The dissipation in lasers and in coherent state

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I. The general process in laser

The general process in lasers is defined in the photon number representation[3].

$$\frac{d\rho_n}{dt} = \mu_0(u - \mu_1 u^2 + \mu_2 u^2 + \mu_3 u^3 - \cdots)\rho_n \tag{1}$$

where u is the matrix change operation[2] $u\rho_n = \rho_{n-1} - \rho_n$, and μ_1, $\mu_2, \cdots$ are the coefficients. In the same way as previous paper[1], we deduced the generating function $G_0(z,t)$ for eq.(1)

$$G_0(z,t) = \sum z^n \rho_n(t) = \exp\left\{\int_0^t (\mu_0(z-1) - \mu_2(z-1)^2 + \cdots)dt\right\} \tag{2}$$

With the aid of generating function $G_0(z,t)$ the mean photon number $< n >_0$ and variance of photon number $< (\Delta n)^2 >_0$ can be evaluated

$$\begin{aligned} < n >_0 &= \int \mu_0(t)dt \\ < (\Delta n)^2 >_0 &= (1 - 2\mu_1) < n >_0 \end{aligned} \tag{3}$$

Now we include the cavity dumping in the treatment, the equation (1) reads

$$\frac{d\rho_n}{dt} = \mu_0(u - \mu_1 u^2 + \cdots) + c(-n\rho_n + (n+1)\rho_{n+1}) \tag{4}$$

After some tedious caculation, finally we arrive at

$$< (\Delta n)^2 > = (1 - \mu_1) < n >_0 \tag{5}$$

Eq.5 shows that when the cavity dissipation is introduced, the variance $<(\Delta n)^2>$ turns out to be smaller by a factor $(1-\mu_1)$ than it would be for a Poisson distribution. However, when the cavity dissipation be moved, the factor should be $(1-2\mu_1)$ according to eq. (3).

We note that in view of the noise reduction the only coefficient evolved is μ_1 in expansion. Three dominant sources of noise contributing to the laser output are pump fluctuations, spontaneous emission, and vaccum fluctuation entering the cavity through the mirror. We may evaluate the function $\mu(z)$ by treating the interaction between atoms and field a closed system first , then take the vaccum fluctuation into account by introducing cavity damping c.

For the atom–field system, if there is any variation in atoms excited $\Delta m = m- <m>$, this must reflect on the photons created $\Delta n = n- <n>$, so that we have

$$\Delta m = \Delta n, \qquad <(\Delta n)^2>=<(\Delta m)^2> \tag{6}$$

For example, the three–level system shown in Fig.1(a), $N_3 \ll N_1, N_2$, the excitation probability p and de–excitation probability q of one atom satisfy the relations of stationary solution

$$p = \frac{N_2}{N_1 + N_2}, \qquad q = \frac{N_1}{N_1 + N_2} \tag{7}$$

The probability of $n = N_1 + N_2$ atoms, m in excited state, $(n-m)$ in the ground state, obeys the binomial distribution

$$p_n(m) = \frac{n!}{m!(n-m)!} p^m q^{n-m} \tag{8}$$

This yields the factorial moment of atoms

$$<(\Delta m)^2>=<m>(1-p) \tag{9}$$

below the threshold, $N_2 \ll (N_1 + N_2), \mu_1 \ll 1$, Poisson
above the threshold, $N_2 \geq N_1, \quad \mu_1 = p/2 \geq 1/4$, sub–Poisson
We have a photon noise reduction factor $1/2 < 1 - \mu_1 \leq 3/4$ (with cavity damping). Similarly for a four–level system (Fig.1(b)) $N_4 \simeq 0, p = N_2/(N_1+N_2) \ll 1, q = N_1/(N_1 + N_2) \simeq 1$, this is essentially a Poisson distribution.

