PHYSICAL MEANING OF THE OPTIMUM MEASUREMENT PROCESS IN QUANTUM DETECTION THEORY

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Abstract

The optimum measurement processes are represented as the optimum detection operators in the quantum detection theory. The error probability by the optimum detection operators goes beyond the standard quantum limit automatically. However the optimum detection operators are given by pure mathematical descriptions. In order to realize a communication system overcoming the standard quantum limit, we try to give the physical meanings of the optimum detection operators.

1 Introduction

The purpose of the quantum detection theory is to realize a communication system with its performance overcoming the standard quantum limit (SQL). Standard quantum limit is often referred as a detection limit achieved by classical detection theory, so that overcoming the SQL is purely quantum mechanical effect. To go beyond the SQL, the quantum measurement process must be generalized to the probability-operator measure (POM) [1, 2]. The optimization of the POM to minimize an error probability results in "the optimum detection operator" which expresses not only a measurement process but also a decision process. However the optimum detection operator works as a mapping from a signal quantum state to a decision result, so that its physical meaning is not evident. In order to realize a communication system whose detection performance is quantum mechanically optimum, investigations into the physical meanings of the optimum detection operators are indispensable.

Recently we have derived some analytical solutions of the optimum detection operators and our group gave the physical example overcoming the SQL by means of the quantum interference [3, 4, 5]. In this paper we would like to interpret the physical meaning of the optimum detection operators as the quantum interference.

2 Summary of Quantum Detection Theory

The significance of the quantum detection theory is the prediction of a receiver whose signal detection performance is superior to the conventional ones optimized by the classical detection theory. The bound between the quantum and classical detection theories is well-known as "the standard quantum limit: SQL" which is rigorously defined as follows:[6]

Definition.1
Standard quantum limit is defined as the minimum error probability achieved by the quantum measurement based on the orthonormal spectrum measure of the signal observable.

Namely, the SQL can be obtained by quantum mechanical re-description of the conventional measurement processes with the optimum decision rule. To go beyond this limit, signal measurement processes must be generalized quantum mechanically. The generalized measurement process is represented by the probability- operator measure (POM), $\hat{\Pi}_j$, which is a non- negative Hermitian operator satisfying the resolution of identity.

$$\hat{\Pi}_j = \hat{\Pi}_j^\dagger \geq 0, \tag{1}$$

$$\sum_{j=1}^{M} \hat{\Pi}_j = \hat{I}. \tag{2}$$

Because of the resolution of identity, POM can include the meaning of a decision process and such a POM is called "a detection operator." Therefore, the measurement of a signal quantum state, $\hat{\rho}_i$, by a detection operator, $\hat{\Pi}_j$, gives a conditional probability, $P(j|i)$, as follows:

$$P(j|i) = \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j. \tag{3}$$

This probability represents the signal decision probability to be 'j' while the received signal is 'i'. The error probability is also given by signal quantum states and detection operators.

$$P_e = 1 - \sum_{i=1}^{M} \xi_i P(i|i) = 1 - \sum_{i=1}^{M} \mathrm{Tr}\xi_i\hat{\rho}_i\hat{\Pi}_i, \tag{4}$$

where ξ_i is a prior-probability for i-th signal.

The quantum detection theory is the optimization theory for these detection operators to minimize the above error probability. There are several formulae to find the optimum detection operators. For example, necessary and sufficient condition for the optimum detection operators based on the quantum minimax strategy is as follows [7]:

$$\mathrm{Tr}\hat{\Pi}_i\hat{\rho}_i = \mathrm{Tr}\hat{\Pi}_j\hat{\rho}_j, \forall i, j, \tag{5}$$

$$\hat{\Pi}_j\left[\xi_j\hat{\rho}_j - \xi_i\hat{\rho}_i\right]\hat{\Pi}_i = 0, \forall i, j, \tag{6}$$

$$\hat{\Gamma} - \xi_i\hat{\rho}_i \geq 0, \forall i, \tag{7}$$

where $\hat{\Gamma}$ is called "the Lagrange operator" defined by

$$\hat{\Gamma} = \sum_{i=1}^{M} \xi_i\hat{\rho}_i\hat{\Pi}_i = \sum_{i=1}^{M} \xi_i\hat{\Pi}_i\hat{\rho}_i. \tag{8}$$

A solution of the above formula goes beyond the SQL automatically. The practical derivation of the optimum detection operators has been carried out for some signal sets consisting of linearly independent quantum states [3]. In the derivation process, the following Lemma by Kennedy plays an important role [1, 8].

Lemma
When the signal quantum states are linearly independent, the optimum POM for the error probability is indeed projection-valued.

