REGULAR AND CHAOTIC QUANTUM DYNAMICS OF TWO-LEVEL ATOMS IN A SELFCONSISTENT RADIATION FIELD

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Abstract

Dynamics of two-level atoms interacting with their own radiation field in a single-mode high-quality resonator is considered. The dynamical system consists of two second-order differental equations, one for the atomic $SU(2)$ dynamical-group parameter and another for the field strength. With the help of the maximal Lyapunov exponent for this set we investigate numerically transitions from regularity to deterministic quantum chaos in such a simple model. Increasing the collective coupling constant $b \equiv 8\pi N_0 d^2/\hbar\omega$ we observed for initially unexcited atoms usual sharp transition to chaos at $b_c \simeq 1$. If we take the dimensionless individual Rabi frequency $a = \Omega/2\omega$ as a control parameter, then a sequence of order-to-chaos transitions has been observed starting with the critical value $a_c \simeq 0.25$ at the same initial conditions.

1 Introduction

When studying field-matter interactions it is usually of interest to consider the possibility of controlling the temporal behavior of the field and/or the atomic subsystems. Say, in resonator quantum electrodynamics, it is important to drive the interaction between atoms, moving through a cavity, and a quantizied field mode in such a way to attain specified states of the electromagnetic field (Fock, coherent, squeezed, and so on) in the cavity and/or desirable states of atoms leaving the cavity. It is inexplicitely supposed that we are able, in principle, to attain any desirable state (which is accessible, of course, in quantum mechanics) under an appropriate control.

However, it has been shown in resent years [1] that beyond the rotating-wave approximation even the simple model, consisting of N two-level atoms interacting with their own radiation field, may demonstrate impredictable temporal behavior in the sense of deterministic chaos. Our previous results [2] have shown that even slight modification in a model, describing this interaction, could create dramatic artifacts. The purpose of this work is to treat the routes to deterministic chaos in the framework of the dynamical-symmetry approach which has been proved to be useful in investigating regular dynamics of a variety of quantum models [3],[4].

2 Dynamical $SU(2)$ model

We consider an ensemble of N identical two-level atoms placed in a single- mode high-quality resonator with the volume V. Each two-level system is described by the $SU(2)$ Hamiltonian

$$H = \hbar\omega R_0 - \hbar\Omega\varepsilon(t)(R_+ + R_-), \tag{1}$$

in which the operators satisfy the usual commutation relations

$$[R_0, R_\pm] = \pm R_\pm,\ [R_+, R_-] = 2R_0, \tag{2}$$

and ω is the atomic transition frequency that coincides with the resonator frequency. The individual Rabi frequency Ω is given by

$$\Omega = \frac{dE_0}{\hbar}, \tag{3}$$

where d is the dipole moment of the atomic transition. Atoms interact self-consistently by dipole interaction with an electric field, whose strength is written in the form

$$E(t) = E_0\varepsilon(t), \tag{4}$$

where E_0 is the constant amplitude and $\varepsilon(t)$ the dimensionless variable, $0 \le \varepsilon \le 1$.

We treat the field *ab initio* semiclassically, assuming that it satisfies the usual Maxwell equation

$$\frac{d^2E}{dt^2} + \omega^2 E = 4\pi\omega^2\mathcal{P}, \tag{5}$$

where $\mathcal{P} = N_0 d < R_+ + R_- >$ is the polarisation created by atoms, $N_0 = N/V$ is the density of atoms in the resonator. Substituting $\tau = \omega t$, we can write the eq.5 in the dimensionless form with the derivative with respect to τ

$$\ddot{\varepsilon} + \varepsilon = \frac{b\omega}{\Omega} < R_+ + R_- > . \tag{6}$$

We have introduced following to [1] the dimensionless constant

$$b = \frac{8\pi N_0 d^2}{\hbar\omega}, \tag{7}$$

characterizing the energy exchange between the atomic ensemble and the field.

In addition another dimensionless constant

$$a = \frac{\Omega}{2\omega} \tag{8}$$

will be used to investigate transitions from order to chaos in our model. The expression (8) is simply the dimensionless individual Rabi frequency.

In the dynamical-symmetry approach, each two-level atom is governed by the following single equation for the $SU(2)$ complex-valued group parameter [5]

$$\ddot{g} - \left(\frac{\dot{\varepsilon}}{\varepsilon} + i\right)\dot{g} + (2a\varepsilon)^2 g = 0,\ g(0) = 1,\ \dot{g}(0) = 0. \tag{9}$$

The derivatives in (10) are also defined with respect to τ.

