EXPONENTIAL FORMULAE AND EFFECTIVE OPERATIONS

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Abstract

One of standard methods to predict the phenomena of squeezing consists in splitting the unitary evolution operator into the product of simpler operations (Yuen [1], Ma and Rhodes [2]). The technique, while mathematically general, is not so simple in applications and leaves some pragmatic problems open. We report an extended class of exponential formulae, which yield a quicker insight into the laboratory details for a class of squeezing operations, and moreover, can be alternatively used to programme different type of operations, as: 1) the free evolution inversion, 2) the soft simulations of the sharp kicks (so that all abstract results involving the kicks of the oscillator potential, become realistic laboratory prescriptions).

1 The manipulation problem

Below, we shall dissent from the orthodox subject of "squeezed states" and dedicate some attention to a more general problem. Suppose, one has a quantum system whose states are represented by vectors in a Hilbert space $\mathcal{H}$. Now, choose any unitary operator

$$U : \mathcal{H} \rightarrow \mathcal{H} \tag{1}$$

Can U be achieved as a realistic evolution operation, performed under the influence of some external fields?

The problem so stated, belongs to the *quantum manipulation theory*, a domain which has progressed quickly in the last decades. The first cases of the dynamical manipulation (for a finite dimensional space of states) achieved wide publicity under the name of the *spin echo* (e.g. [3]).

The general problem of manipulation (control) of quantum states dates from the works of Lamb [4], Lubkin [5] and followers [6, 7, 8, 9, 10]. Quite independently, the subject has been put forward in quantum chemistry where it may soon become crucial [11, 12]. For an infinite dimensional $\mathcal{H} = L^2(\mathbf{R})$ some dynamical operations present a considerable challenge but only one of them has become a "conference subject". We of course refer to the *operation of squeezing*:

$$U = e^{(za^{\dagger 2} - z^* a^2)/2} \qquad \text{(general squeezing)} \qquad z \in \mathbf{C} \tag{2}$$

and/or

$$U = e^{i\lambda(qp+pq)/2} \qquad \begin{array}{l}\text{(scale transformation,} \quad \lambda \in \mathbf{R} \\ \text{coordinate squeezing)}\end{array} \tag{3}$$

Note, that there are several concepts of squeezing in the literature. By choosing (2-3) we ask about the "operatorial squeezing", i.e. the shape transformation which affects all wave packets alike, independently on their initial form. Thus, under the influence of (3) the canonical observables q, p are transformed into

$$\begin{aligned} U^\dagger q U &= e^{-\lambda} q \\ U^\dagger p U &= e^{\lambda} p \end{aligned} \tag{4}$$

and simultaneously all the wave packets $\psi = \{\psi(x)\}$ are deformed as:

$$(U\psi)(x) = \sqrt{k}\ \psi(kx), \quad k = e^{\lambda} \tag{5}$$

As found by Yuen [1], the simplest method of producing such effects in $L^2(\mathbf{R})$ consists in application of variable oscillator potentials with the time dependent Hamiltonians:

$$H(t) = \frac{p^2}{2} + \omega(t)^2 \frac{q^2}{2}; \quad [q, p] = i \tag{6}$$

and the most explicit illustrations of this fact can be found in the *exponential formulae*, which express the evolution operator $U(t)$ [generated by (6)] as the product of simpler exponential operations.

The very subject of the exponential identities has already some antiquity, starting from the papers of Zassenhaus, Baker, Campbell and Haussdorff (BCH) [13]. However, the exponential identities of BCH type involve infinite series and do not offer closed solutions. The key to the techniques of squeezing are the following formulae of Yuen [1] and Ma and Rhodes [2], which might be interpreted as *exactly soluble cases* of BCH and Zassenhauss. If no linear terms in $H(t)$ are present, they read:

$$U(t) \equiv e^{B(t)a^{\dagger 2}} e^{\Omega(t)a^\dagger a} e^{E(t)a^2} \qquad \text{(Yuen, 1976)} \tag{7}$$

and

$$U(t) \equiv e^{[z(t)a^{\dagger 2}-z^*(t)a^2]/2} e^{-i\alpha(t)a^\dagger a} \qquad \text{(Ma and Rhodes, 1988)} \tag{8}$$

