DECOHERENCE AND EXPONENTIAL LAW. A SOLVABLE MODEL

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Abstract

We analyze a modified version of the "AgBr" Hamiltonian, solve exactly the equations of motion in terms of SU(2) coherent states, and study the weak-coupling, macroscopic limit of the model, obtaining an exponential behavior *at all times*. The asymptotic dominance of the exponential behavior is representative of a purely stochastic evolution and can be derived quantum mechanically in the so-called van Hove's limit (which is a weak-coupling, macroscopic limit). At the same time, a temporal behavior of the exponential type, yielding a "probability dissipation" is closely related to dephasing ("decoherence") effects and one can expect a close connection with a dissipative and irreversible behavior. We stress the central relevance of the problem of dissipation to the quantum measurement theory and to the general topic of decoherence.

1 Introduction

Decoherence and dephasing have become very important concepts in quantum theory. Because 'decoherence' technically means the elimination of the off-diagonal elements of the density matrix, a system described by such a diagonal density matrix should exhibit a purely stochastic behavior and we naturally expect a close connection with a dissipative and irreversible behavior.

On the other hand, the temporal evolution of a quantum mechanical system, initially prepared in an eigenstate of the unperturbed Hamiltonian, is known to be roughly characterized by three distinct regions: A Gaussian behavior at short times, a Breit-Wigner exponential decay at intermediate times, and a power law at long times [1]. It is well known that the asymptotic dominance of the exponential behavior is representative of a purely stochastic evolution and can be derived quantum mechanically in the weak-coupling, macroscopic limit (the so-called van Hove's limit) [2]. One may expect a close connection between dissipation and exponential decay. Such a connection has been recently emphasized by Leggett [3]. The Gaussian short-time behavior is in itself of particular significance due, in particular, to the so-called quantum Zeno effect [4, 5].

In this note, an exponential behavior at all times is derived for a solvable dynamical model [6, 7] in the weak-coupling, macroscopic limit [8]. We shall emphasize the important role played by van Hove's diagonal singularity in the present model, together with the central relevance of the problem of dissipation to the quantum measurement theory [9] and to the general topic of decoherence [10]. The present derivation of the exponential behavior differs from the one given in Ref. [8], in that no use is made of scaled variables.

A temporal behavior of the exponential type, yielding a "probability dissipation" is closely related to dephasing effects and is a rather common feature of the interaction between microscopic and macroscopic systems. In this context, the present model is very interesting, because the measurement process is often viewed as a dephasing process and "decoherence" is regarded as a consequence of the interaction with (macroscopic) measuring devices, within the framework of quantum mechanics.

2 The 'AgBr' model

We shall base our discussion on the AgBr model [6], that has played an important role in the quantum measurement problem, and its modified version [7], that is able to take into account energy-exchange processes.

The modified AgBr Hamiltonian [7] describes the interaction between an ultrarelativistic particle Q and a 1-dimensional N-spin array (D- system). The array is a caricature of a linear "photographic emulsion" of AgBr molecules, when one identifies the *down* state of the spin with the undivided molecule and the *up* state with the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential. The total Hamiltonian for the Q+D system reads

$$H = H_0 + H', \qquad H_0 = H_Q + H_D,$$
$$H_Q = c\widehat{p}, \quad H_D = \frac{1}{2}\hbar\omega \sum_{n=1}^{N} \left(1 + \sigma_3^{(n)}\right), \quad H' = \sum_{n=1}^{N} V(\widehat{x} - x_n)\left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\widehat{x}\right) + \text{h.c.}\right], \tag{1}$$

where H_Q and H_D are the free Hamiltonians of the Q particle and of the "detector" D, respectively, H' is the interaction Hamiltonian, $\widehat{p}$ the momentum of the Q particle, $\widehat{x}$ its position, V a real potential, x_n ($n = 1, ..., N$) the positions of the scatterers in the array ($x_n > x_{n-1}$) and $\sigma_{i,\pm}^{(n)}$ the Pauli matrices acting on the nth site. An interesting feature of the above Hamiltonian, as compared to the original one [6], is that we are not neglecting the energy H_D of the array. This enables us to take into account energy-exchange processes between Q and D. The original Hamiltonian [6] is reobtained in the $\omega = 0$ limit.

