature means and dispersions

$$< n_j >= \frac{1}{2}(\sigma_{p_j p_j} + \sigma_{q_j q_j} - 1) + \frac{1}{2}(< p_j >^2 + < q_j >^2). \tag{29}$$

The photon distribution function for transformed state (20) is given by the same formulae (22), (24)–(28) but with changed dispersion matrix

$$\widetilde{\mathbf{M}} = \Lambda^{-1}\mathbf{M}\Lambda^{-1t}, \tag{30}$$

and quadrature means

$$< \widetilde{\mathbf{Q}} >= \Lambda^{-1}(\mathbf{\Delta} - < \mathbf{Q} >). \tag{31}$$

Thus we have a linear transformator of photon statistics suggested in [7].

Let us now introduce a complex 2N–vector $\mathbf{B} = (\beta_1, \beta_2, \ldots, \beta_N, \beta_1^*, \beta_2^*, \ldots, \beta_N^*)$. Then the Q–function is the diagonal matrix element of the density operator in coherent state basis $| \beta_1, \beta_2, \ldots, \beta_N >$. This function is the generating function for matrix elements of the density operator in the Fock basis $|\mathbf{n}>$ which has been calculated in [12]. In notations corresponding to the Wigner function (17) the Q–function is

$$Q(\mathbf{B}) = \mathcal{P}_0 \exp\left[-\frac{1}{2}\mathbf{B}(R + \sigma_{Nx})\mathbf{B} + \mathbf{B}R\mathbf{y}\right]. \tag{32}$$

Thus, if the Wigner function (17) is given one has the Q–function. Also, if one has the Q–function (32), i.e., the matrix R and vector $\mathbf{y}$, the Wigner function may be obtained due to relations

$$\begin{aligned} \mathbf{M} &= \mathbf{U}^*(R + \sigma_{Nx})^{-1}\mathbf{U}^\dagger - 1/2, \\ < \mathbf{Q} > &= \mathbf{U}^*[1 - (R + \sigma_{Nx})^{-1}\sigma_{Nx}]\mathbf{y}. \end{aligned} \tag{33}$$

Multivariable Hermite polynomials describe the photon distribution function for the multimode mixed and pure correlated light [11], [13], [14]. The nonclassical state of light may be created due to nonstationary Casimir effect [15], [16] and the multimode oscillator is the model to describe the behaviour of squeezed and correlated photons.

4 Parametric Oscillator

For the parametric oscillator with the Hamiltonian

$$H = -\frac{\partial^2}{2\partial x^2} + \frac{\omega^2(t)x^2}{2}, \tag{34}$$

where we take $\hbar = m = \omega(0) = 1$, there exists the time-dependent integral of motion found in [2]

$$A = \frac{i}{\sqrt{2}}[\varepsilon(t)p - \dot{\varepsilon}(t)x], \tag{35}$$

where

$$\ddot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0, \qquad \varepsilon(0) = 1, \qquad \dot{\varepsilon}(0) = i, \tag{36}$$

satisfying the commutation relation

$$[A,\ A\dagger] = 1. \tag{37}$$

It is easy to show that packet solutions of the Schrödinger equation may be introduced and interpreted as coherent states [2], since they are eigenstates of the operator A (35), of the form

$$\Psi_\alpha(x,t) = \Psi_0(x,t)\exp\left(-\frac{|\alpha|^2}{2} - \frac{\alpha^2\varepsilon^*(t)}{2\varepsilon(t)} + \frac{\sqrt{2}\alpha x}{\varepsilon}\right), \tag{38}$$

where

$$\Psi_0(x,t) = \pi^{-1/4}\varepsilon(t)^{-1/2}\exp\frac{i\dot{\varepsilon}(t)x^2}{2\varepsilon(t)} \tag{39}$$

is analog of the ground state of the oscillator and α is a complex number.

Variances of the position and momentum of the parametric oscillator in the state (38), (39) are

$$\sigma_x = \frac{|\varepsilon(t)|^2}{2}, \qquad \sigma_p = \frac{|\dot{\varepsilon}(t)|^2}{2}, \tag{40}$$

and the correlation coefficient r of the position and momentum has the value corresponding to minimization of the Schrödinger uncertainty relation [17]

$$\sigma_x\sigma_p = \frac{1}{4}\frac{1}{1-r^2}. \tag{41}$$

If $\sigma_x < 1/2$ $(\sigma_p < 1/2)$ we have squeezing in photon quadrature components.

The analogs of orthogonal and complete system of states which are excited states of stationary oscillator are obtained by expansion of (38) into power series in α. We have

$$\Psi_m(x,t) = \left(\frac{\varepsilon^*(t)}{2\varepsilon(t)}\right)^{m/2}\frac{1}{\sqrt{m!}}\Psi_0(x,t)H_m\left(\frac{x}{|\varepsilon(t)|}\right), \tag{42}$$

and these squeezed and correlated number states are eigenstates of invariant $A^\dagger A$. In case of periodical dependence of frequency on time the classical solution in stable regime may be taken in Floquet form

$$\varepsilon(t) = e^{i\kappa t}u(t), \tag{43}$$

where $u(t)$ is a periodical function of time. Then the states (42) are quasienergy states realizing the unitary irreducible representation of time translation symmetry group of the Hamiltonian and the parameter κ determines the quasienergy spectrum. Unstable classical solutions give continuous spectrum of quasienergy states.

The partial cases of parametric oscillator are free motion ($\omega(t) = 0$), stationary harmonic oscillator ($\omega^2(t) = 1$), and repulsive oscillator ($\omega^2(t) = -1$). The solutions obtained above are described by the function $\varepsilon(t)$ which is equal to $\varepsilon(t) = 1 + it$, for free particle, $\varepsilon(t) = e^{it}$, for usual oscillator, and $\varepsilon(t) = \cosh t + i \sinh t$, for repulsive oscillator.

Another normalized solution to the Schrödinger equation

$$\Psi_{\alpha m}(x,t) = 2N_m \Psi_0(x,t) \exp\left(-\frac{|\alpha|^2}{2} - \frac{\varepsilon^*(t)\alpha^2}{2\varepsilon(t)}\right) \cosh \frac{\sqrt{2}\alpha x}{\varepsilon(t)}, \tag{44}$$

where

$$N_m = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\cosh |\alpha|^2}}, \tag{45}$$

is the even coherent state [18] (the Schrödinger cat male state). The odd coherent state of the parametric oscillator (the Schrödinger cat female state)

$$\Psi_{\alpha f}(x,t) = 2N_f \Psi_0(x,t) \exp\left(-\frac{|\alpha|^2}{2} - \frac{\varepsilon^*(t)\alpha^2}{2\varepsilon(t)}\right) \sinh \frac{\sqrt{2}\alpha x}{\varepsilon(t)}, \tag{46}$$

where

$$N_f = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\sinh |\alpha|^2}}, \tag{47}$$

satisfies the Schrödinger equation and is the eigenstate of the integral of motion A^2 (as well as the even coherent state) with the eigenvalue α^2. These states are one-mode examples of squeezed and correlated Schrödinger cat states constructed in [19]. The experimental creation of the Schrödinger cat states is discussed in [20]. These states belo ng to family of nonclassical superposition states studied in [21], [22].

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