where $B(t)$, $\Omega(t)$, $E(t)$, $z(t)$, $\alpha(t)$ are c-number coefficients and $\equiv$ means the proporcionality of the unitary operators ($U \equiv U' \Rightarrow U = e^{i\alpha}U'$, $\alpha \in \mathbf{R}$). These identities precisely provide the proof that the variable oscillator potentials (more generally: quadratic, time dependent Hamiltonians) can produce the effects of squeezing (2) (or the scale transformation (3)). A particularly simple identity for two oscillator Hamiltonians $H_j = p^2/2 + \omega_j^2 q^2/2$ acting during two time lapses τ_1, τ_2, was detected by Grübl [14]. If the time intervals τ_1, τ_2 are in adequate proportion to the frequencies ω_1, ω_2 (e.g., $\omega_1\tau_1 = \pi/2$, $\omega_2\tau_2 = 3\pi/2$), then:

$$e^{i\lambda(qp+pq)/2} \equiv e^{-i\tau_2 H_2} e^{-i\tau_1 H_1}, \tag{9}$$

where $\lambda = \ln \omega_2/\omega_1$. (Note, that Grübl had no confidence to the operatorial formulae. He has proved (9) implicitely, working with Gaussian packets).

While mathematically complete, (7-8) are not quite easy to apply, due to involved systems of non-linear equations for the c-number coefficients. This explaines a quick development of alternative methods derived from evolution matrices or adiabatic invariants [15, 16]. Yet, the "damped oscillator" of 1940 [17], and the "step-Hamiltonian" of Grübl (9) remain the principal cases solved with all numerical details.

It will be our purpose to show that the trend of the algebraic identities (7- 8) is not at all exhausted! To the contrary, appart of (7-8), it can provide a class of "easy formulae" for the squeezing and for more general control operations.

2 The spin echo without spin

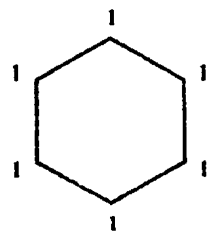
The first "easy formula" (accidentally detected in 1977 [6]) has the form of the "circular identity":

$$\underbrace{e^{-ip^2/2} e^{-iq^2/2} e^{-ip^2/2} \cdots e^{-iq^2/2}}_{12 \text{ terms}} \equiv 1 \tag{10}$$

for the operators q, p in $L^2(\mathbf{R})$, with $[q,p] = i$. Note, that all signs in the exponents are the same: the product (10) is therefore simpler than it could be in the classical case! The formula (10) has an elementary operational sense. Every operator $\exp[-ip^2/2]$ represents the free evolution per unit time of the Schrödinger's particle in $L^2(\mathbf{R})$. Every $\exp[-iq^2/2]$ is the unitary evolution operation caused by infinitely sharp and quick pulse of time dependent oscillator potential (the "δ-like kick" of the ellastic force):

$$e^{-iq^2/2} = \lim_{\epsilon\to 0} e^{-i\epsilon[p^2/2+\epsilon^{-1}q^2/2]} \tag{11}$$

The identity (10) describes a dynamical "evolution loop": the wave packet in $L^2(\mathbf{R})$, manipulated by 6 oscillator kicks and 6 free evolution intervals must return to its initial state (no matter what this state was!). This might be illustrated by the following closed diagramme:



whose sides simbolize the free evolution intervals and vertices the oscillator kicks. An immediate consequence of (10) is:

$$e^{+ip^2/2} = \underbrace{e^{-iq^2/2}e^{-ip^2/2}\cdots e^{-ip^2/2}e^{-iq^2/2}}_{11} \tag{12}$$

The right hand side represesents a sequence of admissible dynamical events (6 kicks and 5 rest intervals), while the left one is the operator inverse to the free evolution. The formula (12) thus tells how to invert the free evolution. Since (12) is an operator identity, the prescription can be applied "in blind": every wave packet in $L^2(\mathbf{R})$, entertained by 11 dynamical events must "go back in time", returning to its past shape, no matter what this shape was [8]. (Compare "Particle Memory" of Brewer and Hahn [18].)