Therefore in the cases of the quantum signal sets with pure states, the optimum detection operators are orthogonal projectors on the signal space.

$$\hat{\Pi}_j = |\omega_j\rangle\langle\omega_j| \quad \text{and} \quad \langle\omega_i|\omega_j\rangle = \delta_{ij}. \tag{9}$$

where $|\omega_j\rangle$ is called "a measurement state." Since the measurement states are the orthonormal bases in the signal space, signal quantum states, $|\psi_i\rangle$: $(\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|)$, can be represented by measurement states.

$$|\psi_i\rangle = \sum_{j=1}^{M} x_{ji}|\omega_j\rangle, \tag{10}$$

where x_{ji} is a parameter defined by

$$x_{ji} \equiv \langle\omega_j|\psi_i\rangle. \tag{11}$$

Then it is possible to represent the relation between the signal quantum states and the measurement states in the matrix form.

$$\begin{bmatrix} |\psi_1\rangle \\ \vdots \\ |\psi_M\rangle \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{M1} \\ \vdots & \ddots & \vdots \\ x_{1M} & \cdots & x_{MM} \end{bmatrix} \begin{bmatrix} |\omega_1\rangle \\ \vdots \\ |\omega_M\rangle \end{bmatrix}. \tag{12}$$

Inversely, the measurement states can be represented by signal quantum states.

$$\begin{bmatrix} |\omega_1\rangle \\ \vdots \\ |\omega_M\rangle \end{bmatrix} = [x_{ji}]^{-1} \begin{bmatrix} |\psi_1\rangle \\ \vdots \\ |\psi_M\rangle \end{bmatrix}. \tag{13}$$

Hence the problems for the optimum detection operators, Eqs.(5–7), are turned into the algebraic equations for parameters $\{x_{ji}\}$.

As an example, let us consider the Binary Phase Shift Keyed (BPSK) signal with coherent states. The signal quantum states are given by $|\psi_1\rangle = |\alpha\rangle, |\psi_2\rangle = |-\alpha\rangle$. The optimum detection operators can be obtained as follows [3]:

$$\begin{cases} \hat{\Pi}_1 = \frac{1}{2(1-\kappa^2)} & \left\{\left(1+\sqrt{1-\kappa^2}\right)|\alpha\rangle\langle\alpha| + \left(1-\sqrt{1-\kappa^2}\right)|-\alpha\rangle\langle-\alpha| \right. \\ & \left. -\kappa\left(|\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|\right)\right\}, \\ \hat{\Pi}_2 = \frac{1}{2(1-\kappa^2)} & \left\{\left(1-\sqrt{1-\kappa^2}\right)|\alpha\rangle\langle\alpha| + \left(1+\sqrt{1-\kappa^2}\right)|-\alpha\rangle\langle-\alpha| \right. \\ & \left. -\kappa\left(|\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|\right)\right\}. \end{cases} \tag{14}$$

The measurements by these optimum detection operators show the error probability going far beyond the SQL. These optimum detection operators look like the Schrödinger Cat states [9] consisting of the signal quantum states, $|\alpha\rangle$ and $|-\alpha\rangle$, so that the quantum interference may be occurred there.

In general cases, the optimum detection operator can be also represented by a coherent superposition state with signal quantum states from Eq.(13).

$$\hat{\Pi}_j = |\omega_j\rangle\langle\omega_j| = \sum_{k=1}^{M}\sum_{\ell=1}^{M} t_{jk}t_{j\ell}^{*}|\psi_k\rangle\langle\psi_\ell|, \tag{15}$$

where t_{ji} is an element of a matrix $[t_{ji}]$ representing the inverse matrix $[x_{ji}]^{-1}$. The conditional probability given in Eq.(3) becomes

$$\begin{aligned} P(j|i) &= \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j \\ &= \mathrm{Tr}\left[|\psi_i\rangle\langle\psi_i|\left(\sum_{k=1}^{M}\sum_{\ell=1}^{M} t_{jk}t_{j\ell}^{*}|\psi_k\rangle\langle\psi_\ell|\right)\right] \\ &= \langle\psi_i|\left(\sum_{k=1}^{M}\sum_{\ell=1}^{M} t_{jk}t_{j\ell}^{*}|\psi_k\rangle\langle\psi_\ell|\right)|\psi_i\rangle. \end{aligned} \tag{16}$$

Here we can see the off-diagonal elements generated from the optimum detection operator. Then there is a question "Can we regard this measurement process as a quantum interference by existence of these off-diagonal elements?" According to the general sense of the quantum interference off- diagonal elements should be generated from a density operator representing a signal quantum state. Hence in the following sections, we verify whether the optimum detection process can be interpreted as a quantum interference.

3 Quantum Interference

To specify what is the quantum interference, we follow the conventional definition [10]

Definition.2
When the quantum probability is affected by the off-diagonal elements of a density operator representation of some coherent superposition state, it is called the quantum interference.

