Coherent states on Riemann surfaces as m-photon states

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Abstract

Coherent states on the the m-sheeted sphere (for the SU(2) group) are used to define analytic representations. The corresponding generators create and annihilate clusters of m photons. Non-linear Hamiltonians that contain these generators are considered and their eigenvectors and eigenvalues are explicitly calculated. The Holstein-Primakoff and Schwinger formalisms in this context are also discussed.

1 Introduction

In recent work [1] we have generalised two-photon states into m-photon states. Previously m-photon states have been considered in [2, 3]. The approach of ref. [2] is related to the Hamiltonian

$$H = \omega a^{+} a + \lambda (a^{+})^{m} + \lambda^{*} a^{m} \tag{1}$$

and is known to have several difficulties. Our m-photon coherent states are more related to those of ref. [3]. Our approach is heavily based on the theory of analytic representations and it goes far beyond previous work [4-7] in the sense that it uses them in the context of Riemann surfaces.

In refs. [1] we have studied m-photon states in connection with the m-sheeted complex plane (for the Heisenberg-Weyl group) and the m-sheeted unit disc (for the $SU(1,1)$ group). In this paper we extend these results to the $SU(2)$ case. Using our formalism we calculate explicitly the eigenvalues and eigenvectors of the Hamiltonian

$$H = \omega J_z + \lambda J_{+}^{(m)} + \lambda^{*} J_{-}^{(m)} \tag{2}$$

where $J_{+}^{(m)}$, $J_{-}^{(m)}$ are $SU(2)$ generators that move an electron up or down by m steps.

From a mathematical point of view the work is a contribution to the study of highly non-linear Hamiltonians. It has been motivated by recent developments in conformal field theory [8], but of course the details are very different here. Only simple cases of m-sheeted Riemann surfaces have been considered so far, but the final goal is to extend this work to more complex Riemann surfaces and solve very large classes of highly non-linear Hamiltonians. We believe that this can become a major tool in the study of non-linear Hamiltonians.

In the context of condensed matter the Hamiltonians considered here describe m-particle clustering. Pairing of particles plays an important role in superfluidity and superconductivity and the more general m-particle clustering studied here, could be useful in the study of new phases in condensed matter.

2 Analytic representations in the extended complex plane ($SU(2)$ group)

$SU(2)$ coherent states in a finite-dimensional Hilbert space H_{2j+1}, are defined in the extended complex plane (which is the stereographic projection of a sphere) as:

$$\begin{aligned} |z\rangle &= (1+|z|^2)^{-j}\sum \delta(j,n) z^{j+n}|j,n\rangle \\ \delta(j,n) &= [(2j)!]^{\frac{1}{2}}[(j+n)!(j-n)!]^{-\frac{1}{2}} \end{aligned} \tag{3}$$

Let $|f\rangle$ be an arbitrary (normalised) state in H_{2j+1}:

$$|f\rangle = \sum_{n=-j}^{j} f_n|j;n\rangle \quad \sum_{n=-j}^{j} |f_n|^2 = 1 \tag{4}$$

Its Bargmann analytic representation in the extended complex plane is the following polynomial (of order 2j):

$$f(z) = (1+|z|^2)^j \langle z^*|f\rangle = \sum_{n=-j}^{j} \delta(j,n) f_n z^{j+n} \tag{5}$$

The scalar product of two such functions is defined as:

$$\langle f|g\rangle = \frac{2j+1}{\pi}\int f^*(z)g(z)(1+|z|^2)^{-2j}d\mu_1(z) \tag{6}$$

$$d\mu_1(z) = (1+|z|^2)^{-2}d^2z \tag{7}$$

The $SU(2)$ generators are represented as:

$$J_- = \partial_z, \quad J_z = z\partial_z - j, \quad J_+ = -z^2\partial_z + 2jz \tag{8}$$

$SU(2)$ transformations on $f(z)$ of equ(5) are implemented through the Mobius conformal mappings:

$$w = \frac{az-b^*}{bz+a^*}; \quad |a|^2+|b|^2 = 1 \tag{9}$$

$$f(z) \to f(w)(bz+a^*)^{2j} = \sum_{n=-j}^{j} f_n\delta(j,n)[az-b^*]^{j+n}[bz+a^*]^{j-n} \tag{10}$$