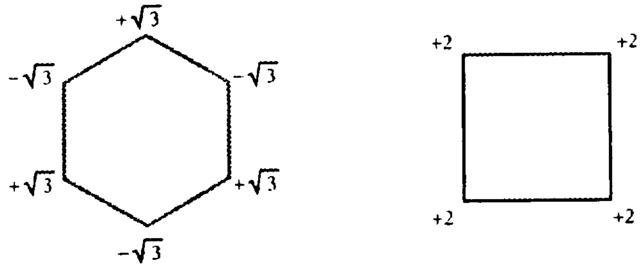
After some consideration, the formula (10) looses a part of mystery: it is just a "discrete imitation" of the oscillator force (the oscillator potential acts only in selected time moments, producing nonetheless a closed dynamical process). Note however the existence of other "circular identities" [8]:

$$\underbrace{e^{-ip^2/2}e^{-i\sqrt{3}q^2/2}e^{-ip^2/2}e^{+i\sqrt{3}q^2/2}\cdots e^{+i\sqrt{3}q^2/2}}_{12} \equiv 1 \tag{13}$$

The left hand side represents a sandwich of the 3 attractive and 3 repulsive pulses interrupted by 6 free evolution intervals. One might expect that the attractive and repulsive shocks will cancel "in average", producing a zig-zag equivalent of the free evolution. However, it is not the case. The whole sequence traps the Schrödinger's packets into a closed dance, with the evolution operator $\equiv 1$. Note furthermore:

$$\left(e^{-ip^2/2}e^{-iq^2}\right)^4 \equiv 1, \quad \text{(4 shocks, 4 free evolutions)} \tag{14}$$

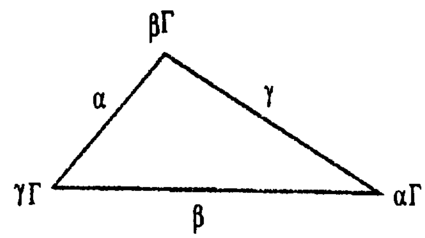
Both formulae (13-14) can be illustrated by graphs:



The simplest loop in $L^2(\mathbf{R})$ must involve at least 3 kicks and 3 rest intervals; its general form is:

$$e^{-i\gamma p^2/2}e^{-i\alpha\Gamma q^2/2}e^{-i\beta p^2/2}e^{-i\gamma\Gamma q^2/2}e^{-i\alpha p^2/2}e^{-i\beta\Gamma q^2/2} \equiv 1 \tag{15}$$

$$\Gamma = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$$



Its "incomplete version":

$$e^{+i\gamma p^2/2} \equiv e^{-i\alpha\Gamma q^2/2}e^{-i\beta p^2/2}e^{-i\gamma\Gamma q^2/2}e^{-i\alpha p^2/2}e^{-i\beta\Gamma q^2/2} \tag{16}$$

permits one to manipulate the free evolution at will. Thus, for $\alpha, \beta, \gamma > 0$, (16) provides a prescription of how to enforce the Schrödinger's wave packet to "go back in time", whereas for $\gamma < 0,\ \alpha, \beta > 0$ one obtains a "time machine" able to slow or accelerate the free evolution [8, 10].

The loop formulae are the obvious analogue of the spin-echo for non-spin states. As far as we could check, the possibility of the (non-adiabatic) loop effects in $L^2(\mathbf{R})$ was first predicted in 1970 (by reinterpreting the transparency phenomenon of the potential wells; see Malkin and Man'ko [19], p. 388), though the subject was later pursued in a different direction. The first kicked system was considered in 1977 [6] and the manipulation of quantum states by potential pulses was systematically studied since 1986 [8, 9, 10].

The exponential identities suggest also how to generate the *scale transformation*. The simplest formula requires again a pair of oscillator pulses of different amplitudes:

$$e^{-i\lambda p^2/2}e^{-i(1+1/\lambda)q^2/2}e^{-ip^2/2}e^{-i(1+\lambda)q^2/2} \equiv e^{i\ln\lambda(qp+pq)/2}\ \mathbf{P} \tag{17}$$

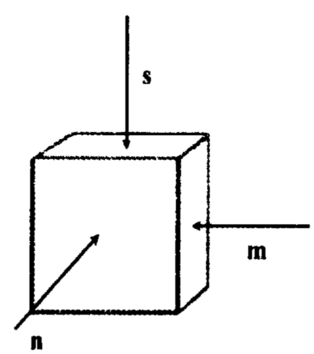
and produces the scale transformation superposed with parity ($\mathbf{P}$). The repetition of the operator sequence of (17) yields the genuine squeezing (without parity: one of the simplest cases of Brown and Carson algorithm [20]). Some more general scenarios for the squeezing operation (2) (multiple kicks on a background of a constant ellastic force) are recently studied [21].

