QUANTIZATION OF ELECTROMAGNETIC FIELDS IN CAVITIES

Kiyotaka Kakazu and Kazunori Oshiro
Department of Physics, University of the Ryukyus, Okinawa 903-01, Japan

Abstract

A quantization procedure for the electromagnetic field in a rectangular cavity with perfect conductor walls is presented, where a decomposition formula of the field plays an essential role. All vector mode functions are obtained by using the decomposition. After expanding the field in terms of the vector mode functions, we get the quantized electromagnetic Hamiltonian.

1 Introduction

Recently we have carried out the field quantization in several rectangular cavities using the vector mode functions [1]. The vector mode functions have been obtained with the help of an orthogonal matrix. However, the procedure developed there has not been applicable to other cavities in a straightforward manner.

To overcome the above difficulty, we have presented another quantization scheme for the field in a circular cylindrical cavity [2]. All vector mode functions have been obtained by using a decomposition formula derived from Maxwell's equations directly. This method is more general than before, because it is also applicable to rectangular and spherical cavities.

In this paper, we applied the above method to a rectangular cavity with perfect conducting walls. Then spontaneous emission of an atom inside the cavity is calculated.

2 Decomposition of Electromagnetic Fields

In this section, we derive the decomposition formula for the field in the Cartesian coordinates. We shall use this result in performing the field quantization in a rectangular cavity in Sec. 3.

Maxwell's equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in free space are given by $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, and

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = 0, \tag{1}$$

where c is the velocity of light in free space and $\partial_t = \partial/\partial t$.

The electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ can be written in the Cartesian coordinates (x, y, z) as $\mathbf{E} = \mathbf{E}_T + \mathbf{E}_z$, $\mathbf{B} = \mathbf{B}_T + \mathbf{B}_z$, where $\mathbf{E}_T = \mathbf{e}_x E_x + \mathbf{e}_y E_y$ is the transverse component of the field and $\mathbf{E}_z = \mathbf{e}_z E_z$. Here $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors in the x, y, and z directions, respectively. For simplicity, the derivative with respect to, for example, x is described as $\partial/\partial x = \partial_x$.

Since $\nabla_z \times \mathbf{E}_z = 0$, the first equation in (1) gives

$$\partial_t \mathbf{B}_z = -\nabla_T \times \mathbf{E}_T, \quad \partial_t \mathbf{B}_T = -\nabla_T \times \mathbf{E}_z - \nabla_z \times \mathbf{E}_T. \tag{2}$$

Similarly, the second equation in (1) leads to

$$\frac{1}{c^2}\partial_t \mathbf{E}_z = \nabla_T \times \mathbf{B}_T, \quad \frac{1}{c^2}\partial_t \mathbf{E}_T = \nabla_T \times \mathbf{B}_z + \nabla_z \times \mathbf{B}_T. \tag{3}$$

Equations (2) and (3) give

$$\triangle_T \mathbf{E} = -\nabla \times \nabla \times \mathbf{E}_z + \nabla \times \partial_t \mathbf{B}_z, \quad \triangle_T \mathbf{B} = -\frac{1}{c^2}\nabla \times \partial_t \mathbf{E}_z - \nabla \times \nabla \times \mathbf{B}_z. \tag{4}$$

To rewrite Eq. (4), we must decompose the components E_z and B_z into two parts. Suppose that the field is in a finite region. Let us expand E_z and B_z in terms of a certain complete system of functions with mode s:

$$E_z(\mathbf{r},t) = \sum_s (E_{zs}(\mathbf{r},t) + \text{c.c.}), \quad B_z(\mathbf{r},t) = \sum_s (B_{zs}(\mathbf{r},t) + \text{c.c.}), \tag{5}$$

where $E_{zs}(\mathbf{r},t) = \tilde{E}_{zs}(\mathbf{r})e^{-i\omega_{s1}t}$ and $B_{zs}(\mathbf{r},t) = \tilde{B}_{zs}(\mathbf{r})e^{-i\omega_{s2}t}$. Here $\omega_{s\sigma}$ ($\omega_{s\sigma} \geq 0$, $\sigma = 1,2$) is determined by using given boundary conditions. Since E_z and B_z satisfy the wave equation, the components E_{zs} and B_{zs} satisfy the Helmholts equations:

$$\triangle E_{zs} = -k_{s1}^2 E_{zs}, \quad \triangle B_{zs} = -k_{s2}^2 B_{zs}(\mathbf{r}), \tag{6}$$

