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A GEOMETRICAL RESULT REGARDING TIME-OF-ARRIVAL LIGHTNING LOCATION

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One reason for investigating Lightning Detection And Ranging (LDAR) is to validate data from the Optical Transient Detector (OTD). A Time-Of-Arrival (TOA) procedure may be used with radio wave portions of lightning signatures. An antenna network is in place at KSC [1].

Algorithms are available [2,3] which provide the advantages of considerably simplifying the numerical estimation of source location and a clear error analysis. These algorithms involve judicious differencing of measurement data which allows information retrieval by performing a linear inversion. Other algorithms [4,5] require numerically finding intersections of hyperboloids. Such algorithms are more involved computationally, and the error analysis is not straightforward. The algorithm in [3] provides for a clearly defined error analysis as given in [6].

Efficiency in the retrieval of lightning location depends on the configuration of the antennas. Certain combinations of source locations and antenna configurations will result in the matrix of the linear system obtained using the algorithm in [3] having small eigenvalues. This will lead to a magnification of error. For example, a system of 4 antennas numbered from 1 to 4 sequentially around the perimeter of a rectangle yields eigenvalues which are proportional to $t_{12} + t_{34} = t_1 - t_2 + t_3 - t_4$, where t_j is the time of arrival of a signal from the lightning event to the j^{th} antenna. Letting R_j denote the distance from the source to the j^{th} antenna and c be the speed of light,

$$t_j = R_j / c. \tag{1}$$

Curved transit paths due to refractive effects in the atmosphere are not considered. For brevity, cable time delays have been neglected. Source locations on either of the 2 planes which are perpendicular to the plane of the rectangle and bisect any line segment connecting 2 consecutively numbered antennas yield $t_{12} + t_{34} = 0$. Such a configuration is "blind" to any sources on these planes.

A better configuration places antennas at the centroid and vertices of an equilateral triangle. The antenna at the centroid will be designated by the index 1; others are numbered 2 through 4. This configuration has been referred to as "ideal" in [7]. The eigenvalues for this configuration are proportional to $t_{12} + t_{13} + t_{14}$. This is equivalent to saying that the configuration is ineffective when the distance from the source to the antenna at the centroid is equal to the average of the distances from the source to the other 3 antennas. It turns out that this is never the case. The question of suitability of this particular arrangement of 4 coplanar antennas has been reduced to a problem in geometry. This report presents a solution.

The desired result is obtained by seeking the extrema of the sum of distances from the vertices to a point on a hemisphere of radius R centered at the centroid of the triangle. Full generality is recovered by allowing R to be arbitrary. We set up a Cartesian coordinate system, with points specified by (x, y, z). Its origin is at the centroid. Vertices are located at

$$\mathbf{D}_2 = D\hat{\mathbf{y}}, \quad \mathbf{D}_3 = -D\left(\frac{\sqrt{3}}{2}\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{y}}\right), \quad \mathbf{D}_4 = D\left(\frac{\sqrt{3}}{2}\hat{\mathbf{x}} - \frac{1}{2}\hat{\mathbf{y}}\right); \tag{2}$$

the distance from the centroid to a vertex is D. For the present, assume $R \geq D$. It will be convenient to consider the distances from the vertices on an orthogonal coordinate system.

Vectors from antennas 1 through 4 to the source will be denoted by **R**, **X**, **Y**, **Z** respectively. Let θ denote the angle between $\mathbf{D}_2$ and **R**. Employing the law of cosines, it may be shown that

$$3R^2 = X^2 + Y^2 + Z^2 - 3D^2. \tag{3}$$

It will be necessary to extremize

$$F(X,Y,Z) = X + Y + Z \tag{4}$$

subject to (3).

Using Lagrange multipliers or any other suitable method, the only extremum in the first octant is $X = Y = Z = \sqrt{R^2 + D^2}$; $F = 3\sqrt{R^2 + D^2} > 3R$. This turns out to be a local maximum. Minima must be found by looking along boundaries. These are not necessarily in any coordinate plane. All of X, Y, Z must be greater than or equal to $R - D$. Eliminating Z reduces the problem;

$$f(X,Y) = X + Y + \sqrt{3\left(R^2 + D^2\right) - X^2 - Y^2} \tag{5}$$

must be minimized requiring that none of X, Y, Z be negative. An additional constraint is obtained from (3),