3 Evolution control in three dimensions

All these techniques concern the Schrödinger's particle in 1 space dimension and are, in fact, only an abstract introduction to physically important problems. It is thus essential to find their analogues in 3 space dimensions. Some results can be already reported.

In the first place, the sequences of sign changing kicks [e.g. (13-14)] can be used to construct sequences of *harmonic pulses* in $\mathbf{R}^3$ generating the loop effect in $L^2(\mathbf{R}^3)$ [8]. This suggests, that the loop effect (state echo) in $\mathbf{R}^3$ might be produced, in principle, by shock waves of *source free external fields*. As the matter of fact, some closed dynamical processes can be induced even *without* any kicks, by a source free, stationary field of an adequately gauged ion trap [9]. A simple scenario of additional potential kicks (electric pulses applied to the trap walls) permits then to generate effects of squeezing upon the charged wave packet retained in the trap interior (see the report by one of us [9]).

What no less important, the effects of positive (attractive) oscillator potentials in $L^2(\mathbf{R})$ traduce themselves immediately into effects of homogeneous magnetic fields in 3 space dimensions. As an example, we have considered a quite simple sequence of identically shaped magnetic pulses in three orthogonal directions: $\mathbf{n}, \mathbf{m}, \mathbf{s}, \mathbf{n}, \mathbf{m}, \mathbf{s} \cdots$. As we have reported on the previous IWSSUR 93, an adequate proportion between time separations and the pulse intensity assures that the sequence must produce the loop effect for the wave packets in $L^2(\mathbf{R}^3)$. Moreover, the same operational scheme, with differently shaped pulses, turns out to work as a "time machine", permitting to accelerate, slow or invert the free evolution operation of the Schrödinger's wave packet [10].



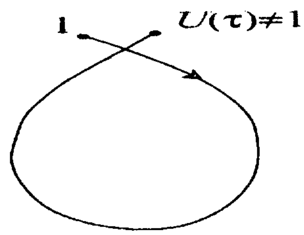
A sequence of homogeneous magnetic pulses from 3 orthogonal directions permits to manipulate the free evolution (see our report in IWSSUR 93).

4 The general manipulation scheme

The most immediate reason why one might be interested in the "evolution loops" is the possibility of controlling the fuzziness (diffraction of the wave packets due to its free evolution), essential for electronic microscopy, programming the non-demolishing measurements etc. (see also Caves et al. [22], Yuen [23], Royer [24]). Also, the original subject of the transparent wells [19] might still bring some surprises [25]. However, the loop phenomenon seems most crucial for the general manipulation methodology.

The class of the dynamical operations induced by stationary fields is rather narrow (for the Schrödinger's particle they are always of the form $\exp[-iH]$ where the exponent H is at most quadratic in $\mathbf{p}$!).

The situation is more interesting for a microobject trapped in an oscillating field of an evolution loop. As long as the loop fields are mantained, the wave packets perform a "periodic dance". A distinct phenomenon occurs, if the loop fields are perturbed or imperfect. Instead of a closed process, the system will then perform, after every loop period τ, a non trivial unitary operation, interpretable as the *loop precession.*



The precession of a distorted loop:
a natural key to the manipulation.

An elementary algebraic argument shows that the precession operations are much more general than the operations stimulated by the stationary fields. In fact, they are the key to solve the manipulation problem: by "adding precessions" an arbitrary unitary operation $U : \mathcal{H} \to \mathcal{H}$ can be approximated [8]. In some cases, already an unsophisticated distortion of the "circular processes" brings interesting results (like e.g. the squeezing or free evolution distortion in "wrong loops"). In principle, every one of the "circular identities" (10,13-15) is a natural starting point for some manipulation procedures solving the inverse evolution problem (1). With one little ammendment, however.

5 The "soft kicks"

The "ellastic kicks", while of undeniable *illustrative* value, are not so easily accessible in laboratories. The difficulty is almost anecdotic if the "δ-like kick" has to be engineered with the help of homogeneous magnetic field acting e.g. inside of a cillindrical solenoid. Since $\omega^2(t)$ of the resulting "magnetic oscillator" is proportional to $B(t)^2$ [10], (where $B(t)$ is the magnetic field intensity), the $B(t)$ in the solenoid would have to model the square root of the Dirac's δ. The request might be promising for the theory of non-linear distributions, but is a nightmare in the laboratory! (Assuming even that the laboratory team would dominate the techniques of approaching $\sqrt{\delta(t)}$, the radiative corrections would probably spoil the effects of the operation).

