A new definition to the phase operator and its properties*

Duan Lu-ming and Guo Guang-can
Department of Physics, University of Science and Techonology of China, Hefei, 230026, P.R.China

By introducing a series of mathematical symbols and the phase quantization condition, we give a new definition of the phase operator, which not only is made directly in infinite state spaces, but also circumvents all difficulties appearing in the traditional approach. Properties of the phase operator and its expressions in some widely-used representations are also given.

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I . Introduction

The phase operator is very important in the quantum optics and field theory. But as was clearly pointed out by Susskind and Glogower[2], there are many difficulties in the traditional definition of the phase operator[1,2]. The traditional approach required that the Hermitian number and phase operator were combined in a polar decomposition of the annihilation operator:

$$\hat{a} = \exp\left(i\hat{\theta}\right)\hat{N}^{\frac{1}{2}}, \tag{1}$$

and supposed that they satisfied the following commutator

$$\left[\hat{\theta}, \hat{N}\right] = -i, \tag{2}$$

But the commutator(2) gives rise to inconsistency when its matrix elements are calculated in a number-state basis and the uncertainty relation $\Delta N \Delta \theta \geq \frac{1}{2}$ derived from Eq.(2) implies that a number state has infinite phase uncertainty which contradicts to the periodic nature of the phase. Furthermore, the exponential operator $\exp\left(i\hat{\theta}\right)$ in this approach is not unitary so it does not define a Hermitian $\hat{\theta}$. Recently, there appeared many developments on this problem[3-8]. Especially Pegg and Barnett defined a phase operator in a finite-dimensional state space[3,4] and the definition has been widely used. This definition circumvents the difficulties in the traditional approach at the price that it is limited to a finite state space, the dimension of which is allowed to tend to infinity only after physically measurable results, such as expectation values, are calculated. It is now often accepted that a well-behaved Hermitian phase operator does not exist in infinite state spaces[2-4]. In this paper we give a new approach to the definition of the phase operator. We have defined a Hermitian phase operator directly in infinite state spaces. By introducing a series of mathematical symbols and the phase quantization condition, we have overcome the above-mentioned difficulties in the traditional approach. As a result of being defined directly in infinite state spaces, the phase operator here has very succinct expressions in some widely-used representations which make it very convenient for use.

II . Definition of the phase operator

We consider the quantized single-mode boson field. In this system, the dimension of the state space determined by the Hamitonian $\hat{H}$ is countable-infinite and all the eigenstates of the number operator $\hat{N}$ make a complete basis. So an operator is well defined if its action on an arbitrary eigenstate of the number operator is given. We define the phase operator as the infinitesimal displacement operator of the number basis. So to determine the phase operator, we first give the following definition of an unitary displacement operator $\hat{D}$ of the number basis:

$$\hat{D}|n\rangle = |n-1\rangle \qquad (\mathrm{n} \neq 0), \tag{3}$$

$$D|0\rangle = |P_\infty - 1\rangle, \tag{4}$$

where $P_\infty = \lim_{m\to\infty} m!$. Some explanation need be added to the definition equation(4). Firstly, one may just suppose $\hat{D}|0\rangle = 0$. But this idea leads to contradiction. It makes $\hat{D}$ not unitary. By intuition, the displacement operator $\hat{D}$ should transform $|0\rangle$ to another eigenstate of $\hat{N}$. Secondly, one may let $\hat{D}|0\rangle = |\infty\rangle$. But the state $|\infty\rangle$ is not well defined because ∞ is not a simple number. Though $P_\infty - 1$ is also infinity, the states $|P_\infty - 1\rangle$ and $|\infty\rangle$ still have discriminations. The state $|\infty\rangle$ just indicates that the eigenvalue of $\hat{N}$ tends to infinity. It does not show the mode of tendency. For example, when $n \to \infty$, the states $|2n\rangle$ and $|2n-1\rangle$ can all be written as $|\infty\rangle$,but these two states are not same because they are orthogonal. Though the discrimination between the states $|P_\infty - 1\rangle$ and $|\infty\rangle$ is not important to the final physical results because the states $|n\rangle$ when $n \to \infty$ have no contribution to usual physical states, it plays a important role in defining a self-consistence Hermitian phase operator because our definition is made directly in infinite state spaces and is not in view of concrete physical states. The mode of tendency to infinity must be determined in this situation. Equation(4) indicates that $\hat{D}$ transforms $|0\rangle$ to an eigenstate of $\hat{N}$ with an eigenvalue tending to infinity and the mode of tendency is given by the sequence $\{n!-1,(n+1)!-1,(n+2)!-1,\cdots\}$. So $\hat{D}$ is completely defined and we will see this definition of $\hat{D}$ makes a good foundation for the definition of a Hermitian phase operator in infinite state spaces.