II. The dissipative cohrent state and quantum interference

The coherent state is defined as the eigenstate of annihiration operator a for a harmonic oscillator, what is the eigenstate of annihiration operator a for the harmonic oscillator with dissipation? If we use the classical solution $a = ae^{-\nu t - i\Omega t}$ for the annihiration operator, evidently the commutation relation $[a, a^\dagger] = 1$ is violated.

$$\frac{da}{dt} = (-i\Omega - \frac{\nu}{2})a + F \tag{1}$$

$$\begin{aligned} a &= a_0 e^{(i\Omega t - \nu/2)t} + \int_0^t F(t')e^{(i\Omega - \nu/2)(t-t')}dt' = a_0 e^{(i\Omega/2 - \nu/2)t} + \beta \\ a^\dagger &= a_0^\dagger e^{(-i\Omega t - \nu/2)t} + \int_0^t F^\dagger(t')e^{(-i\Omega - \nu/2)(t-t')}dt' = a_0^\dagger e^{(-i\Omega/2 - \nu/2)t} + \beta^\dagger \end{aligned} \tag{2}$$

The dissipative coherent state $|\alpha >_d$ corresponding to the dissipative harmonic oscillator may be defined as

$$\begin{aligned} a|\alpha >_d &= (\alpha + \beta)|\alpha >_d \\ {}_d < \alpha|a^\dagger &= (\alpha^* + \beta^\dagger)_d < \alpha| \end{aligned} \tag{3}$$

The states $|\alpha >_d$, ${}_d < \alpha|$ satisfying the definition can be expresed as

$$\begin{aligned} |\alpha >_d &= e^{\beta a^\dagger} e^{-\beta^\dagger a}|\alpha > \\ {}_d < \alpha| &= < \alpha|e^{\beta^\dagger a} e^{-\beta a^\dagger} \end{aligned} \tag{4}$$

Here a, $a^\dagger$, $|\alpha >$, $< \alpha|$ are the usual operators and coherent stats of harmonic oscillator without dissipation, the operators β, $\beta^\dagger$ act on the heat bath only but nothing to do with $|\alpha >$, $< \alpha|$.

$${}_d < \alpha|O(a, a^\dagger)|\alpha >_d = O(\alpha^* + \beta^*, \alpha + \beta) \tag{5}$$

The "quantum interference between two wave packets" studied here we mean that there are two wave packets ψ_1, ψ_2 with it's centers initially located at $x = \pm x_0$, the temporal evolution of ψ_1, ψ_2 assumes[5~7]

$$\begin{aligned} \psi_1(x,t) &= \sqrt{\frac{\alpha}{\pi}} \exp[-\frac{1}{2}(x - x_0 \cos\Omega t)^2 - i(\frac{\Omega}{2}t + x x_0 \sin\Omega t - \frac{x_0^2}{4}\sin 2\Omega t)] \\ \psi_2(x,t) &= \sqrt{\frac{\alpha}{\pi}} \exp[-\frac{1}{2}(x + x_0 \cos\Omega t)^2 - i(\frac{\Omega}{2}t - x x_0 \sin\Omega t - \frac{x_0^2}{4}\sin 2\Omega t)] \end{aligned} \tag{6}$$

The superposition of ψ_1, ψ_2 gives

$$\psi(x,t) = \frac{1}{\sqrt{2}}[\psi_1(x,t) + \psi_2(x,t)] \tag{7}$$

and the probability density $I(x,t)$ is

$$I(x,t) = |\psi(x,t)|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\theta \tag{8}$$

where

$$I_1 = \frac{\alpha}{2\pi}\exp[-(x - x_0\cos\Omega t)^2]$$

$$I_2 = \frac{\alpha}{2\pi}\exp[-(x + x_0\cos\Omega t)^2] \tag{9}$$

$$\theta = 2xx_0\sin\Omega t$$

The density distribution $I(x,t)$ is depicted in Fig.2.

Now we consider the influence on quantum interference when the damping ν is taken into account. In the weak damping limit, i.e. $\nu t \ll 1$, the classical solution $a = a_0 e^{-\nu t/2 - i\Omega t}$ may be use to evaluate the probability $I_c(x,t)$, because the violation of commutation relation $[a, a^\dagger] = 1$ is not seriously.

$$I_c(c,t) = I_{1c} + I_{2c} + 2\sqrt{I_{1c}I_{2c}}\cos\theta_c \tag{10}$$

where

$$I_{1c} = \frac{\alpha}{2\pi}\exp[-(x - x_0 e^{-\nu t/2}\cos\Omega t)^2]$$

$$I_{2c} = \frac{\alpha}{2\pi}\exp[-(x + x_0 e^{-\nu t/2}\cos\Omega t)^2] \tag{11}$$

$$\theta_c = 2xx_0\exp(-\nu t/2)\sin\Omega t$$

If we use the quantum Langevin squation's solution (2) and rewrite a, $a^\dagger$ as

$$a = (a_0 + \tilde{\beta})\exp(-i\Omega t - \nu t/2), \quad \tilde{\beta} = \int_0^t \exp[(i\Omega + \nu/2)t']F(t')dt'$$

$$a^\dagger = (a_0^\dagger + \tilde{\beta}^\dagger)\exp(i\Omega t - \nu t/2), \quad \tilde{\beta}^\dagger = \int_0^t \exp[(-i\Omega + \nu/2)t']F^\dagger(t')dt' \tag{12}$$

