QUANTUM LIMITS IN INTERFEROMETRIC GW ANTENNAS

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Abstract

We discuss a model for interferometric GW antennas illuminated by a laser beam and a vacuum squeezed field. The sensitivity of the antenna will depend on the properties of the radiation entering the two ports and on the optical characteristics of the interferometer components, e.g. mirrors, beam-splitter, lenses.

1 Introduction

An important ingredient for improving the sensitivity of Michelson interferometric gravitational wave detectors (GWD) is using appropriate states for the light beams illuminating its two input ports. In interferometric measurements the quantum noise is due to the fluctuations of the number of photons and to the random motion of the mirrors induced by the radiation pressure. The GW signal is extracted from the spectral density of the output.

Purpose of this paper is to discuss the dependence of the sensitivity of an interferometric GW antenna on the photon-noise and radiation pressure noises. In particular we will consider an interferometer driven by a fluctuating laser beam and a squeezed-vacuum field generated by a degenerate OPO driven by the second harmonic of the laser beam. Particular attention will be paid to the influence of phase and amplitude fluctuations of the laser beam.

2 Michelson interferometer

We consider a Michelson interferometer with two mirrors M_1 and M_2 suspended at the ends of two arms. The vertices of M_1 and M_2 are located on the axes y and x passing through the origin O, while the beam splitter is centered on O (see Fig. 1).

In order to account for aberration effects, we will model the interferometer as a multimode device: we consider two groups of beams entering through the ports P_1, P_2 (Fig. 1), described by

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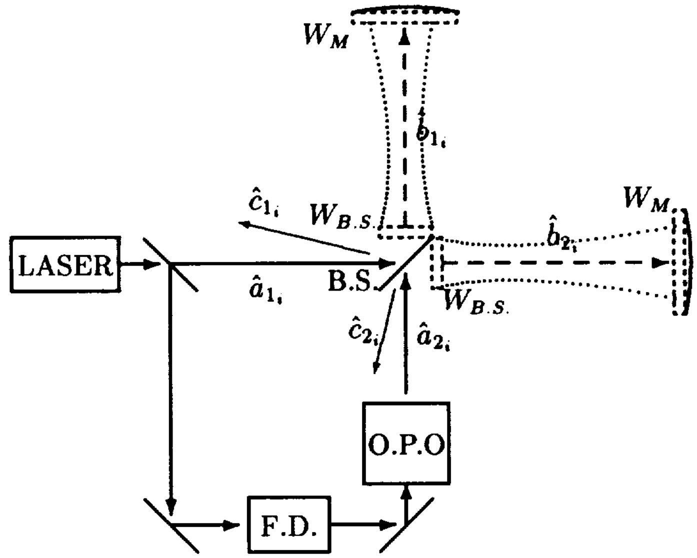

Figure 1: Michelson's interferometer. B.S.= beam-splitter; W_M aberration regions due to the mirrors; $W_{B.S.}$ aberration regions due to the beam-splitter; F.D.= frequency doubler; O.P.O.= Optical Parametric Oscillator; $\hat{a}_{1_i}, \hat{a}_{2_i}$ = in-fields at the port 1,2; $\hat{b}_{1_i}, \hat{b}_{2_i}$ = fields at the mirror 1,2; $\hat{c}_{1_i}, \hat{c}_{2_i}$ = out-fields at the port 1,2.

the operators $(\hat{a}_{j_1}, \hat{a}^\dagger_{j_1}), (\hat{a}_{j_2}, \hat{a}^\dagger_{j_2})$ with $j = 0, 1, \dots, N-1$, acting on a Hilbert space $\mathcal{H}_a = \mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2}$, with $\mathcal{H}_{a_i} = \mathcal{H}^{(0)}_{a_i} \otimes \dots \otimes \mathcal{H}^{(N-1)}_{a_i}$, $i = 1, 2$. More specifically, the modes relative to port P_1 consist of Gauss-Hermite beams travelling along the x-axis with waist in O,

$$u_{lm}(y, z; x) \propto e^{-\frac{y^2+z^2}{2q(x)}} H_l\left(\frac{\sqrt{2}y}{w(x)}\right) H_m\left(\frac{\sqrt{2}z}{w(x)}\right)$$

The pair of indices lm will be denoted by i_1. Analogously, for P_2 we consider a similar family of Gauss-Hermite beams propagating along the y-axis with waist in O (see Fig. 2).