The evolution operator in the interaction picture can be computed exactly [7] as

$$\begin{aligned} U(t,t') &= e^{iH_0t/\hbar}e^{-iH(t-t')/\hbar}e^{-iH_0t'/\hbar} \\ &= \prod_{n=1}^{N} \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} V(\widehat{x} + ct'' - x_n)dt'' \left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\widehat{x}\right) + \text{h.c.}\right]\right), \end{aligned} \tag{2}$$

and a straightforward calculation yields the S-matrix

$$S^{[N]} = \lim_{\substack{t\to\infty \\ t'\to-\infty}} U(t,t') = \prod_{n=1}^{N} S_{(n)} = \exp\left(-i\frac{V_0\delta}{\hbar c}\boldsymbol{\sigma}^{(n)} \cdot \boldsymbol{u}\right) \tag{3}$$

where $\boldsymbol{u} = (\cos(\omega x/c), \sin(\omega x/c), 0)$ and $V_0\delta = \int_{-\infty}^{\infty} V(x)dx$. The "spin-flip" probability, i.e. the probability of dissociating one AgBr molecule, reads

$$q = \sin^2\left(\frac{V_0\delta}{\hbar c}\right). \tag{4}$$

If the initial D state is taken to be the ground state $|0\rangle_N$ (N spins down), and the initial Q state is a plane wave, the final state is

$$S^{[N]}|p,0\rangle_N = \sum_{j=0}^{N} \binom{N}{j}^{1/2} (-i\sqrt{q})^j \left(\sqrt{1-q}\right)^{N-j} |p - j\frac{\hbar\omega}{c}, j\rangle_N. \tag{5}$$

This enables us to compute several interesting quantities, such as the visibility of the interference pattern obtained by splitting an incoming Q wave function into two branch waves, one of which interacts with D, the energy "stored" in D after the interaction with Q, as well as the fluctuation around the average. The final results are

$$\begin{aligned} \mathcal{V} &= (1-q)^{N/2} \to e^{-\overline{n}/2}, \qquad \langle H_D\rangle_F = qN\,\hbar\omega \to \overline{n}\,\hbar\omega, \\ \langle \delta H_D\rangle_F &= \sqrt{\langle (H_D - \langle H_D\rangle_F)^2\rangle_F} = \sqrt{pqN}\,\hbar\omega \to \sqrt{\overline{n}}\,\hbar\omega, \end{aligned} \tag{6}$$

where F stands for final state, $p = 1 - q$, and the trivial trace over the Q particle states is suppressed. The arrows signify the weak- coupling, macroscopic limit $N \to \infty$, $qN = \overline{n} =$ finite [7]. All results are exact. It is worth stressing that $qN = \overline{n}$ represents the average number of excited molecules, so that interference, energy and relative energy fluctuations "gradually" disappear as $\overline{n}$ increases. Observe also that (5) is a generalized [SU(2)] coherent state and becomes a Glauber coherent state in the $N \to \infty$, $qN =$ finite limit.

Our next (and main) task is to study the behavior of the propagator. We start from Eq. (2), set $t' = 0$ for simplicity, and return to the Schrödinger picture by inverting Eq. (2). The exponential is easily disentangled by making use of SU(2) properties. We get

$$e^{-iHt/\hbar} = e^{-iH_0t/\hbar} \prod_{n=1}^{N} \left(e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})} e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}} e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})} \right), \tag{7}$$

where $\alpha_n = \alpha_n(\widehat{x}, t) \equiv \int_0^t V(\widehat{x} + ct' - x_n)dt'/\hbar$. Notice that the evolution operators (2) and (7) as well as the S-matrix (3) are expressed in a factorized form: This is a property of a rather general class of similar Hamiltonians [11].

Let the Q particle be initially located at position $x' < x_1$ (x_1 is the position of the first scatterer in the linear array) and be moving towards the array with speed c. The initial D state is again the ground state $|0\rangle_N$ of the free Hamiltonian H_D (all spins down). This choice of the ground state is meaningful from a physical point of view, because the Q particle is initially outside D.

The propagator

$$G(x, x', t) \equiv \langle x| \otimes {}_N\langle 0|e^{-iHt/\hbar}|0\rangle_N \otimes |x'\rangle, \tag{8}$$

can be easily calculated from eq. (7). We place for simplicity the spin array at the far right of the origin ($x_1 > 0$) and consider the case where potential V has a compact support and the Q particle is initially located at the origin $x' = 0$, i.e. well outside the potential region of D. We get

$$G(x, 0, t) = \delta(x - ct) \prod_{n=1}^{N} \cos\tilde{\alpha}_n(t), \qquad \tilde{\alpha}_n(t) \equiv \int_0^{ct} V(y - x_n)dy/\hbar c. \tag{9}$$