From eq. (12), setting $y_0 = 0$, we derive

$$\bar{x} = x_0 e^{-\nu t/2}\cos\Omega t + \Delta_1 e^{-\nu t/2}\cos\Omega t + \Delta_2 e^{-\nu t/2}\sin\Omega t$$

$$\bar{y} = x_0 e^{-\nu t/2}\sin\Omega t + \Delta_1 e^{-\nu t/2}\sin\Omega t + \Delta_2 e^{-\nu t/2}\cos\Omega t \tag{13}$$

where

$$\bar{x} = \frac{a + a^\dagger}{2}, \qquad \bar{y} = \frac{a - a^\dagger}{-2i}$$

$$\Delta_1 = \frac{\tilde{\beta} + \tilde{\beta}^\dagger}{2}, \qquad \Delta_2 = \frac{\tilde{\beta} - \tilde{\beta}^\dagger}{-2i}$$

Refering to (11), (13), naturally leads to the following formula for quantum Langevin equation's solution.

$$I_q = I_{1q} + I_{2q} + 2\sqrt{I_{1q}I_{2q}}\cos\theta_q$$
$$\begin{aligned} I_{1q} &= \frac{\alpha}{2\pi}\exp[-(x-\bar{x})^2] \\ I_{2q} &= \frac{\alpha}{2\pi}\exp[-(x+\bar{x})^2] \\ \theta_q &= 2x\bar{y} \end{aligned} \tag{14}$$

The mean amplitude and variance of vacuum fluctuation $\Delta_1 e^{-\nu t/2}$, $\Delta_2 e^{-\nu t/2}$ can be find out

$$< \Delta_1 e^{-\nu t/2} > \;=\; < \Delta_2 e^{-\nu t/2} > = 0$$

$$\begin{aligned} < (\Delta_1 e^{-\nu t/2})^2 > &= \frac{e^{-\nu t}}{4} < (\int_0^t F(t')e^{(i\Omega+\nu/2)t'}dt' + \int_0^t F^\dagger(t')e^{(-i\Omega+\nu t/2)t'}dt')^2 > \\ &= \frac{1}{2}(n_\omega + \frac{1}{2})(1-e^{-\nu t}) \\ < (\Delta_2 e^{-\nu t/2})^2 > &= \frac{1}{2}(n_\omega + \frac{1}{2})(1-e^{-\nu t}) \end{aligned} \tag{15}$$

From equ. (15) we write out immediately the distribution functions $f(\Delta_1 e^{-\nu t/2})$, $f(\Delta_2 e^{-\nu t/2})$ as

$$\begin{aligned} f(\Delta_1 e^{-\nu t/2}) &= \frac{1}{\sqrt{\pi(n_\omega+\frac{1}{2})(1-e^{-\nu t})}}\exp\left[-\frac{(\Delta_1 e^{-\nu t/2})^2}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})}\right] \\ f(\Delta_2 e^{-\nu t/2}) &= \frac{1}{\sqrt{\pi(n_\omega+\frac{1}{2})(1-e^{-\nu t})}}\exp\left[-\frac{(\Delta_2 e^{-\nu t/2})^2}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})}\right] \end{aligned} \tag{16}$$

Via $f(\Delta_1 e^{-\nu t/2})$, $f(\Delta_2 e^{-\nu t/2})$ and (14) the expectation value of density operator $< I_q(x,t) >$ can be find out

$$\begin{aligned} < I_q(x,t) > &= \int\int f(\Delta_1 e^{-\nu t/2}) f(\Delta_2 e^{-\nu t/2}) I_q(x,t)\, d\Delta_1 e^{-\nu t/2} d\Delta_2 e^{-\nu t/2} \\ &= I_1(x,t) + I_2(x,t) + I_3(x,t) \end{aligned} \tag{17}$$

