NUMBER-PHASE UNCERTAINTY RELATIONS FOR OPTICAL FIELDS

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Abstract

The Hermitian phase formalism of Pegg and Barnett allows for direct calculations of the phase variance and, consequently, the number-phase uncertainty product. This gives us a unique opportunity, inaccessible before, to study the number-phase uncertainty relations for optical fields in a direct way within a consistent quantum formalism. A few examples of fields generated in nonlinear optical processes are studied from the point of view of their number-phase uncertainty relations.

1 Number-phase uncertainty relations

Pegg and Barnett [1] introduced the Hermitian phase formalism, which is based on the observation that in a finite-dimensional state space the states with well-defined phase exist. Thus they restrict the state space to a finite $(s+1)$-dimensional Hilbert space Ψ spanned by the number states $|0\rangle$, $|1\rangle, \ldots, |s\rangle$. In this space they define a complete orthonormal set of phase states by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)\,|n\rangle, \qquad m = 0, 1, \ldots, s\,, \tag{1}$$

where the values of θ_m are given by

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}. \tag{2}$$

The value of θ_0 is arbitrary and defines a particular basis set of $(s+1)$ mutually orthogonal phase states.

The Pegg-Barnett Hermitian phase operator is defined as

$$\hat{\Phi}_\theta \equiv \sum_{m=0}^{s} \theta_m\,|\theta_m\rangle\langle\theta_m|. \tag{3}$$

Of course, phase states (1) are eigenstates of the phase operator (3) with the eigenvalues θ_m restricted to lie within a phase window between θ_0 and $\theta_0 + 2\pi s/(s+1)$. The Pegg-Barnett prescription is to evaluate any observable of interest in the finite basis (1) and only after that to take the limit $s \to \infty$.

Since the phase states (1) are orthonormal, $\langle\theta_m|\theta_{m'}\rangle = \delta_{mm'}$, the kth power of the Pegg-Barnett phase operator (3) can be written as

$$\hat{\Phi}_\theta^k = \sum_{m=0}^{s} \theta_m^k \, |\theta_m\rangle\langle\theta_m|. \tag{4}$$

Substituting eqs. (1) and (2) into eq. (3) and performing summation over m yields the phase operator explicitly in the Fock basis

$$\hat{\Phi}_\theta = \theta_0 + \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{n\neq n'} \frac{\exp[\mathrm{i}\,(n-n')\theta_0]\,|n\rangle\langle n'|}{\exp[\mathrm{i}\,(n-n')2\pi/(s+1)]-1}\,. \tag{5}$$

It is well apparent that the Hermitian phase operator $\hat{\Phi}_\theta$ has well defined matrix elements in the number state basis and does not suffer from such problems as the original Dirac phase operator. A detailed analysis of the properties of the Hermitian phase operator was given by Pegg and Barnett [1]. As the Hermitian phase operator is defined, one can calculate the expectation value and variance of this operator for a given state of the field $|f\rangle$.

The Pegg-Barnett phase operator (5), expressed in the Fock basis, readily gives the phase-number commutator [1]:

$$[\hat{\Phi}_\theta, \hat{n}] = -\frac{2\pi}{s+1} \sum_{n\neq n'} \frac{(n-n')\exp[\mathrm{i}(n-n')\theta_0]}{\exp[\mathrm{i}(n-n')2\pi/(s+1)]-1} |n\rangle\langle n'|. \tag{6}$$

Equation (6) looks very different from the famous Dirac postulate of the phase-number commutator.