In detail, a coherent superposition state is represented by

$$|\psi\rangle = \sum_n k_n|\phi_n\rangle, \tag{17}$$

where k_n is a normalization constant. Then its density operator representation is as follows:

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_m\sum_n k_m k_n^{*}|\phi_m\rangle\langle\phi_n|. \tag{18}$$

The quantum probability obtained by a certain measurement, $\mathrm{d}E(x)$, results in

$$p(x) = \mathrm{Tr}\hat{\rho}\mathrm{d}\hat{E}(x) = \mathrm{Tr}\sum_m\sum_n k_m k_n^{*}|\phi_m\rangle\langle\phi_n|\mathrm{d}\hat{E}(x). \tag{19}$$

When the off-diagonal elements remain in the quantum probability, it is called "a quantum interference," where the off-diagonal elements are given in the form

$$\mathrm{Tr}|\phi_m\rangle\langle\phi_n|\mathrm{d}\hat{E}(x) \quad m \neq n. \tag{20}$$

Therefore the quantum interference can be in sight by existence of the off-diagonal elements from a density operator. In the case of the optimum detection operators, however, the off-diagonal elements are generated from a measurement process as itself. If the notations of density operators and the optimum detection operators can be exchange, then we can interpret that the physical meaning of the optimum detection operator is the quantum interference.

4 Density Operators and the Optimum Detection Operators

The conditions for an operator to be the optimum detection operator are as follows [1]:

1. Non-negative Hermitian operator (condition to be POM).
$$\hat{\Pi} = \hat{\Pi}^{\dagger} \geq 0. \tag{21}$$

2. Projection on the signal space (after Kennedy's Lemma).
$$\hat{\Pi}^2 = \hat{\Pi}, \quad \text{and} \quad \mathrm{Tr}\hat{\Pi} = 1. \tag{22}$$

On the other hand, the features of density operators are [11]

1. Non-negative Hermitian operator.
$$\hat{\rho} = \hat{\rho}^{\dagger} \geq 0. \tag{23}$$

2. Trace is equal to unit.
$$\mathrm{Tr}\hat{\rho} = 1. \tag{24}$$

As a result, it is possible to exchange the notations of the optimum detection operators and density operators.
$$\left\{ \begin{array}{lcl} \hat{\rho}_i & \Rightarrow & \tilde{\Pi}_i, \\ \hat{\Pi}_j & \Rightarrow & \tilde{\rho}_j. \end{array} \right. \tag{25}$$

Applying this operation to the conditional probability in Eq.(16),
$$\begin{aligned} P(j|i) &= \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j \\ &\Rightarrow \mathrm{Tr}\tilde{\Pi}_i\tilde{\rho}_j \\ &= \mathrm{Tr}\left[|\psi_i\rangle\langle\psi_i|\left(\sum_{k=1}^{M}\sum_{\ell=1}^{M} t_{jk}t_{j\ell}^{*}|\psi_k\rangle\langle\psi_\ell|\right)\right] \\ &= \langle\psi_i|\left(\sum_{k=1}^{M}\sum_{\ell=1}^{M} t_{jk}t_{j\ell}^{*}|\psi_k\rangle\langle\psi_\ell|\right)|\psi_i\rangle, \end{aligned} \tag{26}$$

we can say that the above conditional probability contains the off- diagonal elements from the density operator, $\tilde{\rho}_j$. Hence we can say that the optimum detection process generates the quantum interference. In other words, when the error probability by the optimum detection goes beyond the SQL automatically, the quantum interference is also used there automatically. The optimum measurement state plays an equivalent role of the Schrödinger Cat state as itself.

5 Conclusions

The physical interpretation of the optimum detection operator which represents the optimum measurement process has been investigated. It is the quantum interference caused by the optimum detection operator as itself. Because the optimum detection operator is represented by a coherent superposition state consisting of signal quantum states. While this result is derived under the restriction that signal quantum states are linearly independent, we assume that any optimum detection operator generates the quantum interference as itself and uses it as much as possible to reduce the error probability.

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References

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, NY, 1976).

[2] A. S. Holevo, J. of Mult. Anal. **3**, 337 (1973).

[3] M. Osaki, and O. Hirota, *Proc. of Quantum Communication and Measurement*, 401, (Plenum Publishing, NY, 1995).

[4] T. S. Usuda, and O. Hirota, *Proc. of Quantum Communication and Measurement*, 419, (Plenum Publishing, NY, 1995).

[5] M. Sasaki, T. S. Usuda, and O. Hirota, Phys. Rev. A **51**, 1702 (1995).

[6] O. Hirota, *Annals of New York Academy of Science* **755**, 863, (Plenum Publishing, NY, 1995).

[7] O. Hirota, and S. Ikehara, The Trans. of IECE Japan **E65**, 627 (1982).

[8] R. S. Kenndy, M.I.T. Res. Lab. Electron. Quart. Progr. Rep. **110**, 142 (1973).

[9] B. Yurke, and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).

[10] K. Blum, *Density Matrix Theory and Applications* (Plenum Publishing, NY, 1981).

[11] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley & Sons, NY, 1973).