Solutions of the Quantum Yang – Baxter Equations Assocated with (1 – 3/2)$^{-}$D Re presentations of SU_q (2)

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Abstract

The solutions of the spectral independent QYBE associated with (1 – 3/2) – D representations of SU_q (2) are derived, based on the weight conservation and extended Kauffman diagrammatic technique. It is found that there are nonstandard solutions.

1 Introduction

It is well known that the quantum Yang – Baxter equations (QYBE) play an important role in various theoretical and mathematical physics, such as completely integrable systems in (1 + 1) dimensions, exactly solvable models in statistical mechanics, the quamtum inverse scatteringmethod and the conformal filed theories in 2 – dimensions. [1]-[7] Recently, much remarkable progress has been made in construction the solutions of the QYBE associated with the representations of Lie algebras. [8]-[9] In this paper we derive the solutions of the spectral independent QYBE associated with (1 – 3/2) – D representations of SU_q (2) , based on the weight conservation and extended Kauffman diagrammatic technique. It is found that there are nonstandard solutions.

2 Braid relations of (1 – 3/2) – D representations of SU_q(2)

We know that there is the relation for Universal R – matrix:

$$R_{12}^{j_1j_2}\ R_{13}^{j_1j_3}\ R_{23}^{j_2j_3} = R_{23}^{j_2j_3}\ R_{13}^{j_1j_3}\ R_{12}^{j_1j_2} \qquad (2.1)$$

We define the new R – matrix:

$$\bar{R}^{j_2j_1} = PR^{j_1j_2} \qquad (2.2)$$

Where P is the transposition $(P:\ V^{j_1} \otimes V^{j_2} \rightarrow V^{j_2} \otimes V^{j_1})$

Then the eq. (2 – 1) can be rewritten ax follows

$$\bar{R}_{12}^{j_1j_2}\ \bar{R}_{23}^{j_1j_3}\ \bar{R}_{12}^{j_2j_3} = \bar{R}_{23}^{j_2j_3}\ \bar{R}_{12}^{j_1j_3}\ \bar{R}_{23}^{j_1j_2} \qquad (2.3)$$

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For the $(1-3/2)-D$ representation of $SU_q(2)$, $(j_1, j_2, j_3) \in (1, 1, 3/2)$, then eq. (2.3) gives the following relations

$$\bar{R}^{11}_{12} \bar{R}^{1\,3/2}_{23} \bar{R}^{1\,3/2}_{12} = \bar{R}^{1\,3/2}_{23} \bar{R}^{1\,3/2}_{12} \bar{R}^{11}_{23} \quad (2.4-1)$$

$$\bar{R}^{3/2\,1}_{12} \bar{R}^{11}_{23} \bar{R}^{11}_{12} = \bar{R}^{11}_{23} \bar{R}^{3/2\,1}_{12} \bar{R}^{3/2\,1}_{23} \quad (2.4-2)$$

$$\bar{R}^{1\,3/2}_{12} \bar{R}^{11}_{23} \bar{R}^{3/2\,1}_{12} = \bar{R}^{3/2\,1}_{23} \bar{R}^{11}_{12} \bar{R}^{1\,3/2}_{23} \quad (2.4-3)$$

These are the braid relations associated $(1-3/2)-D$ representations of $SU_q(2)$. We suppose that the $\bar{R}$ satisfies the C−P invarance, then eq. (2.4−1) is equal to eq. (2.4−2).

3 The weight conservation and the solutions of QYBE

To determine the structure for the solutions, We consider the weight conservation

$$(\bar{R})^{ab}_{cd} = 0 \qquad \text{unless } a+b=c+d \qquad (3.1)$$

where

$$\bar{R} = \bar{R}^{1\,3/2}, \quad \bar{R}^{3/2\,1}, \quad \bar{R}^{11}$$

$$a, b, c, d \in (\pm 3/2, \pm 1/2, \pm 1, 0)$$

It is well known that $\bar{R}^{11}$ which satisfied the conditions of c−p invarance and eq. (3.1) be written as

$$\bar{R}^{11} = \sum_a u_a E_{aa} \otimes E_{aa} + \sum_{a<b} W^{(a\ b)} E_{ab} \otimes E_{ab} + \sum_{a \pm b} P^{(a,\ b)} E_{ab} \otimes E_{ba}$$

$$+ \sum_{a \leqslant b} q^{(a,\ c)}_{a+b} (E_{ab} \otimes E_{cd} + E_{cd} \otimes E_{ab}) \qquad (3,\ 2)$$