Passing through the beam-splitter, the input beams transform in two fields at M_1, M_2 described by $(\hat{b}_{j_1}, \hat{b}^\dagger_{j_1}), (\hat{b}_{j_2}, \hat{b}^\dagger_{j_2})$ with $j = 0, 2, \dots, N-1$, acting on a Hilbert space $\mathcal{H}_b = \mathcal{H}_{b_1} \otimes \mathcal{H}_{b_2}$, such that $\mathcal{H}_{b_i} = \mathcal{H}^{(0)}_{b_i} \otimes \dots \otimes \mathcal{H}^{(N-1)}_{b_i}$ and two outgoing beams described at P_1, P_2 by $(\hat{c}_{j_1}, \hat{c}^\dagger_{j_1}), (\hat{c}_{j_2}, \hat{c}^\dagger_{j_2})$ with $j = 0, 1, \dots, N-1$, acting on a Hilbert space $\mathcal{H}_c = \mathcal{H}_{c_1} \otimes \mathcal{H}_{c_2}$, where, again, $\mathcal{H}_{c_i} = \mathcal{H}^{(0)}_{c_i} \otimes \dots \otimes \mathcal{H}^{(N-1)}_{c_i}$.

For the sake of notational convenience, we introduce the bold symbols $\hat{\mathbf{a}}_i = \hat{a}_{i_1} \otimes \hat{a}_{i_2} = \begin{pmatrix} \hat{a}_{i_1} \\ \hat{a}_{i_2} \end{pmatrix}$, for indicating the pair of i-th modes relative to P_1 and P_2 respectively. Analogously, we introduce the vector $A \equiv \begin{pmatrix} \hat{\mathbf{a}}_0 \\ \vdots \\ \hat{\mathbf{a}}_{N-1} \end{pmatrix}$. With the same meaning, we will introduce the vectors $\mathbf{B}$ and $\mathbf{C}$ for the operators $\hat{b}_j$ and $\hat{c}_j$, relative respectively to the mirrors and the output ports.

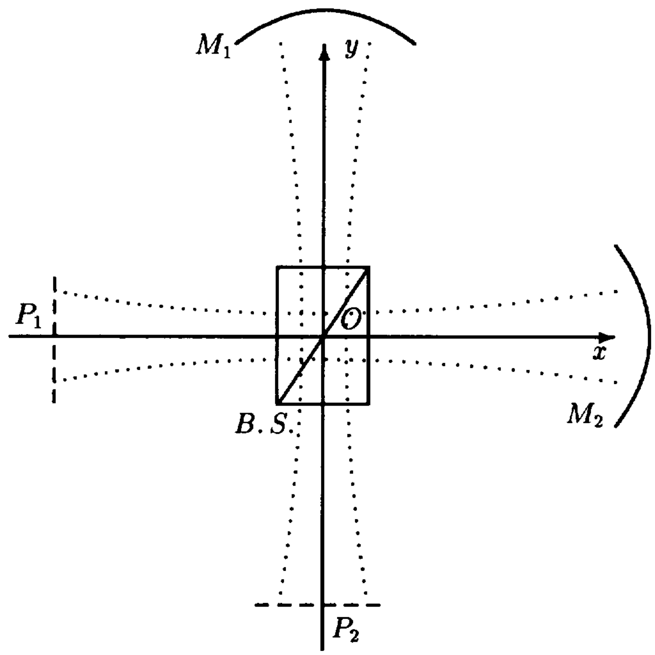

Figure 2: Schematic of Gauss-Hermite beams.

Assuming the fraction of energy lost during the passage through the interferometer be indipendent of the mode considered, **B** and **C** can be redefined as vectors proportional to the actual ones and carrying the same energy of **A**. In view of the energy conservation, the linearity and time invariance of the antenna, the outgoing vector **C** can be related to the ingoing one **A** by the unitary matrix **U**,

$$C = U \cdot A, \tag{1}$$

with

$$U \equiv \begin{pmatrix} U_{0,0} & U_{0,1} & \cdots & U_{0,N-1} \\ U_{1,0} & U_{1,1} & \cdots & U_{1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ U_{N-1,0} & U_{N-1,1} & \cdots & U_{N-1,N-1} \end{pmatrix}, \tag{2}$$

where each U_{ij} is a 2×2 matrix. Moreover, to preserve the bosonic commutation relations, U must be a symplectic matrix, that is $U \in Sp(2N, R)$ and $U_{ij} \in Sp(2, R)$.