Thus we have two coupled oscilators (9) and (6) describing the self-consistent interaction between two-level atoms and a single-mode classical field. Rewriting (9) and (6) in the equivalent first-order form, we obtain the following nonlinear dynamical system

$$\begin{aligned} \dot{x}_1 &= 2a\varepsilon y_2, \\ \dot{x}_2 &= -2a\varepsilon y_1, \\ \dot{y}_1 &= -y_2 + 2a\varepsilon x_2, \\ \dot{y}_2 &= y_1 - 2a\varepsilon x_1, \\ \dot{\varepsilon} &= -\mathcal{P} \\ \dot{\mathcal{P}} &= \varepsilon \mp \tfrac{b}{2a}(x_1 y_1 + x_2 y_2). \end{aligned} \tag{10}$$

Signs $-$ and $+$ in the last equation of (10) refer to the initially unexcited and excited atoms, respectively.

The atom-field system (10) obeys two conservation laws

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1, \tag{11}$$

$$\pm \frac{b}{4a^2}(x_1^2 + x_2^2 - y_1^2 - y_2^2) - (\varepsilon^2 + \mathcal{P}^2) \pm \frac{b}{a}\varepsilon(x_1 y_1 + x_2 y_2) = const. \tag{12}$$

It should be noted that the variables $x_1 \equiv Re\, g$ and $x_2 \equiv Im\, g$ are not independent [5]. Therefore we have three independent real variables, that is the minimum required for chaos [6].

For two-level atoms the dynamical system (10) is equivalent to the usually adopted Maxwell-Bloch equations. Let us introduce the components of the Bloch vector

$$\begin{aligned} u(t) &= c_1^* c_2 + c_1 c_2^*, \\ v(t) &= i(c_1^* c_2 - c_1 c_2^*), \\ w(t) &= |c_2|^2 - |c_1|^2, \end{aligned} \tag{13}$$

where c_1 and c_2 are the probability amplitudes of lower and upper states respectively. On the other hand these components can be expressed in terms of the variables x and y as follows

$$\begin{aligned} u &= 2(x_1 y_1 + x_2 y_2), \\ v &= 2(x_1 y_2 - x_2 y_1), \\ w &= {y_1}^2 + {y_2}^2 - {x_1}^2 - {x_2}^2. \end{aligned} \tag{14}$$

Thus we can rewrite (10) in the standard Maxwell-Bloch form

$$\begin{aligned} \dot{u} &= -v, \\ \dot{v} &= u + 4a\varepsilon w, \\ \dot{w} &= -4a\varepsilon v, \\ \dot{\varepsilon} &= -\mathcal{P}, \\ \dot{\mathcal{P}} &= \varepsilon - \tfrac{b}{4a}u. \end{aligned} \tag{15}$$

3 Numerical results

Our model possesses two control parameters a and b. We will numerically treat here transitions from order to chaos varying one of them in a certain range and keeping another constant. Chaos will be diagnosed with the help of the maximal Lyapunov exponent λ, which is a quantitative characteristic of deterministic chaos describing the mean exponental rate of divergence of two initially adjacent trajectories in a phase space [6]. The sign of λ gives up a reliable criterion to distinguish between regular and chaotic dynamics of a system in question. When it is neglibly small the motion is said to be regular. If λ becomes positive for a certain range of values of a control parameter a system is chaotic for this range. Chaos may also be confirmed by continuous power spectra.

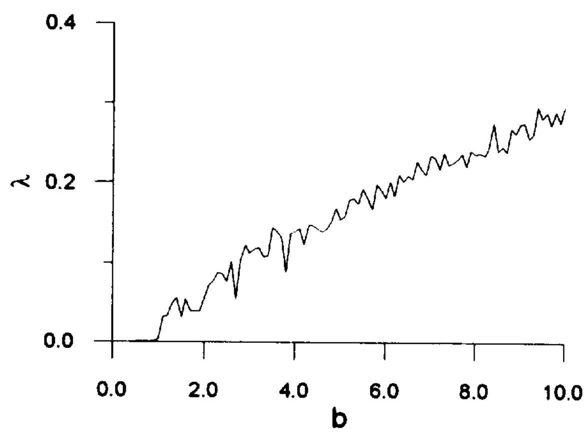

Fig.1 The maximal Lyapunov exponent as a function of the control parameter b for initially unexcited atoms, $a=0.25 \cdot 10^{-6}$.