where

$$
\begin{aligned}
I_1(x,t) &= \frac{\alpha}{2\pi\sqrt{1+(n_\omega+1/2)(1-e^{-\nu t})}} \exp\left[-\frac{(x-x_0e^{-\nu t/2}\cos\Omega t)^2}{1+(n_\omega+1/2)(1-e^{-\nu t})}\right] \\
I_2(x,t) &= \frac{\alpha}{2\pi\sqrt{1+(n_\omega+1/2)(1-e^{-\nu t})}} \exp\left[-\frac{(x+x_0e^{-\nu t/2}\cos\Omega t)^2}{1+(n_\omega+1/2)(1-e^{-\nu t})}\right] \\
I_3(x,t) &= \frac{\alpha}{\pi\sqrt{1+(n_\omega+1/2)(1-e^{-\nu t})}} \exp\left\{-[1+(n_\omega+\frac{1}{2})(1-e^{-\nu t})]x^2\right\} \\
&\times \exp\left[-\frac{x_0^2e^{-\nu t}\cos^2\Omega t}{1+(n_\omega+\frac{1}{2})(1-e^{-\nu t})}\right]\cos(2xx_0e^{-\nu t/2}\sin\Omega t)
\end{aligned}
\tag{18}
$$

If the vaccum is squeezed to a degree of $\ln\mu$,the variance of $\Delta_1e^{-\nu t/2}$, $\Delta_2e^{-\nu t/2}$ reads as

$$
\begin{aligned}
&< (\Delta_1e^{-\nu t/2})^2 >= \frac{\mu}{2}(n_\omega+\frac{1}{2})(1-e^{-\nu t}) \\
&< (\Delta_2e^{-\nu t/2})^2 >= \frac{1}{2\mu}(n_\omega+\frac{1}{2})(1-e^{-\nu t})
\end{aligned}
\tag{19}
$$

The expectation value for squeezed vaccum fluctuation $< I_s(x,t) >$ assumes a similar formula as (17)

$$
< I_s(x,t) = I_{1s}(x,t) + I_{2s}(x,t) + I_{3s}(x,t) \tag{20}
$$

where

$$
\begin{aligned}
I_{1s}(x,t) &= \frac{\alpha\sqrt{\mu}}{2\pi\sqrt{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}} \\
&\times \exp\left[-\frac{\mu(x-x_0e^{-\nu t/2}\cos\Omega t)^2}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}\right] \\
I_{2s}(x,t) &= \frac{\alpha\sqrt{\mu}}{2\pi\sqrt{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}} \\
&\times \exp\left[-\frac{\mu(x+x_0e^{-\nu t/2}\cos\Omega t)^2}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}\right]
\end{aligned}
\tag{21a}
$$

$$
\begin{aligned}
I_{ss}(x,t) &= \frac{\alpha\sqrt{\mu}}{\pi\sqrt{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}} \\
&\times \exp\left\{-\frac{x^2\mu[1+(n_\omega+\frac{1}{2})^2(1-e^{-\nu t})^2]}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}\right\} \\
&\times \exp\left\{-\frac{x^2(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\mu^2+\sin^2\Omega t+\mu^4\cos^2\Omega t)}{(\sin^2\Omega t+\mu^2\cos^2\Omega t)[(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu]}\right\} \\
&\times \exp\left[-\frac{\mu x_0^2 e^{-\nu t}\cos^2\Omega t}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}\right] \\
&\times \cos\left\{\frac{[(n_\omega+\frac{1}{2})(1-e^{-\nu t})+\mu]2xx_0e^{-\nu t/2}\sin\Omega t}{(n_\omega+\frac{1}{2})(1-e^{-\nu t})(\sin^2\Omega t+\mu^2\cos^2\Omega t)+\mu}\right\}
\end{aligned}
\tag{21b}
$$

The calculation results for $I_s(x,t)$ are shown in Fig.3 and a comparison between I_s and I_q, I_c shown in Fig.4.

References

[1] Weihan Tan, Phys. Lett. A. 190 (1994), 13.

[2] Yu.M.Golubev and I.V.Sokolov, Zh. Eksp. Teor. Fiz. 87, 408(1984) [Sov. Phys -JETP 60, 234(1984)].

[3] Tan Weihan, The General Process in Lasers. to be published in Opt. Comm.

[4] Tan Weihan, Wang Xue–Wen, Xie Cheng–gong, Zhang Guan–Mei, Acta Physica Sinnica 31(1982), 1569.