where $k_{s\sigma}^2 = \omega_{s\sigma}^2/c^2$. We assume that the components satisfy

$$\partial_z^2 E_{zs} = -h_{s1}^2 E_{zs}, \quad \partial_z^2 B_{zs} = -h_{s2}^2 B_{zs}, \tag{7}$$

where $h_{s\sigma}^2$ is determined by the boundary conditions. Then we have two dimensional Helmholtz equations:

$$\triangle_T E_{zs}(\mathbf{r}) = -g_{s1}^2 E_{zs}(\mathbf{r}), \quad \triangle_T B_{zs}(\mathbf{r}) = -g_{s2}^2 B_{zs}(\mathbf{r}), \tag{8}$$

where $g_{s\sigma}^2 = k_{s\sigma}^2 - h_{s\sigma}^2$.

Here we define two functions F_σ from E_{zs} and B_{zs} with $g_{s\sigma}^2 \neq 0$ as

$$F_\sigma(\mathbf{r},t) = \sum_{g_{s\sigma}^2 \neq 0} [F_{s\sigma}(\mathbf{r},t) + \text{c.c.}] = \sum_{g_{s\sigma}^2 \neq 0} [\tilde{F}_{s\sigma}(\mathbf{r})\, e^{-i\omega_s t} + \text{c.c.}], \tag{9}$$

where

$$F_{s1} = E_{zs}/g_{s1}^2, \quad F_{s2} = B_{zs}/g_{s2}^2, \qquad (g_{s\sigma}^2 \neq 0). \tag{10}$$

The functions F_σ and their components $F_{s\sigma}$ satisfy the same equations as E_z and E_{zs}, respectively. The component F_σ is a solution of the Poisson equation. On the other hand, if there is a component E_{zs} or B_{zs} with $g_{s\sigma}^2 = 0$, Eq. (8) reduces to two dimensional Laplace equation.

The functions $\triangle_T F_\sigma$ satisfy

$$-\triangle_T F_1 = \sum_{g_{s1}^2 \neq 0} g_{s1}^2 F_{s1} = \sum_{g_{s1}^2 \neq 0} E_{zs}, \quad -\triangle_T F_2 = \sum_{g_{s2}^2 \neq 0} g_{s2}^2 F_{s2} = \sum_{g_{s2}^2 \neq 0} B_{zs}. \tag{11}$$

Here define E_{0z} and B_{0z} as

$$E_{0z} = E_z + \triangle_T F_1 = \sum_{g_{s1}^2 = 0} E_{zs}, \quad B_{0z} = B_z + \triangle_T F_2 = \sum_{g_{s2}^2 = 0} B_{zs}, \tag{12}$$

which satisfy $\triangle_T E_{0z} = 0$ and $\triangle_T B_{0z} = 0$. Then we have useful formulas for E_z and B_z:

$$E_z = -\triangle_T F_1 + E_{0z}, \quad B_z = -\triangle_T F_2 + B_{0z}. \tag{13}$$

Using Eq. (13) and defining $\mathbf{F}_\sigma$ as $\mathbf{F}_\sigma = \mathbf{e}_z F_\sigma$, we can rewrite Eq. (4) as

$$\triangle_T\Big(\mathbf{E} - \nabla \times \nabla \times \mathbf{F}_1 + \nabla \times \partial_t \mathbf{F}_2\Big) = -\nabla \times \nabla \times \mathbf{E}_{0z} + \nabla \times \partial_t \mathbf{B}_{0z}, \tag{14}$$

$$\triangle_T\left(\mathbf{B} - \frac{1}{c^2}\nabla \times \partial_t \mathbf{F}_1 - \nabla \times \nabla \times \mathbf{F}_2\right) = -\frac{1}{c^2}\nabla \times \partial_t \mathbf{E}_{0z} - \nabla \times \nabla \times \mathbf{B}_{0z}, \tag{15}$$

where $\mathbf{E}_{0z} = \mathbf{e}_z E_{0z}$ and $\mathbf{B}_{0z} = \mathbf{e}_z B_{0z}$. Define $\mathbf{E}_0$ and $\mathbf{B}_0$ as the quantities in the parentheses at the left hand side in Eqs. (14) and (15), respectively. The results of this section is summarized in the following theorem.