3 Analytic representations in the m-sheeted extended complex plane

The formalism developed in the previous section is generalised here by replacing z by z^m. In order to have one-to-one mappings we introduce appropriate Riemann surfaces: an m-sheeted complex plane and an m-sheeted extended complex plane. The point $z = 0$ is a branch point of order $m-1$ in all three cases. We also have cuts along the lines

$$\begin{aligned} z &= r\omega^l; \quad l = 0, 1, \ldots (m-1) \\ \omega &= \exp\left[i\frac{2\pi}{m}\right] \end{aligned} \tag{11}$$

We shall call sheet number s(z) of a complex number z the

$$s(z) = \mathrm{IP}\left(\frac{m\arg(z)}{2\pi}\right) \tag{12}$$

where IP stands for the integer part of the number. s(z) takes the integer values from 0 to m-1 (modulo m). The Hilbert space is $(2j+1)$-dimensional and we only consider cases where the $2j+1$ is an integer multiple of m

$$2j + 1 = m(2k+1) \tag{13}$$

The states $|jn\rangle$ can also be relabeled as:

$$|jn\rangle = |ml; kh\rangle \tag{14}$$

$$h = \mathrm{IP}\left[\frac{j+n}{m}\right] \tag{15}$$

$$l = \mathrm{REM}\left[\frac{j+n}{m}\right] \tag{16}$$

where IP and REM stand for the integer part and remainder of the indicated division, correspondingly. The Hilbert space H_{2j+1} can be decomposed as:

$$H_{2j+1} = \sum_{l=0}^{m-1} H_l \tag{17}$$

$$H_l = \{|ml; kh\rangle; \quad -k \le h \le k\} \tag{18}$$

The SU(2) coherent states of equ(3) are generalised into coherent states on an m-sheeted covering of the SU(2) group, defined as follows:

$$|z; m\rangle = (1 + |z|^{2m})^{-k} \sum_{h=-k}^{k} \delta(k,h)(z^m)^{k+h}|m, s(z); k, h\rangle \tag{19}$$

They are SU(2) coherent states within the Hilbert subspace $H_{s(z)}$. A resolution of the identity in terms of these states is written as follows:

$$\frac{2k+1}{\pi}\int_C |z; m\rangle\langle z; m| d\mu_m(z) = 1 \tag{20}$$

$$d\mu_m(z) = (1+|z|^{2m})^{-2} m^2 |z|^{2(m-1)} d^2 z \tag{21}$$

The metric $d\mu_m(z)$ comes from the metric of equ(7) with z replaced by z^m. Using the states (19) we define the extended Bargmann representation in the m-sheeted extended complex plane of the arbitrary state $|f\rangle$ of equ(4) as:

$$f(z;m) = (1+|z|^{2m})^k \langle z^*; m|f\rangle = \sum_{h=-k}^{k} \delta(k,h)(z^m)^{k+h} f_{h,s(z)} \tag{22}$$

f(z;m) is a polynomial of order 2km=2j-(m-1) and is analytic at the interior of each sheet. The scalar product is given as

$$\langle f|g\rangle = \frac{2k+1}{\pi} \int_C f^*(z;m) g(z;m)(1+|z|^{2m})^{-2k} d\mu_m(z) \tag{23}$$

Substitution of z by z^m in (8) gives the operators:

$$\begin{aligned}
J_+^{(m)} &= -m^{-1} z^{1+m} \partial_z + 2kz^m & (24)\\
J_-^{(m)} &= m^{-1} z^{1-m} \partial_z & (25)\\
J_z^{(m)} &= m^{-1} z \partial_z - k & (26)\\
\left[J_z^{(m)}, J_+^{(m)}\right] &= J_+^{(m)} & (27)\\
\left[J_z^{(m)}, J_-^{(m)}\right] &= -J_-^{(m)} & (28)\\
\left[J_+^{(m)}, J_-^{(m)}\right] &= 2J_z^{(m)} & (29)\\
J_+^{(m)}|ml;kh\rangle &= [k(k+1) - h(h+1)]^{\frac{1}{2}} |m,l;k,h+1\rangle & (30)\\
J_-^{(m)}|ml;kh\rangle &= [k(k+1) - h(h-1)]^{\frac{1}{2}} |m,l;k,h-1\rangle & (31)\\
J_z^{(m)}|ml;kh\rangle &= h|ml;kh\rangle & (32)
\end{aligned}$$