From now on we will consider an interferometer illuminated by two TEM_{00} gaussian modes on ports P_1 and P_2 respectively. This amounts to considering an input state vector of the form

$$|\Psi>=|\psi_0>|0_1>\ldots|0_N-1>, \tag{3}$$

where $|0_i>=|0_{i_1}>|0_{i_2}>$ indicates that the modes i_1, i_2 are unexcited ground state. As a result of the propagation through the imperfect interferometer, the states of all these modes will be mixed up to some extent. So that, a mode initially in the ground state will be partially excited at the output ports.

In view of (3), it is worth splitting $\mathcal{H}_a$ in the product $\mathcal{H}_a = \mathcal{H}_{a_0} \otimes \mathcal{H}_{a_\beta}$, whith $\mathcal{H}_{a_0}$ relative to the fundamental modes entering the two ports, and $\mathcal{H}_{a_\beta}$ relative to the remaining $2N-2$ modes.

In the same manner we will write $A \equiv \begin{pmatrix} \hat{a}_0 \\ \hat{a}_\beta \end{pmatrix}$, where $\hat{a}_0 = \begin{pmatrix} \hat{a}_{0_1} \\ \hat{a}_{0_2} \end{pmatrix}$, $\hat{a}_\beta = \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_2 \end{pmatrix}$, and so for **B** and **C**.

Analogously to (1), the fields **B** at the mirrors will depend linearly on **A**

$$B = V \cdot A,$$

whith **V** a unitary linear transformation $V \equiv \begin{pmatrix} V_0 & V_{0\beta} \\ V_{\beta 0} & V_\beta \end{pmatrix}$. Physically, **V** describes reflection and transmission at the beam-splitter, followed by propagation through the interferometer arms. Then, it can be expressed as the product

$$V = \Phi \cdot \Phi_{(BS)} \cdot K,$$

where **K** describes the aberration-free beam-splitter, $\Phi_{(BS)}$ is the aberration matrix relative to the beam-splitter itself and Φ is the interferometer arm delay matrix.

Introducing the $N \times N$ matrices $1_{ij} = \delta_{i_1 j_1}\delta_{i_2 j_2}$, $\overline{1}_{ij} = \delta_{i_1 j_2}\delta_{i_2 j_1}$, and the 2×2 matrices k_1, k_2, we can write **K**, whose elements are 2×2 matrices, as

$$K = e^{i\phi}(cos\ \gamma\ \sigma_0 1 + sin\ \gamma\ \sigma_1 \overline{1}). \tag{4}$$

with σ_0, σ_1 Pauli matrices. The aberrations of the beam splitter are modeled by including at the two output faces two transparencies characterized by the aberration eykonals $W_{(BS)}(x, z)$ and $W_{(BS)}(y, z)$ for the faces perpendicular to y- and x-axes respectively. $\Phi_{(BS)}$, representing the aberration eykonal phase factor, is symmetric with respect to the exchange of the pair of indices i_1, i_2 with j_1, j_2,

$$\Phi_{(BS)} = \begin{pmatrix} \Phi_{(BS)0,0} & \Phi_{(BS)0,1} & \cdots & \Phi_{(BS)0,N-1} \\ \Phi_{(BS)1,0} & \Phi_{(BS)1,1} & \cdots & \Phi_{(BS)1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{(BS)N-1,0} & \Phi_{(BS)N-1,1} & \cdots & \Phi_{(BS)N-1,N-1} \end{pmatrix}, \tag{5}$$

where

$$\Phi_{(BS)ij} = \begin{pmatrix} e^{iW_{(BS)i_1 j_1}} & 0 \\ 0 & e^{iW_{(BS)i_2 j_2}} \end{pmatrix}. \tag{6}$$

The symmetry of the $\Phi_{(BS)}$ matrix is a consequence of the identities $W_{(BS)i_1,j_1} = W_{(BS)j_1,i_1}$ and $W_{(BS)i_2,j_2} = W_{(BS)j_2,i_2}$.