Having defined the Hermitian operators for the number and phase variables and knowing their commutator, we can easily test the number-phase Heisenberg uncertainty relation for any given field with known number state decomposition.

$$\Delta\hat{\Phi}_\theta \Delta\hat{n} \geq \frac{1}{2}|\langle[\hat{n}, \hat{\Phi}_\theta]\rangle| \tag{7}$$

For physical states the number-phase commutator can be considerably simpified [1], and its expectation value in the physical state $|p\rangle$ can be expressed in terms of the phase distribution function $P(\theta_0)$, which makes calculations of this quantity pretty simple.

$$\langle p|\,[\hat{\Phi}_\theta, \hat{n}]\,|p\rangle = -i[1-(s+1)|\langle p|\,|\theta_0\rangle|^2] \tag{8}$$
$$\rightarrow -i[1-2\pi P(\theta_0)] \tag{9}$$

In the next Sections we give a few examples of the number-phase uncertainty relations calculated using the above formulas.

2 Examples

2.1 Anharmonic oscillator model

The anharmonic oscillator model is described by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\kappa\hat{a}^{\dagger 2}\hat{a}^2, \tag{10}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the field mode, and κ is the coupling constant, which is real and can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium if the anharmonic oscillator is used to describe propagation of laser light (with right or left circular polarization) in a nonlinear Kerr medium. If the state of the incoming beam is a coherent state $|\alpha_0\rangle$, the resulting state of the outgoing beam is given by

$$|\psi(\tau)\rangle \;=\; \hat{U}(\tau)|\alpha_0\rangle \;=\; \exp(-|\alpha_0|^2/2)\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\exp\left[\mathrm{i}\frac{\tau}{2}n(n-1)\right]|n\rangle, \tag{11}$$

where $\tau = -\kappa t$.

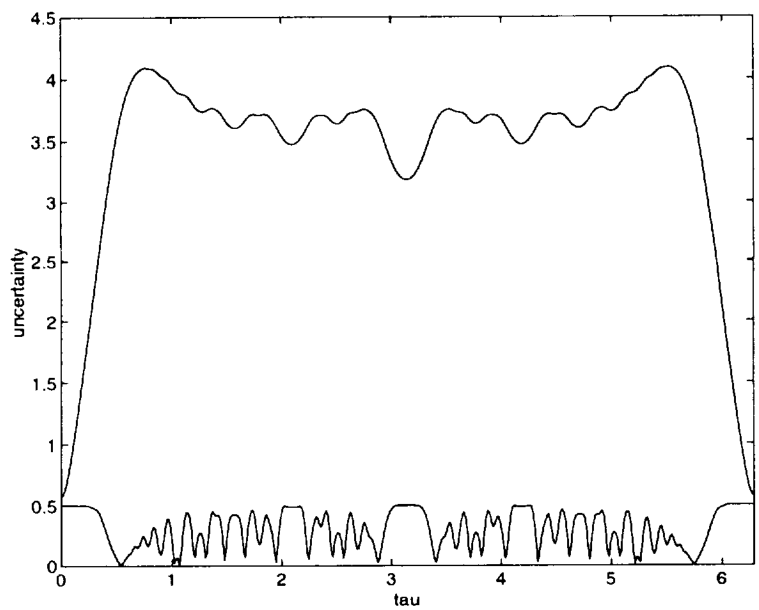

FIG. 1. Evolution of the uncertainty product (lhs of eq. (7) – upper curve) and its lower bound (rhs of eq. (7) – lower curve) for the anharmonic oscillator state with $|\alpha_0|^2 = 4$.

The appearance of the nonlinear phase factor in the state (11) modifies essentially the properties of the field represented by such a state with respect to the initial coherent state $|\alpha_0\rangle$. It was shown by Tanaś [2] that a high degree of squeezing can be obtained in the anharmonic oscillator model. Squeezing in the same process was later considered by Kitagawa and Yamamoto [3] who used the name *crescent squeezing* because of the crescent shape of the quasiprobability distribution contours obtained in the process.

The Pegg-Barnett Hermitian phase formalism has been applied for studying the phase properties of the states (11) by Gerry [4], who discussed the limiting cases of very low and very high light intensities, and by Gantsog and Tanaś [5], who gave a more systematic discussion of the exact results.