They act as $SU(2)$ generators within H_l and therefore they move the state $|jn\rangle$ upwards or downwards by m steps. SU(2) transformations on the f(z;m) of equ(22) are implemented as generalised Mobius conformal mappings:

$$w = \left[\frac{az^m - b^*}{bz^m + a^*}\right]^{\frac{1}{m}}; \quad |a|^2 + |b|^2 = 1 \tag{33}$$

$$f(z;m) \to f(w;m)(bz^m + a^*)^{2k} \tag{34}$$

4 Applications to m-photon states

We consider the Hamiltonian:

$$H = \omega J_z + \lambda J_+^{(m)} + \lambda^* J_-^{(m)} \tag{35}$$

Its eigenvectors and eigenvalues are:

$$HU_m(\theta,\phi)|ml;kh\rangle = \{[l - \frac{1}{2}(m-1)]\omega + \tau h\} U_m(\theta,\phi)|ml;kh\rangle \tag{36}$$

$$U_m(\theta,\phi) = \exp\left[-\frac{1}{2}\theta e^{-i\phi}J_+^{(m)} + \frac{1}{2}\theta e^{i\phi}J_-^{(m)}\right] \tag{37}$$

$$\tau = [(\omega m)^2 + |\lambda|^2]^{\frac{1}{2}} \tag{38}$$

$$\phi = \arg(\lambda) \tag{39}$$

$$\cos(\theta) = \omega m\sigma^{-1} \tag{40}$$

5 Holstein-Primakoff and Schwinger formalisms

The operators $J_+^{(m)}$, $J_-^{(m)}$, $J_z^{(m)}$ studied in this paper can be connected with the creation and annihilation operators of m-photons $a_m^\dagger$, a_m studied explicitly in [1], through the Holstein-Primakoff and Schwinger formalisms. In the Holstein-Primakoff case:

$$\begin{aligned} J_+^{(m)} &= \left[(2k+1) - a_m^\dagger a_m\right]^{\frac{1}{2}} a_m^\dagger \\ J_-^{(m)} &= a_m\left[(2k+1) - a_m^\dagger a_m\right]^{\frac{1}{2}} \\ J_z^{(m)} &= a_m^\dagger a_m - k \end{aligned} \tag{41}$$

In the Schwinger case the operators $J_+^{(m)}$, $J_-^{(m)}$, $J_z^{(m)}$ are expressed in terms of two modes as:

$$\begin{aligned} J_+^{(m)} &= a_{mA}^\dagger a_B \\ J_-^{(m)} &= a_{mA} a_B^\dagger \\ J_z^{(m)} &= (a_{mA}^\dagger a_{mA} - a_B^\dagger a_B)/2 \end{aligned} \tag{42}$$

$a_{mA}^\dagger$, a_{mA} are m-photon creation and annihilation operators for the mode A; and $a_B^\dagger$, a_B are ordinary creation and annihilation operators for the mode B. Terms like $a_{mA}^\dagger a_B$ describe the conversion of one B-photon into m A-photons. Inserting (41), (42) into the Hamiltonian (35) we get other Hamiltonians whose eigenvalues and eigenvectors we can calculate.

6 Discussion

Previous work on coherent states in the m-sheeted extended complex plane (for the Heisenberg-Weyl group) [1], has been extended to the m-sheeted sphere (for the $SU(2)$). They have been used to define analytic representations and study highly non-linear Hamiltonians that describe m-photon clustering. Further work should be directed to more complicated Riemann surfaces and their possible use in the study of even more general classes of non-linear Hamiltonians.

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8 References

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