This result is *exact.* Notice that the "spin-flip" probability (4) is $q = \sin^2\tilde{\alpha}_n(\infty) = \sin^2(V_0\Omega/\hbar c)$. We consider again the weak-coupling, macroscopic limit

$$q \simeq \left(\frac{V_0\Omega}{\hbar c}\right)^2 = O(N^{-1}), \tag{10}$$

and set

$$x_n = x_1 + (n-1)\Delta, \qquad L = x_N - x_1 = (N-1)\Delta. \tag{11}$$

The following derivation is different from the one given in Ref. [8]. We keep L finite and consider the continuous limit $\Delta/L \to 0$ as $N \to \infty$. A summation over n is then replaced by a definite integration

$$q\sum_{n=1}^{N} f(x_n) \to \frac{q}{\Delta}\int_{x_1}^{x_N} f(x)dx \simeq \frac{\overline{n}}{L}\int_{x_1}^{x_N} f(x)dx. \tag{12}$$

For the sake of simplicity, we restrict our attention to the case of δ-shaped potentials, by setting $V(y) = (V_0\Omega)\delta(y)$. We get

$$\begin{aligned} G \;&\propto\; \exp\left(\sum_{n=1}^{N}\ln\left\{\cos\int_{-x_n}^{ct-x_n}(V_0\Omega/\hbar c)\delta(y)dy\right\}\right) = \exp\left(\sum_{n=1}^{N}\ln\left\{\cos\left[(V_0\Omega/\hbar c)\theta(ct-x_n)\right]\right\}\right) \\ &\to\; \exp\left(-\frac{q}{\Delta}\int_{x_1}^{x_n}\theta(ct-x)dx\right) = \exp\left(-\frac{\overline{n}}{2}\left[\frac{ct-x_1}{L}\theta(x_N-ct)\theta(ct-x_1)+\theta(ct-x_N)\right]\right), \end{aligned} \tag{13}$$

where θ is the step function and the arrow denotes the weak- coupling, macroscopic limit (10). This brings about an exponential regime *as soon as the interaction starts*: Indeed, if $x_1 < ct < x_N$,

$$G \propto \exp\left(-\overline{n}\frac{c(t-t_0)}{2L}\right), \tag{14}$$

where $t_0 = x_1/c$ is the time at which the Q particle meets the first potential. Notice that there is *no* Gaussian behavior at short times and *no* power law at long times. Observe that $|G|^2$ is nothing but the probability that Q goes through the spin array *and* leaves it in the ground state.

It is well known [1, 4] that deviations from exponential behavior at short times are a consequence of the finiteness of the mean energy of the initial state. If the position eigenstates in eq. (8) are substituted with wave packets of size a, a detailed calculation shows that the exponential regime is attained a short time after t_0, of the order of a/c, which, in the present model, can be made arbitrarily small. Moreover, a detailed calculation (by H. Nakazato), making use of square potentials of strenght V_0 and width b yields, for $x_1 + \frac{b}{2} < ct < x_N - \frac{b}{2}$,

$$G \propto \exp\left(-\overline{n}\frac{c(t-t_0)}{2L} + \frac{\overline{n}b}{12L}\right). \tag{15}$$

In this case, the exponential regime is attained a short time after t_0, of the order of the width of the potential V. The regions $t \sim t_0 + O(a/c)$ and/or $t \sim t_0 + O(b/c)$ may be viewed as a possible residuum of the short-time Gaussian-like behavior. For this reason, the temporal behavior derived in this Letter is not in contradiction with some general theorems [1, 4].

What causes the occurrence of the exponential behavior displayed by our model ? This is a delicate problem. Our analysis suggests that the exponential behavior is mainly due to the locality of the potentials V and the factorized form of the evolution operator U. On the other hand, there are also profound links between the limiting procedure considered in this letter and van Hove's "$\lambda^2 T$" limit [2]. Work is in progress in order to clarify different aspects of this issue. Let us briefly discuss them. First of all, the evolution operators (2), (7) and the S-matrix (3) are expressed in a factorized form: This shows that the interactions between Q and adjacent spins of the array are independent, and the evolution "starts anew" at every step. This suggest the presence of a sort of Markovian process, which would justify the purely dissipative behavior (14). At the same time, the role played by the energy gap ω deserves to be clarified: ω plays undoubtedly an important role by guaranteeing the consistency of the physical framework, as discussed in [8]. On the other hand, the connection between the exponential "probability dissipation" (14) and the (practically irreversible) energy-exchange between the particle and the "environment" (our spin system) is a very open problem and should be investigated in detail. Leggett's remark [3], about the central relevance of the problem of dissipation to the quantum measurement theory makes the above topic very interesting: Indeed, in our opinion, the temporal behavior derived in this note is certainly related to dephasing ("decoherence") effects of the same kind of those encountered in quantum measurements.