[5] L.I.Schiff, Quantum Mechanics, Third Edition (1955), MeGrAW–Hill Book Company, New York.

[6] A.O.Caldeira, A.J.Leggett, Phys. Rev. A, 31 (1985), 1059.

[7] D.F.Wall, G.J.Milburn, Phys. Rev. A, 31 (1985), 2405.

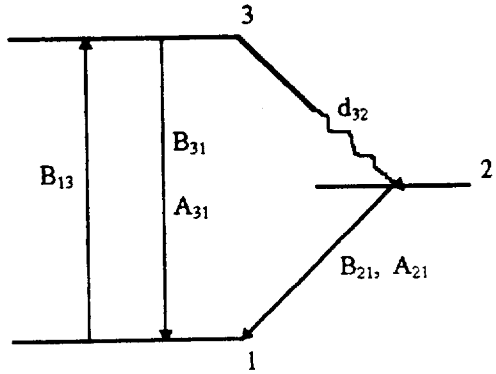

Fig.1(a) Three - Level System

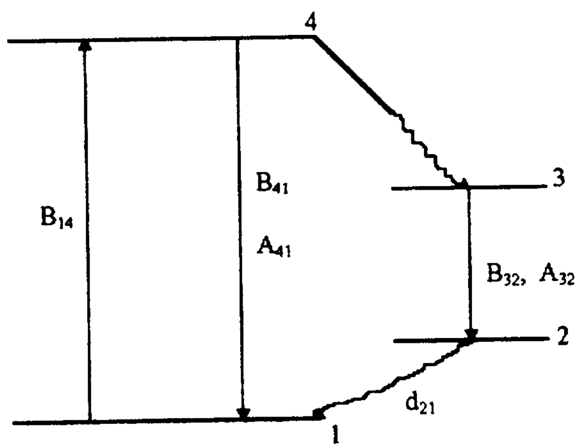

Fig.1(b) Four - Level System

