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ON A RELATION BETWEEN QUANTUM INTERFERENCE AND STANDARD QUANTUM LIMIT

R. Momose, M. Osaki, M. Ban†, M. Sasaki, and O. Hirota
Research Center for Quantum Communications,
Tamagawa University, Tokyo, JAPAN

†*Advanced Reseach Laboratory, Hitachi Ltd., Saitama, JAPAN*

Abstract

This paper discusses a physical meaning of the standard quantum limit (SQL) in quantum decision theory. It will be shown that a necessary condition for overcoming the SQL is quantum interference.

1 Introduction

The problem of finding the best quantum measurement process in order to distinguish quantum states is called quantum decision theory which was devised extensively by Helstrom, Yuen and Holevo as quantum aspect of communication theory. In this theory, the measurement process is treated as a black box, and it is described by a probability operator measure as a simple mathematical generalization of the Born statistical postulate[1]. The discrimination among quantum states is one of the interesting topics in quantum optics and related fields, because they require the control of the quantum measurement process to find better measurement apparatus. So it is interesting to clarify the relation between the abstract description of quantum measurement processes and its physical correspondence.

Recently, Usuda and Hirota[2] pointed out that the performance of the decision error probability for binary pure state signals can be improved by means of received quantum state control consisting of the Kerr medium and the conventional homodyne system. Then Sasaki, Usuda, and Hirota[3] verified that the improvement of the performance is caused by quantum interference effect. Thus, quantum decision theory has predicted a possibility of overcoming the standard quantum limit. However, we have not yet understood what it means in general. We shall clarify in the present paper the physical meaning of the improvement of the decision error probability by control of the quantum measurement process.

2 Quantum interference

According to the quantum mechanics, any state vector represents a realizable physical state. When the state is represented by a linear superposition, we can find the quantum interference between the superposed states as follows:

$$N|\langle x|(|\Psi_1\rangle + |\Psi_2\rangle)|^2 \\ = N[|\langle x|\Psi_1\rangle|^2 + |\langle x|\Psi_2\rangle|^2 + 2\mathrm{Re}\{\langle x|\Psi_1\rangle\langle x|\Psi_2\rangle^*\}] , \tag{1}$$

where N is a normalization constant. The third term represents the quantum interference. This corresponds to the fact that the quantum probability is affected by off-diagonal elements of the density operator of a coherent superposition state. On the other hand, in the quantum measurement process, if the measurement process itself generates the superposition effect from a standard basis $\{|y\rangle\}$,

$$N|(\langle y| + \langle \delta y|)|\Psi\rangle|^2 \\ = N[|\langle y|\Psi\rangle|^2 + |\langle \delta y|\Psi\rangle|^2 + 2\mathrm{Re}\{\langle y|\Psi\rangle\langle \delta y|\Psi\rangle^*\}] , \tag{2}$$

then the resulting interference term represents the macroscopic quantum interference effect by the quantum measurement itself. Here the macroscopic means that the interference term is clearly observed.

3 Decision problem for quantum states

We first give a brief survey of quantum decision theory. The theory is formulated on the basis of the quantum probability describing the quantum measurement processes. According to quantum probability theory, measurement processes can be classified as standard and generalized processes. The standard quantum measurement process is described by the spectral theorem of von Neumann as follows:

$$\begin{cases} \hat{A}|x\rangle = x|x\rangle \\ p(x)\mathrm{d}x = \mathrm{Tr}\hat{\rho}|x\rangle\langle x|\mathrm{d}x , \end{cases} \tag{3}$$

where $\hat{\rho}$: density operator, $\hat{A}$: observable in the quantum system.
Any observable $\hat{A}$ and state $\hat{\rho}$ induce a mapping from a quantum state to a classical probability measure. On the other hand, the generalized quantum measurement process is described by the probability operator measure (POM) $\mathrm{d}\hat{\Pi}(x)$ which satisfies to the following conditions[1]:

$$\hat{\mathrm{I}} = \int \mathrm{d}\hat{\Pi}(x) \quad \text{and} \quad \mathrm{d}\hat{\Pi}(x) \geq 0 . \tag{4}$$