In Fig. 1 we show the evolution of the number-phase uncertainty product as given by the lhs of ineq. (7) (upper curve) and its lower bound as given by the rhs of ineq. (7) (lower curve) for the state (11) of the anharmonic oscillator assuming that the mean number of photons $|\alpha_0|^2 = 4$. It is seen that the number-phase uncertainty product rapidly increases at the early stage of the evolution, which is due to the rapid randomization of the phase, since the photon statistics remain all the time Poissonian with the number of photons variance equal to the mean number of photons $|\alpha_0|^2$. This is a typical behavior for mean numbers of photons greater than unity. It is also seen that the states generated in the anharmonic oscillator model are never the minimum uncertainty or intelligent states. The level of noise is much bigger than its lower bound allowed by quantum mechanics. Since the dynamics is periodic, after time $\tau = 2\pi$ the system returns back to its initial state. It can be shown [5] that for $|\alpha_0|^2 \gg 1$ the number and phase uncertainty product takes the approximate analytical form

$$\langle(\Delta)^2\rangle\hat{\Phi}_\theta\langle(\Delta)^2\rangle\hat{n} \;=\; (\frac{1}{4} + |\alpha_0|^2\tau^2), \tag{12}$$

explicitly showing rapid increase of the uncertainty product from the value 1/4 known for the coherent state.

2.2 Jaynes-Cummings model

The model is described by the Hamiltonian (at exact resonance)

$$\hat{H} \;=\; \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{R}^z) + \hbar g(\hat{R}^\dagger\hat{a} + \hat{R}\hat{a}^\dagger), \tag{13}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the field mode; the two-level atom is described by the raising, $\hat{R}^\dagger$, and lowering, $\hat{R}$, operators and the inversion operator $\hat{R}^z$, and g is the coupling constant.

To study the phase properties of the field mode we have to know the state evolution of the system. After dropping the free evolution terms, which change the phase in a trivial way, and assuming that the atom is initially in its ground state and the field is in a coherent state $|\alpha_0\rangle$, the state of the system is found to be

$$|\psi(t)\rangle \;=\; \sum_{n=0}^{\infty} b_n \exp(in\vartheta_0)\left[\cos(\sqrt{n}\,gt)|n,g\rangle - \mathrm{i}\sin(\sqrt{n}\,gt)|n-1,e\rangle\right], \tag{14}$$

where $|g\rangle$ and $|e\rangle$ denote the ground and excited states of the atom, the coefficients b_n are the Poissonian weighting factors of the coherent state $\alpha_0\rangle$ and ϑ_0 is the coherent state phase (phase of α_0). The main oscillations of the uncertainty product reflect the oscillations of the phase variance, which has its extrema for the revival times (in the figure time $T = gt/(2\pi|\alpha_0|)$ is scaled in the revival times). Small oscillations seen on the figure stem from the oscillations of the photon number variance and have only minor effect on the overall behavior. They are associated with the revivals of the rapid Rabi oscillations in the model. However, this is the phase variance that smoothly oscillates in the time scale of the subsequent revivals. In this way, the well known phenomenon of collapses and revivals has obtained clear interpretation in terms of the cavity mode phase [6].