What one needs are the *soft analogues* of oscillator kicks (11), and they are not so difficult to programme with the help of exponential formulae. Below, we shall report a quite simple "exponential experiment".

Consider first of all the product of three operators:

$$W = e^{i\lambda(qp+pq)/2} e^{-i\gamma q^2/2} e^{-i\alpha(p^2+q^2)/2} \tag{18}$$

Let's ask the question: can one choose λ, γ, α to be three functions of time in such a way that the product W fulfills a physically interpretable evolution equation

$$\frac{dW}{dt} = -iH(t)W(t) \tag{19}$$

with $H(t)$ having the oscillator form (6)? To simplify the problem, we shall first determine λ and γ as functions of α, $\lambda = \lambda(\alpha)$ and $\gamma = \gamma(\alpha)$, and only afterwards we shall look for $\alpha = \alpha(t)$. Each term in (18) is easily differentiable:

$$\begin{aligned} i\frac{dW}{d\alpha} = \Big(& -\dot{\lambda}(\alpha)\tfrac{qp+pq}{2} + \dot{\gamma}(\alpha)e^{i\lambda\frac{qp+pq}{2}}\tfrac{q^2}{2}e^{-i\lambda\frac{qp+pq}{2}} + \\ & + e^{i\lambda\frac{qp+pq}{2}}e^{-i\gamma\frac{q^2}{2}}H_0 e^{i\gamma\frac{q^2}{2}}e^{-i\lambda\frac{qp+pq}{2}}\Big)W(\alpha) \end{aligned} \tag{20}$$

where $H_0 = p^2/2 + q^2/2 = a^\dagger a + 1/2$. Due to the transformation rule (4) and:

$$e^{-i\gamma\frac{q^2}{2}}H_0 e^{i\gamma\frac{q^2}{2}} = \frac{(p+\gamma q)^2}{2} + \frac{q^2}{2} \tag{21}$$

one easily finds:

$$i\frac{dW}{d\alpha} = \mathcal{H}(\alpha)W(\alpha) = \left((-\dot{\lambda}+\gamma)\frac{qp+pq}{2} + e^{-2\lambda}\frac{p^2}{2} + (\dot{\gamma}+\gamma^2+1)e^{2\lambda}\frac{q^2}{2}\right)W(\alpha) \tag{22}$$

To assure that the term with $(qp+pq)/2$ vanish it suffices to put:

$$\gamma(\alpha) = \dot{\lambda}(\alpha) \tag{23}$$

thus obtaining:

$$\mathcal{H}(\alpha) = e^{-2\lambda}\frac{p^2}{2} + (\dot{\gamma}+\gamma^2+1)e^{2\lambda}\frac{q^2}{2} \tag{24}$$

If now $\alpha = \alpha(t)$, the differential equation for W in terms of t reads:

$$\frac{dW}{dt} = \left(\frac{d\alpha}{dt}\right)^{-1}\mathcal{H}(\alpha)W(t), \tag{25}$$

and if in addition $\alpha(t)$ is determined by:

$$\frac{d\alpha}{dt} = e^{-2\lambda(\alpha)} \tag{26}$$

then (25) acquires the familiar form:

$$\frac{dW}{dt} = \left[\frac{p^2}{2} + g(t)\frac{q^2}{2}\right] W(t) \tag{27}$$

where $\alpha(t)$ is given by (26) and

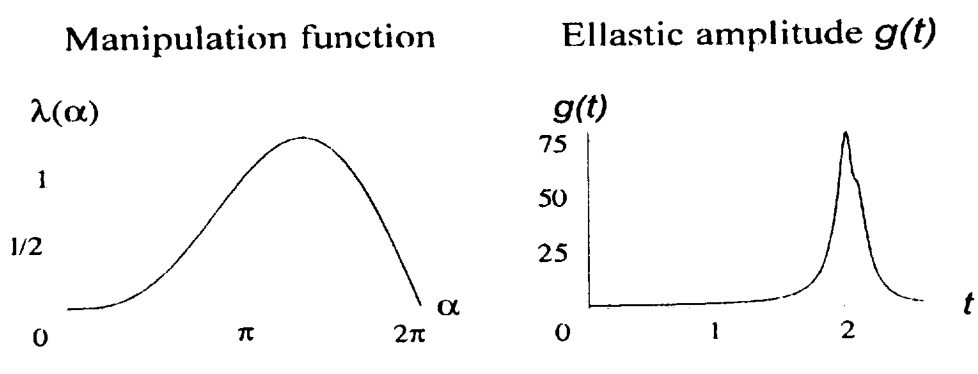
$$g(t) = (\dot{\gamma} + \gamma^2 + 1)e^{4\lambda(\alpha(t))} \tag{28}$$

If now $W(t)$ satisfies (27), then $U(t) = W(t)W(0)^{-1}$ solves the evolution problem (6) with $\omega(t)^2 = g(t)$ and with the initial condition $U(0) = 1$.

Adopting $\lambda(\alpha)$ defined in $[0, 2\pi]$ as our arbitrary "manipulation function", we can model at will the desired properties of $W(t)$ [and consistently, of $U(t)$]. Thus, if $\lambda(0) \neq \lambda(2\pi)$ but $\dot{\lambda}(0) = \dot{\lambda}(2\pi)$, U at $\alpha = 2\pi$ becomes the scale operator of form (4) and the function $g(t)$ defined by (28) gives the prescription of how the effect can be generated. If, however, $\lambda(0) = \lambda(2\pi)$ but $\gamma_1 = \dot{\lambda}(2\pi) \neq \dot{\lambda}(0) = 0$, the same product (18) reduces to the single non trivial term

$$W(t) = e^{-i\gamma_1 q^2/2} \tag{29}$$

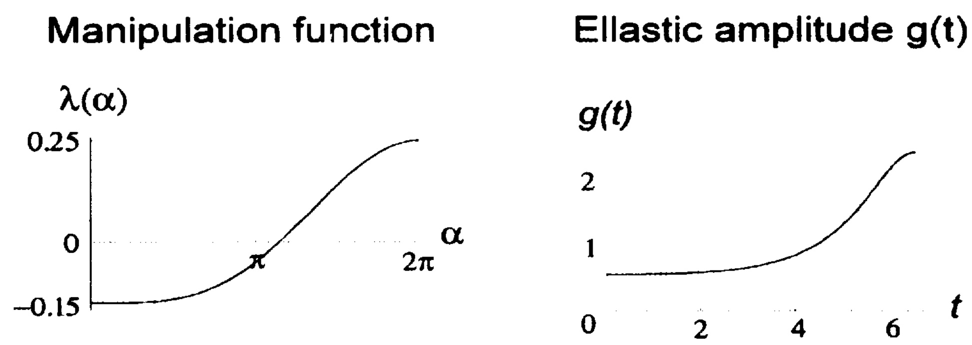
and henceforth, U imitates the effects of the δ-like kick of the oscillator force. As an example, we report two simple computer simulations where the manipulation function $\lambda(\alpha)$ yields either the "soft imitation" of the oscillator kick or the coordinate squeezing (see below). It seems pertinent to notice, that if an authentic kick were to be applied in the laboratory e.g. by creating a very short and sharp magnetic pulse, then in the first place it could never be exact (nor well approached: the δ-functions are not truly accessible in labs!) In our scenario below, this difficulty is absent:



$\lambda(\alpha) = \sin(\alpha/2) - \sin(\alpha)/2$

Effective Operation: $e^{iq^2/2}$ (repulsive kick)

The kick effect can be exact (produced with unlimited accuracy), even if negative, and is achieved by *softly varying fields* awaking little radiative response. By the same, all previous results involving the oscillator kicks [6-9,23] can be interpreted as realistic laboratory prescriptions. Note also the squeezing scenario based on the same formula (18):



$\lambda(\alpha) = 0.05\cos\alpha - 0.2\cos(\alpha/2)$

Effective Operation: $e^{i(0.4)(pq+qp)/2}$

The shape of $g(t)$ agrees with the observation that the squeezing is caused by an increase of the ellastic constant [14, 17].

The story does not end up here; it hardly starts. The method of distorted loops makes possible much more sophisticated manipulations of quantum degrees, which will be probably the daily routine of the experimental physics in a predictable future.

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