Theorem 1: If the components E_{zs} and B_{zs} satisfy Eq. (7), the field can be decomposed into three components as follows:

$$\mathbf{E} = \nabla \times \nabla \times \mathbf{F}_1 - \nabla \times \partial_t \mathbf{F}_2 + \mathbf{E}_0, \quad \mathbf{B} = \frac{1}{c^2}\nabla \times \partial_t \mathbf{F}_1 + \nabla \times \nabla \times \mathbf{F}_2 + \mathbf{B}_0, \tag{16}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfy

$$\triangle_T \mathbf{E}_0 = -\nabla \times \nabla \times \mathbf{E}_{0z} + \nabla \times \partial_t \mathbf{B}_{0z}, \quad \triangle_T \mathbf{B}_0 = -\frac{1}{c^2}\nabla \times \partial_t \mathbf{E}_{0z} - \nabla \times \nabla \times \mathbf{B}_{0z}. \tag{17}$$

Theorem 1 plays a central role in performing the field quantization in this paper. It is worth emphasizing that each term in Eq. (16) is a solution to Maxwell's equations.

3 Vector Mode Functions and Field Quantization

The cavity we treat here is enclosed by rectangular walls having sides L_x, L_y, and L_z in the x, y, and z directions, respectively: $0 < x < L_x$, $0 < y < L_y$, and $0 < z < L_z$. We assume that the cavity has perfectly conducting walls. The tangential component of the electric field $\mathbf{E}|_{\tan}$ and the normal component of the magnetic field $\mathbf{B}|_{\mathrm{norm}}$ must accordingly vanish at the boundaries of the cavity.

The above boundary condition reduces to that for the z components

$$E_z = 0, \quad \partial_x B_z = 0, \qquad (x = 0,\ L_x), \tag{18}$$
$$E_z = 0, \quad \partial_y B_z = 0, \qquad (y = 0,\ L_y), \tag{19}$$
$$B_z = 0, \quad \partial_z E_z = 0, \qquad (z = 0,\ L_z). \tag{20}$$

The solution to the Helmholts equation (6) for the components E_{zs} and B_{zs} under the above boundary conditions is given by

$$\begin{aligned} E_{zs}(\mathbf{r},t) &= C_{s1}(t)\sin(\ell\pi x/L_x)\sin(m\pi y/L_y)\cos(n\pi z/L_z), \\ B_{zs}(\mathbf{r},t) &= C_{s2}(t)\cos(\ell\pi x/L_x)\cos(m\pi z/L_z)\sin(n\pi z/L_z), \end{aligned} \tag{21}$$

where the mode index is $s = (\ell, m, n)$ $(\ell, m, n = 0, \pm 1, \pm 2, \cdots)$.

From the solution (21) we have

$$\begin{aligned} g_{s\sigma}^2 &\equiv g_s^2 = (\ell\pi/L_x)^2 + (m\pi/L_y)^2, \\ k_{s\sigma}^2 &\equiv k_s^2 = (\ell\pi/L_x)^2 + (m\pi/L_y)^2 + (n\pi/L_z)^2. \end{aligned} \tag{22}$$

Consequently, we can use Theorem 1 in the preceding section. Although it follows from Eq. (22) that $g_{s\sigma}^2 \geq 0$, we can prove that $g_{s\sigma}^2 > 0$, which results from the following lemma. We omit its proof.

Lemma: $\mathbf{E}_0$, $\mathbf{B}_0 = 0$ and $g_{s\sigma}^2 > 0$. As a result, the term with $\ell = m = 0$ in Eq. (21) cannot be used.

Let us next obtain the functions F_σ, whose definitions are given in Eqs. (9) and (10). That is, the functions are given by

$$\begin{aligned} F_{s1}(\mathbf{r},t) &= \frac{E_{zs}(\mathbf{r},t)}{g_s^2} \equiv i\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0}}a_{s1}(t)\psi_{s1}(\mathbf{r}), \\ F_{s2}(\mathbf{r},t) &= \frac{B_{zs}(\mathbf{r},t)}{g_s^2} \equiv i\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0}}a_{s2}(t)\psi_{s2}(\mathbf{r}), \end{aligned} \tag{23}$$

where $\omega_{s\sigma} \equiv \omega_s$ and we have introduced $a_{s\sigma}(t) = a_{s\sigma}(0)e^{-i\omega_s t}$ and $\psi_{s\sigma}$ given by

$$\psi_{s1}(\mathbf{r},t) = c_{s1}\sin(\ell\pi x/L_x)\sin(m\pi y/L_y)\cos(n\pi z/L_z), \tag{24}$$
$$\psi_{s2}(\mathbf{r},t) = c_{s2}\cos(\ell\pi x/L_x)\cos(m\pi y/L_y)\sin(n\pi z/L_z), \tag{25}$$