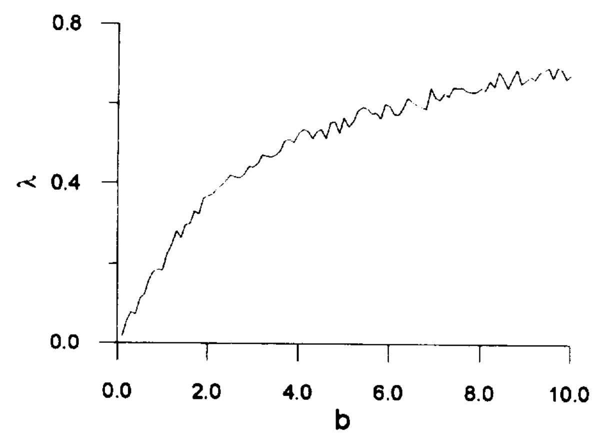

Fig.2 The maximal Lyapunov exponent as a function of the control parameter b for initially excited atoms, $a=0.25 \cdot 10^{-6}$.

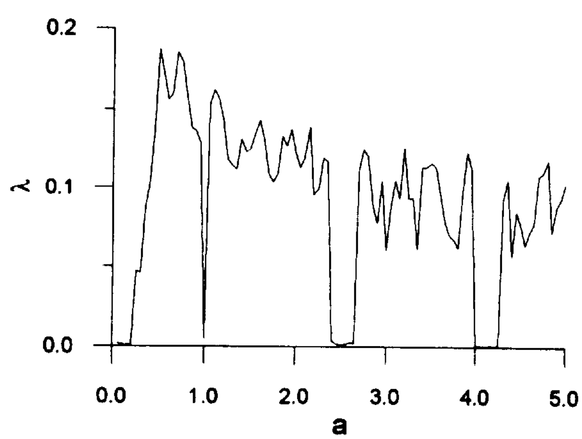

Fig.3 The maximal Lyapunov exponent as a function of the control parameter a for initially unexcited atoms, b=1.

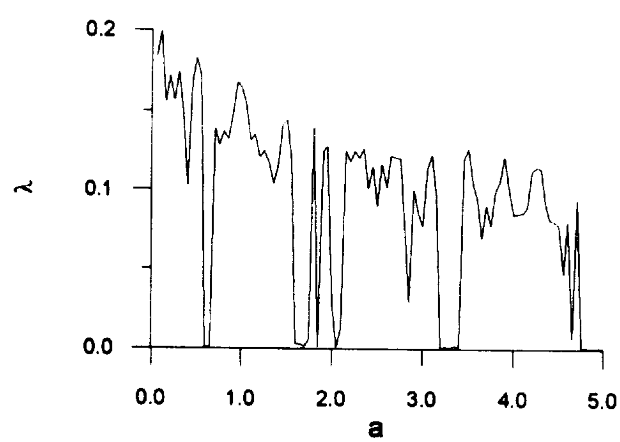

Fig.4 The maximal Lyapunov exponent as a function of the control parameter a for initially excited atoms, b=1.

By varying the collective coupling parameter b we, in fact, change the density of atoms N_0 in a cavity. Numerical integration shows that the maximal Lyapunov exponent λ becomes positive when b exceeds a critical value b_c. Its magnitude depends essentially on initial conditions. It is seen from Fig.1 that $b_c \simeq 1$ for initially unexcited atoms. For initially excited atoms (Fig.2) the maximal Lyapunov exponent becomes positive for much smaller critical value of b.

We have observed a quite different transition to chaos with chaotic regimes alternating among regular regimes when varying the individual dimensionless Rabi frequency a and fixing the parameter b. Fig.3 and Fig.4 demonstrate such a behavior for initially unexcited and excited atoms, respectively.

4 Outlooks

We have demonstrated two possible routes to chaos in the interaction of two-level atoms with their own radiation field. From a more general points of view, we have observed numerically order-to-chaos transitions in the system of two coupled nonlinear oscillators (6) and (9). At last, from an abstract point of view, we have treated such transitions in a system consisting of the "driven" $SU(2)$ group treated as a nonlinear dynamical system. Thus, the results, obtained in this work, are applicable in a more general context. They may be applied with slight modifications to any driven physical (chemical, biological, ecological, etc) system with the underlying $SU(2)$ dynamical symmetry.

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