In general, $\mathrm{d}\hat{\Pi}(x)$ is not a projection-valued measure (PVM). Then the measurement probability is given by

$$p(x)\mathrm{d}x = \mathrm{Tr}\hat{\rho}\mathrm{d}\hat{\Pi}(x) .$$

Based on the above formulas, one can define the decision operator for decision among the quantum states. Let $\{\hat{\rho}_i\}$ be a set of quantum states representing M signals. The probability of decision is

$$P(j|i) = \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j \quad , \quad i,j \in M , \tag{5}$$

where $\hat{\Pi}_j$ is called the decision operator. This is a probability operator measure (POM) as follows:

$$\hat{\mathrm{I}} = \sum_j \hat{\Pi}_j \quad \text{and} \quad \hat{\Pi}_j \geq 0\,. \tag{6}$$

This is the discrete case of the generalized resolution of identity: Eq(4). The optimization in the quantum decision problem is formulated as follows:

$$P_{\mathrm{e}} = \min_{\{\hat{\Pi}_j\}} \left\{ 1 - \sum_j \xi_j \mathrm{Tr}(\hat{\rho}_j \hat{\Pi}_j) \right\}\,. \tag{7}$$

If the decision operators consist of PVM of the *signal observable* given by a specific basis,

$$\begin{cases} \hat{\Pi}_{j(\mathrm{SQL})} = \int f_j(x_{\mathrm{d}})|x\rangle\langle x|\mathrm{d}x \\ \sum_j \hat{\Pi}_{j(\mathrm{SQL})} = \hat{\mathrm{I}} \quad \text{and} \quad \hat{\Pi}_{j(\mathrm{SQL})} \geq 0\,, \end{cases} \tag{8}$$

where $f_j(x_{\mathrm{d}})$ is a Wald's decision function, then they are called the standard decision operators[4]. In this case, the optimization is only for Wald's decision function, and we do not need quantum decision theory. Decision operator based on different observations of the signal is called "generalized decision operator." In this general case, the role of decision and measurement process is embedded into a decision operator, and we do not separate out an observable. In general, we have PVM or POM. The whole process is treated as a black box, and this is called Helstrom-Holevo formalism[1].

4 Standard quantum limit in decision theory

Here we give the definition of the standard quantum limit. Suppose we fix a single signal observable and generate the M different signals with different quantum state. Basically, modulation scheme will be set as such a way. Minimum error probability based on the standard decision operator of the signal observable will be called the Standard Quantum Limit (SQL)[4]. If the signal observable is a set of non-commuting observables, then the minimum error probability based on the simultaneous measurement for such non-commuting observables is called the SQL. Or it is equivalent to that based on standard decision operator on the Naimark extension space. In this case, the standard decision operator is constructed by PVM of corresponding signal observables on the extended space.

Our definition is convenient to evaluate how new scheme is different from it as conventional one in the measurement process. In this definition, signal quantum state does not play so important role. We emphasize that the SQL is given for each system with various quantum states.

Let us give some examples. For a single observable, the binary PSK with coherent states is a typical example. In this case, the signal observable corresponds to the quadrature amplitude $\hat{X}_{\mathrm{c}}$ or $\hat{X}_{\mathrm{s}}$. The SQL is given by a homodyne receiver corresponding to $|x_{\mathrm{c}}\rangle\langle x_{\mathrm{c}}|$ or $|x_{\mathrm{s}}\rangle\langle x_{\mathrm{s}}|$. However, if we send more than two classical phase information of light wave, for instance, ternary PSK and quarternary PSK, then the SQL is given by a heterodyne receiver or an optical costas-loop system based on homodyne. When the quantum state is squeezed state, the SQL is for the squeezed state. But the measurement process which give the SQL is the same homodyne receiver. So we