with $c_{s1} = [8/(Vk_s^2 g_s^2)]^{1/2}$ and $c_{s2} = [8/(V\omega_s^2 g_s^2)]^{1/2}$. The functions $\psi_{s\sigma}$ has the orthonormality property

$$\int_{\mathrm{c}} d\mathbf{r}\, \psi_{s\sigma}^*(\mathbf{r})\psi_{s'\sigma}(\mathbf{r}) = \frac{1}{8}|c_{s\sigma}|^2 V\delta_{ss'}, \tag{26}$$

where $\int_{\mathrm{c}} d\mathbf{r} = \int_{\mathrm{cavity}} dxdydz$ and V is the cavity volume. Here the quantity $\cos x$ in Eqs. (24) and (25) must be changed to $1/\sqrt{2}$ when $x = 0$.

Substituting the functions F_σ in Eq. (23) into $\mathbf{E}$ in Eq. (16), we find

$$\mathbf{E}(\mathbf{r},t) = i\sum_{s\sigma}\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0}}\Big[a_{s\sigma}(t)\mathbf{u}_{s\sigma}(\mathbf{r}) - a_{s\sigma}^*(t)\mathbf{u}_{s\sigma}^*(\mathbf{r})\Big], \tag{27}$$

where the vector mode functions $\mathbf{u}_{s\sigma}$ are given by

$$\mathbf{u}_{s1} = \nabla \times \nabla \times \mathbf{e}_z \psi_{s1}, \quad \mathbf{u}_{s2} = i\omega_s \nabla \times \mathbf{e}_z \psi_{s2}. \tag{28}$$

The vector mode functions satisfy $\nabla \cdot \mathbf{u}_{s\sigma} = 0$, $(\Delta + k_{s\sigma}^2)\mathbf{u}_{s\sigma} = 0$, and at the boundaries $\mathbf{u}_{s\sigma}|_{\tan} = 0, \nabla \times \mathbf{u}_{s\sigma}|_{\mathrm{norm}} = 0$. They also satisfy the orthonormality property needed for quantization:

$$\int_c d\mathbf{r}\, \mathbf{u}_{s\sigma}^*(\mathbf{r}) \cdot \mathbf{u}_{s'\sigma'}(\mathbf{r}) = \delta_{ss'}\, \delta_{\sigma\sigma'}. \tag{29}$$

To get the quantized field, the functions $a_{s\sigma}(t)$ are regarded as annihilation operators satisfying the commutation relation $[a_{s\sigma}(t), a_{s'\sigma'}^\dagger(t)] = \delta_{ss'}\delta_{\sigma\sigma'}$. Then we get the following theorem.

Theorem 2: The quantized field and the Hamiltonian are given by

$$\mathbf{E}(\mathbf{r},t) = i\sum_{s\sigma}\sqrt{\frac{\hbar\omega_s}{2\varepsilon_0}}\Big[a_{s\sigma}(t)\mathbf{u}_{s\sigma}(\mathbf{r}) - a_{s\sigma}^\dagger(t)\mathbf{u}_{s\sigma}^*(\mathbf{r})\Big], \tag{30}$$

$$\mathbf{B}(\mathbf{r},t) = \sum_{s\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_s}}\Big[a_{s\sigma}(t)\nabla \times \mathbf{u}_{s\sigma}(\mathbf{r}) + a_{s\sigma}^\dagger(t)\nabla \times \mathbf{u}_{s\sigma}^*(\mathbf{r})\Big], \tag{31}$$

$$H_R = \sum_{s\sigma}\frac{1}{2}\hbar\omega_s\left(a_{s\sigma}^\dagger a_{s\sigma} + a_{s\sigma}a_{s\sigma}^\dagger\right) = \sum_{s\sigma}\hbar\omega_s\left(a_{s\sigma}^\dagger a_{s\sigma} + \frac{1}{2}\right). \tag{32}$$

4 Spontaneous Emission

As an application, we consider the transition rates of an atom in the cavity, using the dipole approximation. The Hamiltonian is given by $H = H_A + H_R + H_I$, where H_A is the free Hamiltonian for the atom, H_R for the field which is given in Eq. (32), and $H_I = e\mathbf{D} \cdot \mathbf{E}(\mathbf{R})$ ($-e\mathbf{D}$: the total electric dipole moment of the atom; $\mathbf{R} = (X, Y, Z)$: the position of the atom).