Finally, the time-delay matrix is diagonal

$$\Phi = \begin{pmatrix} \Phi_0 & 0 & \dots & 0 \\ 0 & \Phi_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Phi_{N-1} \end{pmatrix} \tag{7}$$

with $\Phi_i = \begin{pmatrix} e^{i\phi_{i_1}} & 0 \\ 0 & e^{i\phi_{i_2}} \end{pmatrix}$, $\phi_{i_{1,2}}$ representing the phase delay of the $i_{1,2}$-th Gauss-Hermite mode hitting $M_{1,2}$. In particular,

$$\phi_{i_{1,2}} = kL_{1,2} - \delta\phi_{i_{1,2}} + \delta\phi_{GW_{1,2}} + \delta\phi_{(sus)_{1,2}} + \delta\phi_{(mir)_{1,2}} + \delta\phi_{(pres)_{1,2}} + \delta\phi_{(rp)_{1,2}} \tag{8}$$

where $\delta\phi_{i_{1,2}}(> 0)$ stands for the delay of the i-th Gauss-Hermite mode with respect the phase delay $kL_{1,2}$ of a plane wave. $\delta\phi_{GW1,2}$ represents the gravitational wave ($\delta\phi_{GW_1} = -\delta\phi_{GW_2}$). The other terms stand for : (i) $\delta\phi_{(susp)}$=noise transmitted to the mirrors through the suspensions, (ii) $\delta\phi_{(mir)}$= noise caused by the vibration modes of the mirrors, (iii) $\delta\phi_{(pres)}$= pressure fluctuations in the partially evacuated pipes of the interferometer arms, and (iv) $\delta\phi_{(rp)}$= radiation pressure noise.

As a result of the reflection on M_1 and M_2, the different modes propagate toward the exit ports, by retracing the same paths followed before. Then,

$$U = -K \cdot \Phi_{(BS)} \cdot \Phi \cdot \Phi_{(M)} \cdot \Phi \cdot \Phi_{(BS)} K \tag{9}$$

where $\Phi_{(M)}$ is the mirror aberration matrix.

3 Interferometer output

The interferometer output is proportional to the expectation value of the difference I between the photocurrents detected at the ports 1 and 2 respectively

$$I = \sum_i (c^\dagger_{i_1} c_{i_1} - c^\dagger_{i_2} c_{i_2}) = C^\dagger \cdot \sigma_3 \cdot C = A^\dagger \cdot S \cdot A \tag{10}$$

where **S** is the unitary self-adjoint operator $S = U^\dagger \cdot \sigma_3 \cdot U$.

Introducing the quantity $K_3 \equiv K^* \cdot \sigma_3 \cdot K = cos(2\gamma)\sigma_3 \; 1 - sin(2\gamma)\sigma_2 \; \bar{1}$ (see Eq. 4) and assuming an input state of the form (3), it yelds

$$S = K^* \cdot \Phi^*_{(BS)} \cdot \Phi^* \cdot \Phi^*_{(M)} \cdot \Phi^* \cdot \Phi^*_{(BS)} \cdot K_3 \cdot \Phi_{(BS)} \cdot \Phi \cdot \Phi_{(M)} \cdot \Phi \cdot \Phi_{(BS)} \cdot K = S_{(0)} + S_{(ab)} \tag{11}$$

where $S_{(0)} \equiv -sin\;\tilde{\phi}\sigma_1 - cos\;\tilde{\phi}\sigma_3$, with $\tilde{\phi} \equiv 2(\phi_{0_2} - \phi_{0_1})$, is the matrix in absence of aberrations and $\delta\gamma = 0$, while $S_{(ab)} \equiv \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$ describes the effects of the aberrations and the deviation from the condition of exact equipartition of the incident intensity between the two B.S. outputs.