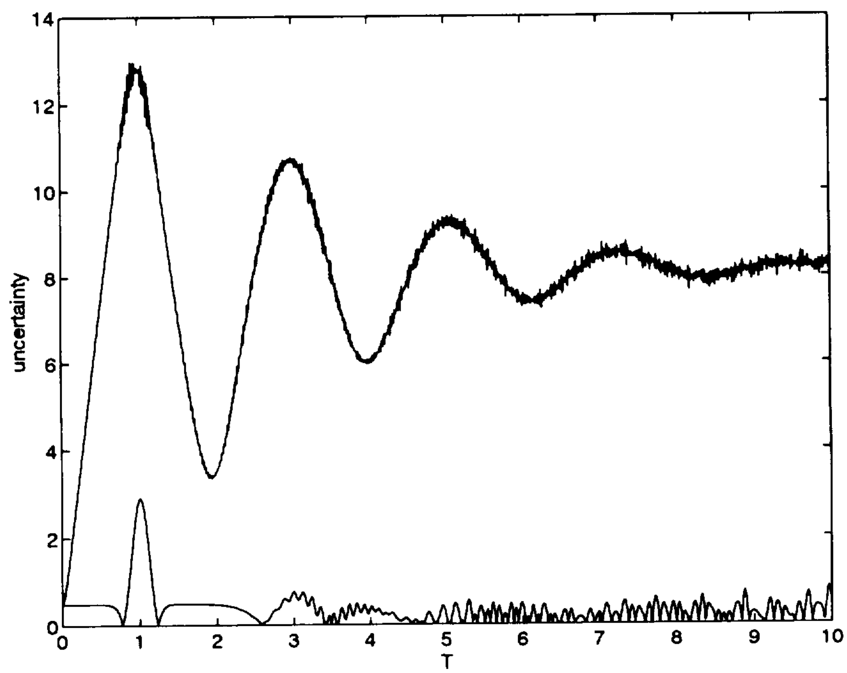

FIG. 2. Same as fig. 1 but for the Jaynes-Cummings model with $|\alpha_0|^2 = 20$.

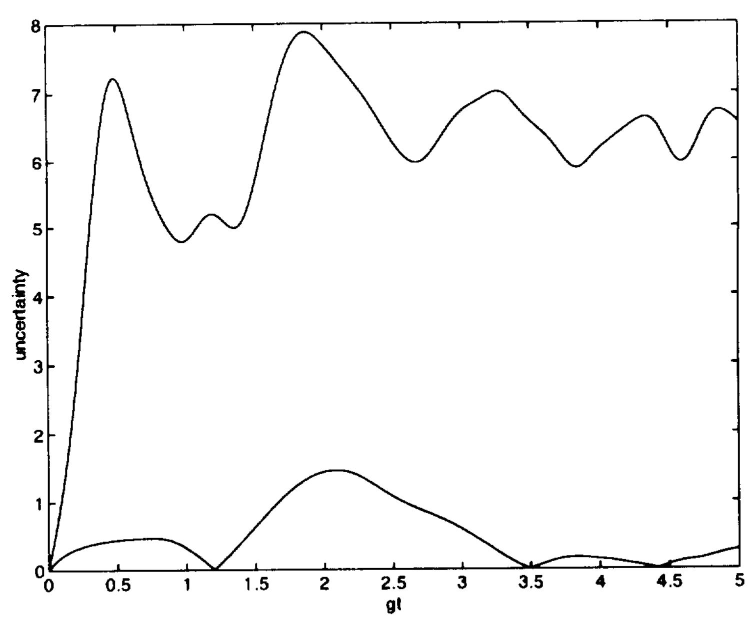

FIG. 3. Same as fig. 1 but for the down conversion with quantum pump with the initial mean number of photons equal to 4.

2.3 Down conversion with quantum pump

The parametric down conversion with quantum pump is governed by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm I} = \hbar\omega\hat{a}^\dagger\hat{a} + 2\hbar\omega\hat{b}^\dagger\hat{b} + \hbar g(\hat{b}^\dagger\hat{a}^2 + \hat{b}\hat{a}^{\dagger 2}), \tag{15}$$

where $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) are the annihilation (creation) operators of the signal mode of frequency ω and the pump mode at frequency 2ω, respectively. The coupling constant g, which is real, describes the coupling between the two modes.

Phase properties of this system have been described by Gantsog et al. [7] and Tanaś and Gantsog [8] and the details of the calculations can be found there. Here, in Fig. 3, we show, as in previous examples, the evolution of the number phase uncertainty product and its lower bound for this process. For finite initial mean number of photons the number-phase uncertainty product remains finite during the evolution contrary to the parametric approximation under which it rapidly explodes to infinity.

3 Conclusions

All above examples, are typical examples of the fields generated in nonlinear optical processes, and they show clearly that nonlinear processes typically evolve to quantum states which are far from being the minimum uncertainty or intelligent states.

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