say the SQL is for a system with squeezed state. If the state for the fixed modulation scheme is different, we will say it is the SQL for that state. Our problem is that when the signal observable or modulation signal is prepared, by controlling the measurement process we get performance better than expected in classical communication theory. Here we are concerned with the physical meaning of overcoming the SQL. We would prove the next conjecture:
"In order to overcome the SQL, the quantum interference effect by the quantum measurement process is necessary."
The proof is following: The SQL average error probability is

$$P_{\mathrm{e(SQL)}} = 1 - \sum_j \xi_j \mathrm{Tr}(\hat{\rho}_j \hat{\Pi}_{j(\mathrm{SQL})}) \,. \tag{9}$$

Here, from Eq(8), the SQL means bounds when quantum fluctuation can be treated as a classical noise and a classical decision theory is applied to them. As a result, a generation of a quantum effect is required by different measurements to get result better than the SQL. To overcome the SQL, for $P_{\mathrm{e}} < P_{\mathrm{e(SQL)}}$, one has

$$\sum_j \xi_j \mathrm{Tr}\hat{\rho}_j (\hat{\Pi}_{j(\mathrm{SQL})} - \hat{\Pi}_j) < 0 \,. \tag{10}$$

Since the decision operators can involve a classical effect, we should choose operators representing a quantum effect from the various measurement schemes. From this point, to choose the different schemes from the standard decision process, which give a quantum effect, has a possibility to bring a result better than the SQL. That is, we can say that the quantum effect which does not have classical interpretation is an essential requirement to overcome the SQL. However it is clear that the different measurement schemes from the standard do not mean better measurements. ¿From now on, we discuss what kind of quantum effect is necessary. If $\left[\hat{\Pi}_j, \hat{\Pi}_{j(\mathrm{SQL})}\right] = 0$, then $\hat{\Pi}_j$ can be represented by the same PVM as the signal observable. Since $\hat{\Pi}_{j(\mathrm{SQL})}$ is the optimum among the class of decision operators consisting of the PVM of the signal observable, we require $\left[\hat{\Pi}_j, \hat{\Pi}_{j(\mathrm{SQL})}\right] \neq 0$ to overcome the SQL. The detail logic of the proof was given by Ban[5]. We check physical meaning of the above statement. Let us discuss here only case that the signal observable is a single one and the non-commutativity of standard decision operators and new operators can be described by applying a certain unitary operator as follows:

$$\begin{aligned} \hat{\Pi}_1 &= \hat{U}^\dagger \hat{\Pi}_{1(\mathrm{SQL})} \hat{U} \,, \\ \hat{\Pi}_2 &= \hat{U}^\dagger \hat{\Pi}_{2(\mathrm{SQL})} \hat{U} \,. \end{aligned} \tag{11}$$

Here we require from Ban's result

$$\left[\hat{U}^\dagger \hat{\Pi}_{j(\mathrm{SQL})} \hat{U} \,,\, \hat{\Pi}_{j(\mathrm{SQL})}\right] \neq 0 \,. \tag{12}$$

It means that $\hat{U}$ must be generated by operators which do not commute with the signal observable and also commutation relation of the generator of $\hat{U}$ and $\hat{A}$ is not c-number. The unitary operator is represented from Stone's theorem by

$$\hat{U} = \int \exp[\mathrm{i}g(y)]\mathrm{d}\hat{E}(y) = \int K(y)\mathrm{d}\hat{E}(y)\,. \tag{13}$$

Then

$$\begin{aligned} \hat{\Pi}_2 &= \int f_2(x_\mathrm{d})\left\{\int K^*(y)\langle y|x\rangle \mathrm{d}y|y\rangle \int K(y)\langle x|y\rangle\langle y|\mathrm{d}y\right\}\mathrm{d}x \\ &= \int f_2(x_\mathrm{d})\left\{\int K^*(y)h^*(x,y)\mathrm{d}y|y\rangle \int K(y)h(x,y)\langle y|\mathrm{d}y\right\}\mathrm{d}x\,, \end{aligned} \tag{14}$$

where $h(x,y) = \langle x|y\rangle$.