At $t = 0$, the atom is in an energy state $|i_0>$ (with energy E_{i_0}) and the field is in the vacuum. Then the probability per second of finding the atom in a state (with energy E_i) at sufficiently large time t is given by

$$w = \frac{e^2\pi}{\varepsilon_0\hbar}\sum_{s\sigma}|\mathbf{u}_{s\sigma}(\mathbf{R})\cdot <i_0|\mathbf{D}|i>|^2\frac{\omega_s\sin(\omega_s - \omega_0)t}{\pi(\omega_s - \omega_0)}, \tag{33}$$

where $\hbar\omega_0 = E_{i_0} - E_i$.

Let us take the average of the coordinates Y and Z and take $L_z \to \infty$. The transition rates w_y and w_z vanish (w_i indicates the rate where the dipole moment is along with the i direction). The rate w_x is given by

$$w_x/w_0 = \frac{6}{\pi\xi_y}\sum_m(1 - m^2/\xi_y^2)^{-1/2}\theta(1 - m^2/\xi_y^2), \tag{34}$$

where $\xi_i = \omega_0 L_i/c\pi$, $\xi_x = 1/2$, $w_0 = e^2| <i_0|\mathbf{D}|i>|^2\omega_0^3/(3\pi\hbar\varepsilon_0 c^3)$, $\theta(x) = 1$ $(x > 0)$, and

$\theta(x)$ 0 ($x < 0$), which is shown in FIG. 1.

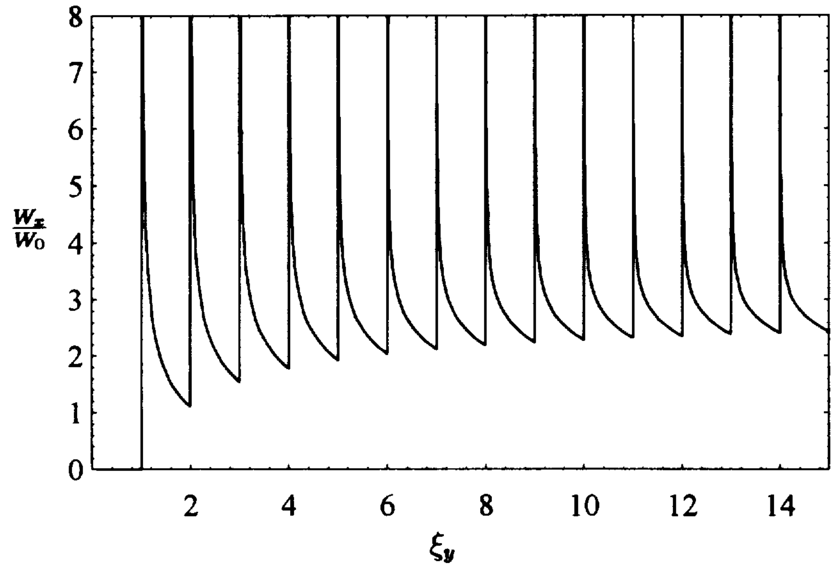

FIG. 1. Transition rate w_x for $L_z \to \infty$, where the dipole moment is along with the x direction.

Setting here $\omega_0 = 10^{13}$ Hz, we have $L_x = 4.7 \times 10^{-2}$ mm, so that the cavity is quite narrow. Also, FIG. 1 shows that the transition is forbidden when $\xi_y < 1$, i.e., $L_y < 9.4 \times 10^{-2}$ mm., where the cavity is a thin tube in this case.

5 Conclusions

The quantization for the field in the cavity has been performed as follows: obtain the decomposition formula (16) in the Cartesian coordinates; solve the Helmholts equations (6) for the components E_{zs} and B_{zs} under the boundary conditions; determine the functions F_σ, substitute them into the decomposition formula (16), and obtain the vector mode functions satisfying the orthonormality property (29); then we arrive at the quantized field and Hamiltonian. In the whole process of quantization, the decomposition formula in Theorem 1 plays an important role.

Acknowledgments

We would like to thank Prof. M. Namiki, Prof. S. Matsumoto, Dr. A. Vourdas, and Dr. S. Kudaka for numerous valuable discussions.

References

[1] K. Kakazu and Y. S. Kim, Phys. Rev. A **50**, 1830 (1994).

[2] K. Kakazu and Y. S. Kim, preprint DPUR-84.