Now introducing the quantities $A_1 = a_0{}^\dagger \cdot \sigma_1 \cdot a_0$, $A_2 = a_0^\dagger \cdot \sigma_2 \cdot a_0$, $A_3 = a_0^\dagger \cdot \sigma_3 \cdot a_0$ and considering an interferometer operating on a dark fringe, we can express the photocurrent I as $I \equiv I_{(d)} + I_N$, that is as the sum of a deterministic part $I_{(d)} =< A_3 > [\delta\phi_{GW} + \alpha_3]$ depending on the GW signal and the aberrations, and a noise depending part

$$\begin{aligned} I_N &\approx A_1(-1+\alpha_1) + A_2\alpha_2 + < A_3 > \delta\phi_N + (A_3 - < A_3 >)\alpha_3 \\ &= I_{N_{(pn)}} + I_{N_{(sus)}} + I_{N_{(mir)}} + I_{N_{(pres)}} + I_{N_{(rp)}} + I_{N_{(ab)}} \end{aligned} \tag{12}$$

In particular, as regards to the radiation pressure noise $I_{N_{(rp)}}$, the mirrors M_1 and M_2 can be considered as multiple damped pendula driven by known time dependent pressure forces,

$$y_1(t), x_2(t) = \int_{-\infty}^{t} \left(b_{1,2}^{\dagger}(t') b_{1,2}(t') + \frac{1}{2} \right) \Gamma_M(t-t') dt' \tag{13}$$

with $\Gamma_M(t)$ the impulse response of the mirrors. Accordingly $\delta\phi_{(rp)}(t) = k(y_1(t) - x_2(t)) = \Gamma_M \star (b^{\dagger} \cdot \sigma_3 \cdot b)$, having indicated with $\Gamma_M \star$ the convolution integral (13). So That $I_{N_{(rp)}} = (-1)^{k+1}(\Gamma_M \star A_2) < A_3 >$.

4 Fourier analysis of the interferometer output

In most GW antennas the signal is extracted from the frequency spectrum of the photocurrent $I = I_{(d)} + I_N$. Therefore, the sensitivity of the interferometer depends on the autocorrelation of I,

$$<: I(\tau), I(0) :>=<: I_{(d)}, I_{(d)} :> + <: I_N, I_N :> \tag{14}$$

having considered $I_{(d)}$ and I_N as indipendent.

The limiting sensitivity of the antenna will be obtained by equating the Fourier component $S_{GW}(\omega)$ of $<: I_{(d)}, I_{(d)} :>$ at the frequency of the gravitational wave to the noise component, $S_{GW}(\omega) = S_N(\omega) / < A_3 >^2$

The noise terms $I_{N_{(sus)}}, I_{N_{(mir)}}, I_{N_{(pres)}}$ are mutually independent, so that

$$\begin{aligned} S_{GW}(\omega) &= S_{(sus)}(\omega) + S_{(mir)}(\omega) + S_{(pres)} + S_{(rp)}(\omega) + \frac{\alpha_1^2 S_1(\omega) + \alpha_2^2 S_2(\omega) + \alpha_3^2 S_3(\omega)}{< A_3 >^2} \\ &+ \frac{S_1(\omega)}{< A_3 >^2} + |H_M(\omega)|^2 S_2(\omega) + \frac{S_{12}(\omega) + S_{21}(\omega)}{< A_3 >} \end{aligned} \tag{15}$$

where S_1, S_2, S_3 are the Fourier transforms of the convolutions $<: A_1, A_1 :>, <: A_2, A_2 :>, <: \delta A_3, \delta A_3 :>$, while S_{12} and S_{21} represent the Fourier transforms of the convolutions $<: A_1, \Gamma_M \star A_2 :>$ and $<: \Gamma_M \star A_2, A_1 :>$ respectively.

The beam $\hat{a}_2$, entering the port 2 of our interferometer, is generated by a degenerate parametric oscillator (OPO) excited by a pump beam $\hat{a}_p$, obtained by duplicating the laser beam $\hat{a}_1$, entering P_1. In the following we will treat $\hat{a}_1(t) = e^{i\phi(t)}\sqrt{n_l(t)}$ as a classical field (semiclassical analysis) whose instantaneous phase $\phi(t)$ and intensity fluctuations $\delta n_l(t) = n_l(t) - < n_l >_l$ will be assumed to be both Gaussian and mutually independent stationary processes, with autocorrelations $< (\phi(\tau) - \phi(0))^2 >= \sigma_\phi^2(|\tau|)$, $< \delta n_l(\tau), \delta n_l(0) >= \sigma_l^2 C_l(|\tau|)$ with $\sigma_l^2 =< (\delta n_l)^2 >$.