Second, it is worth discussing the link between the weak-coupling, macroscopic limit $qN = \overline{n} =$ finite considered above and van Hove's "$\lambda^2 T$" limit [2], leading to the master equation. The interaction Hamiltonian H' has nonvanishing matrix elements only between those eigenstates of H_0 whose spin-quantum numbers differ by one. As discussed in [8], this causes van Hove's so-called diagonal singularity, because for each diagonal matrix element of H'^2, there are N intermediate-state contributions: For example

$$\langle 0,\ldots,0|H'^2|0,\ldots,0\rangle = \sum_{j=1}^{N} |\langle 0,\ldots,0|H'|0,\ldots,0,1_{(j)},0,\ldots,0\rangle|^2. \tag{16}$$

On the other hand, at most 2 states can contribute to each off-diagonal matrix element of H'^2. This ensures that only the diagonal matrix elements are kept in the weak-coupling, macroscopic limit, $N \to \infty$ with $qN < \infty$, which is the realization of diagonal singularity in our model. The link with the $\lambda^2 T$ limit is easily evinced from the following reasoning: The free part of the Hamiltonian is $H_Q = c\widehat{p}$, so that the particle travels with constant speed c, and interacts with the detector for a time $T = L/c$, where $L \simeq N\Delta$ is the total length of the detector. Since the coupling constant $\lambda \equiv g \propto V_0\Omega$, one gets $\lambda^2 T = g^2 N\Delta/c \propto qN$. Notice that the "lattice spacing" Δ, the inverse of which corresponds to a density in our 1-dimensional model, can be kept finite in the limit. (In such a case, we have to express everything in terms of scaled variables.) As a final remark, we stress that the limit $N \to \infty$ with $qN < \infty$ considered in this note is physically very appealing, in our opinion, because it corresponds to a *finite* energy loss of the Q particle after interacting with the D system.

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References

[1] E.J. Hellund, Phys. Rev. **89**, 919 (1953); M. Namiki and N. Mugibayashi, Prog. Theor. Phys. **10**, 474 (1953).

[2] L. van Hove, Physica **21**, 517 (1955). See also E.B. Davies, Commun. Math. Phys. **39**, 91 (1974); G.L. Sewell, *Quantum theory of collective phenomena*, (Clarendon Press, Oxford, 1986).

[3] A.J. Leggett, Proc. 4th Int. Symp. *Foundations of Quantum Mechanics*, eds. M. Tsukada et al. (Phys. Soc. Japan, Tokyo, 1993) p.10.

[4] L.A. Khalfin, Zh. Eksp. Teor. Fiz. **33**, 1371 (1958) [Sov. Phys. JETP **6**, 1053 (1958)]; A. DeGasperis, L. Fonda and G.C. Ghirardi, Nuovo Cimento **A21**, 471 (1974); B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[5] S. Pascazio, M. Namiki, G. Badurek and H. Rauch, Phys. Lett. A **179**, 155 (1993); S. Pascazio and M. Namiki Phys. Rev. A **50**, 4582 (1994); M. Namiki and S. Pascazio, these proceeedings.

[6] K. Hepp, Helv. Phys. Acta **45**, 237 (1972); J.S. Bell, Helv. Phys. Acta **48**, 93 (1975); S. Machida and M. Namiki, Proc. Int. Symp. *Foundations of Quantum Mechanics*, eds. S. Kamefuchi et al. (Phys. Soc. Japan, Tokyo, 1984) p.136; S. Kudaka, S. Matsumoto and K. Kakazu, Prog. Theor. Phys. **82**, 665 (1989). See also other references in [7].

[7] H. Nakazato and S. Pascazio, Phys. Rev. Lett. **70**, 1 (1993); Phys. Rev. **A48**, 1066 (1993); K. Hiyama and S. Takagi, Phys. Rev. **A48**, 2568 (1993).

[8] H. Nakazato, M. Namiki and S. Pascazio, Phys. Rev. Lett. **73**, 1063 (1994).

[9] J. von Neumann, *Die Mathematische Grundlagen der Quantenmechanik* (Springer Verlag, Berlin, 1932); *Quantum Theory and Measurement*, eds. J.A. Wheeler and W.H. Zurek (Princeton University Press, 1983); P. Busch, P.J. Lahti and P. Mittelstaedt, *The quantum theory of measurement* (Springer Verlag, Berlin, 1991).

[10] S. Machida and M. Namiki, Prog. Theor. Phys. **63** 1457 (1980); 1833 (1980). M. Namiki, Found. Phys. **18** 29 (1988); M. Namiki and S. Pascazio, Phys. Rev. A **44**, 39 (1991); Physics Reports **232**, 301 (1993).

[11] C.P. Sun, Phys. Rev. **A48**, 898 (1993).