If we want to overcome the SQL, at least each term of error probabilities must satisfy the following inequality:

$$\begin{aligned} \langle\psi_1|\hat{\Pi}_2|\psi_1\rangle &= \langle\psi_1|\hat{U}^\dagger\hat{\Pi}_{2(\mathrm{SQL})}\hat{U}|\psi_1\rangle \\ &= \int f_2(x_\mathrm{d})\left|\int K(y)h(x,y)\langle y|\psi_1\rangle\mathrm{d}y\right|^2\mathrm{d}x \\ &< \int f_2(x_\mathrm{d})\left|\int h(x,y)\langle y|\psi_1\rangle\mathrm{d}y\right|^2\mathrm{d}x \\ &= \int f_2(x_\mathrm{d})|\langle x|\psi_1\rangle|^2\mathrm{d}x\,. \end{aligned} \tag{15}$$

and

$$\begin{aligned} \langle\psi_2|\hat{\Pi}_1|\psi_2\rangle &= \langle\psi_2|\hat{U}^\dagger\hat{\Pi}_{1(\mathrm{SQL})}\hat{U}|\psi_2\rangle \\ &= \int f_1(x_\mathrm{d})\left|\int K(y)h(x,y)\langle y|\psi_2\rangle\mathrm{d}y\right|^2\mathrm{d}x \\ &< \int f_1(x_\mathrm{d})\left|\int h(x,y)\langle y|\psi_2\rangle\mathrm{d}y\right|^2\mathrm{d}x \\ &= \int f_1(x_\mathrm{d})|\langle x|\psi_2\rangle|^2\mathrm{d}x\,. \end{aligned} \tag{16}$$

These inequalities are the requirement for the new decision process to get below the SQL. We can see $\left[\hat{U}^\dagger\hat{\Pi}_{j(\mathrm{SQL})}\hat{U}\,,\ \hat{\Pi}_{j(\mathrm{SQL})}\right] \neq 0$ in order to obtain the error probability below the SQL, because if it is commutative operator, the inequality becomes inverse. Thus the requirement to be the non-commutativity is clear. Furthermore, in order to hold the inequalities, the probability of the overlapped region of the both signals must be reduced. It is possible by only quantum interference effect (see Ref.[6]). The $\int K(y)h(x,y)\langle y|\psi_1\rangle\mathrm{d}y$ in Eq(15) is, in general, regarded as the superposition on the coordinate of y. The superposition has a potential to give a quantum interference, because this corresponds to Eq(2). By the square of the absolute value of the above term, the modified measurement probability of the original probability: $|\langle\psi_1|x\rangle|^2$ based on the quantum interference may be obtained. We can easily understand, however, that even if the decision operator is non-commuting with $\hat{\Pi}_{j(\mathrm{SQL})}$, we cannot always obtain the macroscopic quantum interference. For example, even if $\{x\,\mathrm{and}\,y\}$ are physical quantities with non c-number commutator, it does not always give the macroscopic quantum interference which shows reduction and increase of probability on the standard basis. That is, it sometimes provides only a kind of transformation of function. In this case, we have no hope to overcome the SQL. This means that we must find a decision operator which gives the macroscopic quantum interference from non-commuting decision operators.

5 Conclusions

We have clarified the followings:

1. The physical meaning of the SQL is given.
2. To overcome the SQL is caused by the quantum interference effect in the quantum measurement process.
3. A physical meaning of the POM involves the quantum interference in the quantum measurement process, though it has been regarded as unsharp measurements like the random decision, convolution effect or cross correlation effect with other uncertainty[7, 8].

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