In the number representation $\hat{D}$ defined by Eqs.(3)(4) has the matrix form

$$D = \lim_{n\to\infty} \underbrace{\left.\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\} n!}_{n!}. \tag{5}$$

Its eigenvalues have the expression

$$d = e^{i\theta} = e^{i2\pi r}. \tag{6}$$

When $n \to \infty$, the value of r is limited to rational numbers and also $e^{i2\pi r}$ with any rational r is the eigenvalue of $\hat{D}$. Writing the eigenstate of $\hat{D}$ with the eigenvalue $e^{i\theta}$ as $|\theta\rangle$,we have

$$\hat{D}|\theta\rangle = e^{i\theta}|\theta\rangle, \tag{7}$$

where θ satisfies the discrete condition

$$\theta = 2\pi r \quad (r \in \mathrm{R}) \tag{8}$$

and R is the rational number set. Combining Eq.(7)(8) with Eqs.(3)(4), we get the transition function between the number and the phase representation

$$\langle\theta|n\rangle = Ae^{-in\theta}, \tag{9}$$

where A is a normalization constant.

Before giving the correct normalization of the states $|\theta\rangle$, we make two preparations. Firstly, from the countability of the rational number set, all θ between θ_a and θ_b satisfying the discrete condition (8) can be numbered as $\theta_1,\theta_2,\cdots,\theta_i,\cdots$. We introduce a symbol called discrete integration indicated by $\int d_r\theta$ to represent the mean value of the function $f(\theta)$ over all θ between θ_a and θ_b satisfying the discrete condition, et.al. ,

$$\frac{1}{\theta_b-\theta_a}\int_{\theta_a}^{\theta_b} f(\theta)d_r\theta \overset{define}{=} \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\theta_i) = \overline{f(\theta)}, \tag{10}$$

The definition domain of the function $f(\theta)$ can be extended analytically to the real number set (the function after analytical extension is unique and will be still indicated by $f(\theta)$.) Because the rational number set is dense in the real number set, it is evident that

$$\frac{1}{\theta_b-\theta_a}\int_{\theta_a}^{\theta_b} f(\theta)d_r\theta = \overline{f(\theta)} = \frac{1}{\theta_b-\theta_a}\int_{\theta_a}^{\theta_b} f(\theta)d\theta \tag{11}$$

So the discrete integration can be expressed by the real integration. Secondly, we introduce a periodic δ – function (indicated by δ_T). The definition of δ_T is

$$\delta_T(\theta) = \delta_T(\theta+2\pi), \tag{12}$$

$$\delta_T(\theta) = 0 \quad (\theta \neq 2\mathrm{k}\pi, \mathrm{k} \in \mathrm{Z}), \tag{13}$$

$$\int_{-\pi}^{\pi} \delta_T(\theta)d\theta = 1. \tag{14}$$

The symbol Z in Eq.(13) represents the integer set. If the definition domain of the function $\delta_T(\theta)$ is limited to all θ satisfying the discrete condition, it becomes the periodic discrete δ – function and the integration in Eq.(14) should be replaced by the discrete integration.

Having these preparations, we can prove that the states $|\theta\rangle$ satisfy the normalization

$$\langle\theta_1|\theta_2\rangle = \delta_T(\theta_1-\theta_2) \tag{15}$$

and the complete equation

$$\int_{\theta_0}^{\theta_0+2\pi} |\theta\rangle\langle\theta| d_r\theta = \hat{I} \tag{16}$$

when we make the normalization constant

$$A = \frac{1}{\sqrt{2\pi}}. \tag{17}$$

Equation (15) and (16) suggest that the states $|\theta\rangle$ in an arbitrary 2π interval $[\theta_0, \theta_0 + 2\pi]$ satisfying the discrete condition make an orthogonal and normalized complete basis.

With above preparations, the phase operator now can be easily given. $\hat{\theta}$ corresponds to the infinitesimal displacement operator of the number basis and it has the following relation with the displacement operator $\hat{D}$

$$\hat{\theta} = \frac{1}{i}\ln \hat{D}. \tag{18}$$

So from Eq.(7) in the phase representation $\hat{\theta}$ can be expressed as

$$\hat{\theta} = \int_{\theta_0}^{\theta_0+2\pi} \theta|\theta\rangle\langle\theta| d_r\theta, \tag{19}$$

where θ_0 is arbitrary. The arbitrary θ_0 is merely the reflection of the periodic nature of the phase.