Where

$$u_0 = 1, \quad u_{\pm 1} = q^2, \quad p^{(0,\ 1)} = p^{(1,\ 0)} = 1, \quad p^{(+1,\ +1)} = q^{-2}$$

$$w^{(0,\ 1)} = w = q^2 - q^{-2}, \quad w^{(-1,\ 1)} = (1-q^{-2})\ w, \quad q_0^{(-1,\ 0)} = q_0^{(0,\ -1)} = q^{-1} w \qquad (3.3)$$

By the weight conservation $\bar{R}^{1\,3/2}$ can be constructed in the form

$$\bar{R}^{1\,3/2} = \sum_{a,b} p^{a,b}_{a+b} \; E_{ab} \otimes E_{ba} + \sum_{\substack{a<d \\ c<d}} q^{a,c}_{a+b} \; E_{ac} \otimes E_{bd} \tag{3.4}$$

Where

$a, b \in (\pm 1, 0)$; $b, c \in (\pm 3/2, \pm 1/2)$ $P^{(a\,b)}_{a+b}$ and $q^{(a,\,c)}_{a+b}$

are the determined parameters.

Substituting eq. (3.2), (3.4) into eq. (2.4-1.3), We obtain the unknown parameters by extended Karffman diagrammatic techique.

$$P^{(1\;\;3/2)}_{5/2} = q^3, \quad P^{(-1,\,-3/2)}_{-5/2} = P^{(1,\,3/2)}_{5/2} Q = q^3 Q$$

$$P^{(-1,-1/2)}_{-3/2} = P^{(1,\,1/2)}_{3/2} Q = qQ, \quad P^{(0,\,b)}_{b} = Q^{1/2} \tag{3.5-1}$$

$$q^{(0,1/2)}_{3/2} = Q^{-1/2} q^{(-1,-3/2)}_{3/2} = (1-q^{-2})\, q^{3/2} \; ([3]!)^{1/2} \, Q^{1/4}$$

$$q^{(-1,\,1/2)}_{1/2} = Q^{1/2} \; q^{(0,-\,3/2)}_{-1/2} = (1-q^{-2})\, q^{-1/2} \; ([3]!)^{1/2} \, Q^{1/4}$$

$$q^{(0,-1/2)}_{1/2} = Q^{-1/2} \; q^{(-1,-1/2)}_{1/2} = (1-q^{-2})\, q^{1/2} \; ([2]!)^{3/2} \, Q^{-1/4}$$

$$q^{(-1,-1/2)}_{1/2} = q^{(-1,-3/2)}_{-1/2} = (1-q^{-2})\, q \; ([2]![3]!)^{1/2} Q^{3/4} \tag{3.5-2}$$

Where

$$[u] \equiv \frac{q^u - q^{-u}}{q - q^{-1}}, \quad [u]! \equiv [u]\,[u-1]....[1], \quad [0]! \equiv 1 \tag{3.5-3}$$

Substituting eq.(3.5) into eq.(3.4), we obtain the solutions $\bar{R}^{1\,3/2}$. And we obtain the solutions $\bar{R}^{3/2\;1}$ by employing the c-p invarance.

We have derived the solutions of the spectral independent QYBE associated with (1-3/2)-D representations. It is easy to see that there is a new arbitary parameter, Q, then there are new solutions. In fact when Q=1, the solutions is Universal R-matrix of SU_q (2).

$$(\bar{R}^{j_2 j_1})^{m_2 m_1}_{m_1 m_2} = \delta^{m_1'+m_2'}_{m_1+m_2} \frac{[\,(1-q^{-2})^{m_1'-m_1}\,]}{[m_1'-m_1]\,!} \, q^{m_1 m_2' + m_2 m_1' \; -1/2 \; (m_1'-m_1)(m_1'-m-1)}$$

$$\left\{ \frac{(j_1+m_1')!\;(j_1-m_1)!\,(j_1-m_2')\;!\;(j_2+m_2)!}{(j_1-m_1')!\;(j_1+m_1)!\;(j_2+m_2')!\;(j_2-m_2)!} \right\}^{1/2} \tag{3.6}$$

Standard solutions . When $Q \neq 1$, there are new solutions .

Referene

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