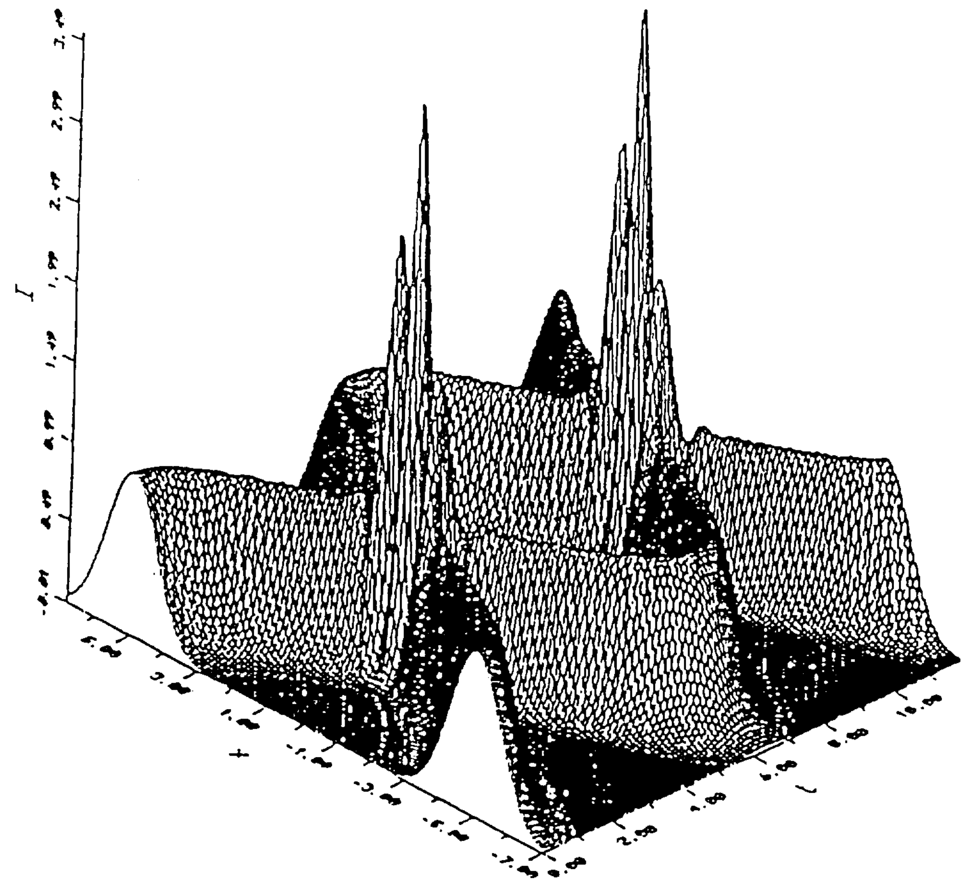

Fig.2 $I(x,t)$, no damping.
$x_0 = 5.0,\ \Omega = 0.5$

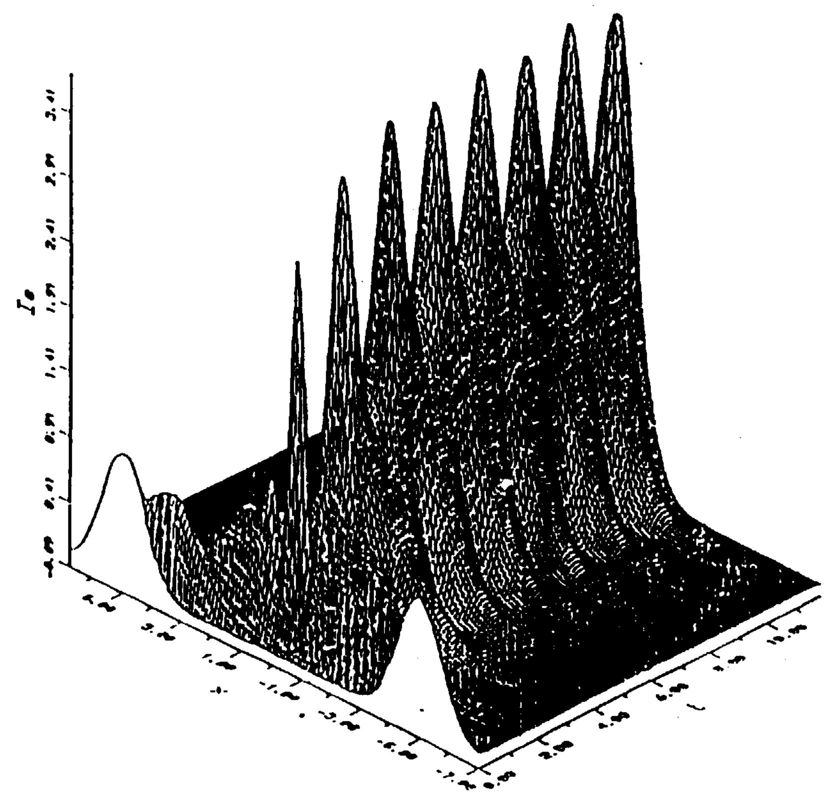

Fig.3 Quantum solution with the vaccum squeezed.

$x_0 = 5.0,\ \Omega = 2.0,\ \nu = 1.0,\ \mu = 4.0$

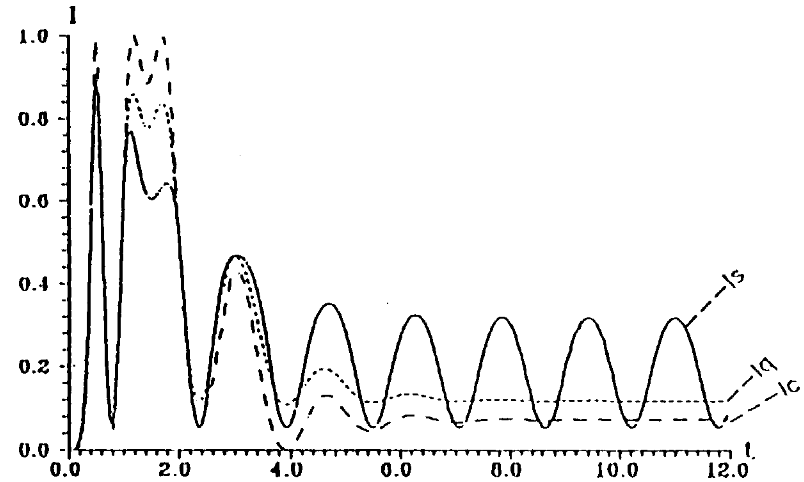

Fig.4 A comparison between I_c, I_q and I_s.

$x_0 = 5.0,\ \Omega = 2.0,\ \nu = 1.0,\ \mu = 4.0,\ x = 2.0$