From the definition , we know the eigenvalues of $\hat{\theta}$ cannot be any real number, it must be 2π times a rational number. This confiction can be called phase quantization condition and its explicit form is given here for the first time. This condition suggests that the eigenvalues of $\hat{\theta}$ cannot change continuely though their change can be infinitesimal. This picture is different from that given by the classical phase, but it is natural and necessary. Here the phase operator is defined in a countable-infinite state space, in which the number of independent vectors cannot be beyond countable-infinite, but the eigenvectors of $\hat{\theta}$ with different eigenvalues are orthogonal and independent, so the eigenvalues of $\hat{\theta}$ cannot be continue and at most be countable-infinite. This leads to the phase quantization condition. The condition is very important for a self-consistence definition of the phase operator in infinite state spaces.

III. Phase-number commutator and expressions of the phase operator in some widely-used representations

Starting from the phase operator defined in the above section, we can give the expression of the number operator in the phase representation and the phase-number commutator. Firstly we introduce a symbol called discrete differentiation

$$\frac{\partial}{\partial\theta_r} f(\theta_r) \overset{define}{=\!=} \frac{\partial}{\partial x} f(x)\Big|_{x=\theta_r}, \tag{20}$$

where $f(x)$ is the analytical extension of $f(\theta_r)$ to the real number set . Then the number operator in the phase representation has a succinct form:

$$\hat{N} = \int_{\theta_0}^{\theta_0+2\pi} |\theta\rangle i\frac{\partial}{\partial\theta}\langle\theta| d_r\theta. \tag{21}$$

From Eq.(21) , we get the phase-number commutator

$$\left[\hat{N}, \hat{\theta}\right] = i\left[\hat{I} - 2\pi\delta_T\left(\hat{\theta} - \theta_0\right)\right]. \tag{22}$$

If we limit the phase value to $[\theta_0, \theta_0 + 2\pi]$ in the classical case , the commutator given by Eq.(22) just equals $i\hbar$ times the classical Possionian bracket[4]. This fact shows that the definition here is reasonable. The mean value of Eq.(22) over a physical state $|p\rangle$ gives the result obtained in Ref.[4]

$$\langle p|[\hat{N},\hat{\theta}]|p\rangle = i[1-2\pi P(\theta_0)], \tag{23}$$

where $P(\theta_0)=|\langle\theta_0|p\rangle|^2$ is the probability that the phase of the state is θ_0. The phase-number uncertainty relation is

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta\hat{\theta})^2\rangle \geq \frac{1}{4}[1-2\pi P(\theta_0)]^2. \tag{24}$$

Further we give direct expressions of the phase operator in the number and coherent representations. They have succinct and useful expressions which benefit from the fact that we have defined the phase operator directly in infinite state spaces.

In the number representation the phase operator have the following expression

$$\hat{\theta}=\sum_n |n\rangle\frac{1}{i}\frac{\partial}{\partial n}\langle n|, \tag{25}$$

where the symbol $\frac{\partial}{\partial n}$ represents the discrete diferentiation defined by Eq.(20). Noticing that $f(n)$ is equivalent to $f(n)e^{i2k\pi n}$ $(k \in Z)$ and $\frac{1}{i}\frac{\partial}{\partial n}f(n)$ is different from $\frac{1}{i}\frac{\partial}{\partial n}[f(n)e^{i2k\pi n}]$ with a difference $2k\pi$, we know that after $\frac{1}{i}\frac{\partial}{\partial n}$ acts on a function $f(n)$ there may appears a difference $2k\pi$. This fact also results from the periodic nature of the phase and we avoid the arbitary $2k\pi$ by limiting the mean value of $\hat{\theta}$ in Eq.(25) to $[\theta_0,\theta_0+2\pi]$.

Equation.(25) is very convenient for use because usual physical states are easy expanded by Fock states and then using Eq.(25) we can analyse phase properties of the states by simple differentiation.

Now we give an approximate form of the phase operator expressed by the annihilation and cretion operators $\hat{a},\hat{a}^+$ when the mean photon number $\langle\hat{N}\rangle >> 1$. The result is

$$\hat{\theta}\approx\frac{1}{i}:\ \ln\hat{a}-\ln(\hat{a}^+\hat{a}+1):, \tag{26}$$

where the symbol : : represents normal product. Using Eq.(26) we get the approximate expression of the phase operator in the coherent representation when the mean photon number is large.

$$\hat{\theta}=\frac{1}{\pi}\int|\alpha\rangle\frac{1}{i}[\ln\alpha-\ln(|\alpha|^2+1)]\langle\alpha|d^2\alpha. \tag{27}$$

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