The evolution of the the field operator $\hat{a}_2$ has been derived by Collett and Gardiner [13] for a classical coherent pump. We have integrated C.-G. equation of motion of $\hat{a}_2$ by representing the pump as $\hat{a}_p = \eta e^{2i\theta + 2i\phi} n_l$ and applying the WKB method.

The expectation values of the Eq.(14) have been obtained by averaging over the noise entering the OPO and the laser field amplitude and phase.

In particular, as a consequence of the classsical approximation for $\hat{a}_p$ we can write

$$A_1 = \hat{a}^{(1)\dagger}\hat{a}^{(2)} + \hat{a}^{(2)\dagger}\hat{a}^{(1)} \equiv a_l X_\phi^{(2)}, \ A_2 = a_l X_{\phi+\pi/2}^{(2)}, \ A_3 = n_l - n^{(2)} \tag{16}$$

with $n^{(2)} = a^{(2)\dagger}a^{(2)}$.

References

[1] *The Detection of Gravitational Waves*, edited by D.G. Blair (Cambridge University Press, Cambridge, England, 1991), and the references therein.

[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981); Phys. Rev. Lett., **45**, 75 (1980).

[3] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[4] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963); Phys. Rev. **131**, 2766 (1963).

[5] A. Luis and L. I. Sanchez-Soto, J. Mod. Opt. **38**, 971 (1991).

[6] R. Loudon, Phys. Rev. Lett., **47**, 815 (1981)

[7] D. Schoemaker, R. Schilling, L. Schnupp, W. Winkler, K. Maischberger and A. Rudiger, Phys. Rev. D **38**, 423 (1988)

[8] A. Rudiger, R. Schilling, L. Schnupp, W. Winkler, H. Billing and K. Maischberger, *Gravitation Wave Detection by Laser Interferometry,* MPQ Report-68 (1983)

[9] A. Brillet, Ann. Phys. Fr. **10**, 219 (1985)

[10] J. Gea-Banacloche and G. Leuchs, J. Opt. Soc. Am. **B4**, 1667 (1987); J. Mod. Opt. **34**, 793 (1987); J. Mod. Opt. **36**, 1277 (1989).

[11] W. G. Unruh, in "Quantum optics experimental gravitation and measurement theory", eds. P. Meystre and M.O. Scully, Plenum Press, N.Y. (1983) p. 647

[12] M. T. Jackel and S. Reynaud, Europhysics Lett. **13**, 301 (1991)

[13] M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).

[14] C. W. Gardiner and C. M. Savage, Opt. Commun. **50**, 173 (1984).

[15] S. Solimeno, F.Barone, C. de Lisio, L. Di Fiore, L. Milano, and G. Russo, Phys. Rev. **A** **43**,6227 (1991).

[16] N.A. Ansari, L. Di Fiore, M.A. Man'ko, V.I. Man'ko, S. Solimeno and F. Zaccaria, Phys. Rev. A**49**, 2151 (1994).

[17] Nadeem A. Ansari, L. Di Fiore, M. Man'ko, V. Man'ko, R. Romano, S. Solimeno and F. Zaccaria, in *Technical Digests of EQEC '93*, (Firenze, 1993), Ed. P. De Natale, R. Meucci, and S. Pelli, Vol. 2, 688 (1993); 2nd Workshop on Harmonic Oscillator, Cocoyoc, Mexico, March (1994) (to be published as NASA reports)

[18] J.N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).

[19] V.V. Dodonov, I.A. Malkin, and V.I. Man'ko, Physica **72**, 597 (1974).

[20] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).

[21] C. Fabre, E. Giacobino, A. Heidmann, L. Lugiato, S. Reynaud, M. Vadacchino and Wang Kaige, Quantum Optics **2**, 159 (1990);

[22] A. S. Lane, M. D. Reid, D. F. Walls, Phys. Rev. A **38**, 788 (1988).

[23] J. Mertz, T. Debuisschert, A. Heidmann, C. Fabre, and E. Giacobino, Opt. Lett. **16**, 1234 (1991).

[24] T. Debuisschert, A. Sizmann, E. Giacobino, and C. Fabre, J.O.S.A. **B10**, 1668 (1993).