$$X^2 + Y^2 \le 2\left(R^2 + D^2 + RD\right). \tag{6}$$

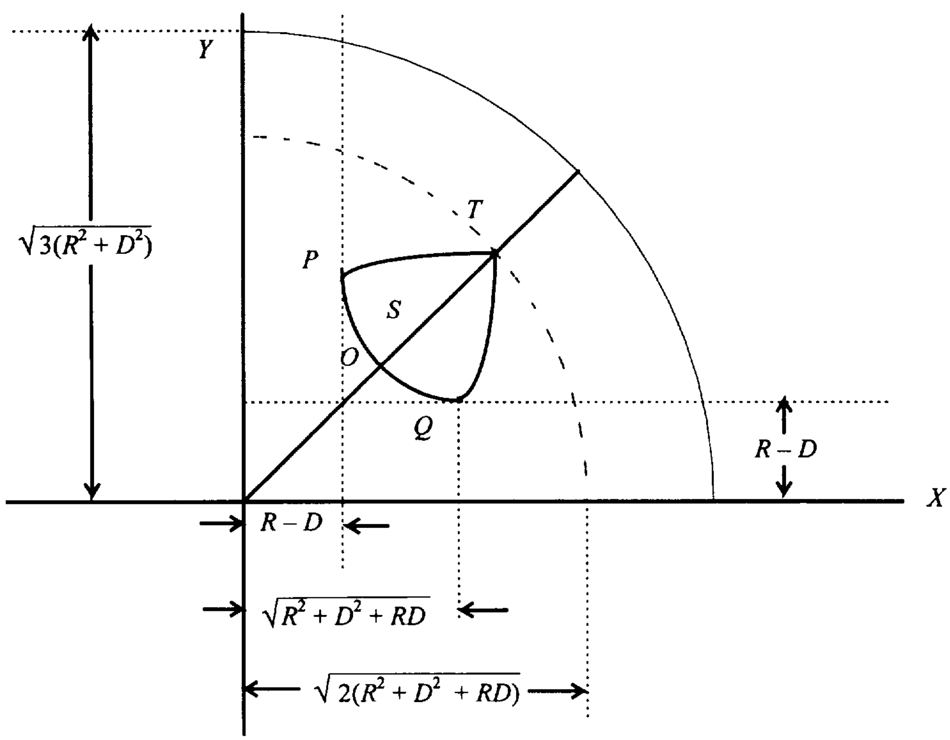

Fig. 1

This constraint does not sufficiently limit f; X and Y cannot both attain their minimum values of $R - D$ at the same time. Boundaries are shown in Fig. 1. The smallest values of Y for a fixed X

occur when $z = 0$ (the lightning event is in the plane of the triangle). This may be demonstrated by writing

$$Y^2 = R^2 + D^2 - 2\mathbf{R} \cdot \mathbf{D}_3. \tag{7}$$

Elevating the lightning event from the plane of the triangle while keeping X and R constant is equivalent to rotating $\mathbf{X}$ and $\mathbf{R}$ about $\mathbf{D}_2$. Here, $\mathbf{D}_3$ has no z component and y is constant. The value of Y is minimized when x is as small as possible (at $z = 0$).

The slope of a tangent to $\overline{POQ}$ is given by

$$\frac{dY}{dX} = -\frac{\sin(2\pi/3 - \theta)}{\sin\theta}\frac{X}{Y}. \tag{8}$$

It is never positive and goes from 0 at $\left(\sqrt{R^2 + D^2 + RD},\, R - D\right)$ to $-\infty$ at $\left(R - D,\, \sqrt{R^2 + D^2 + RD}\right)$. At the point O, $X = Y$, $\theta = \pi/3$, and $dY/dX = -1$. The coordinates of this point are $X = Y = \sqrt{R^2 + D^2 - RD}$.

The gradient

$$\nabla f(X, Y) = (1 - X/Z)\hat{\mathbf{X}} + (1 - Y/Z)\,\hat{\mathbf{Y}} \tag{9}$$

points in the direction of greatest increase of f. The region in Fig. 1 can be thought of as a mountain with its summit at $X = Y = \sqrt{R^2 + D^2}$ (the point labeled S). On the square region $0 \le X \le \sqrt{R^2 + D^2}$, $0 \le Y \le \sqrt{R^2 + D^2}$, we have $Z \ge X$ and $Z \ge Y$. Except at S, the inequality is strict. The gradient points up and to the right. Below the line segment $\overline{OS}$, $\pi/3 \le \theta \le 2\pi/3$, and $X > Y$. The opposite is true above $\overline{OS}$. The mountain has a ridge along this line segment. Below and to the left of S, it is clear that as X and Y decrease, Z increases (the components of the gradient remain positive). Values of f beneath or to the left of $\overline{POQ}$ are smaller than values on this curve.

It will not be necessary to investigate the values of f along $\overline{PT}$ or $\overline{QT}$. These curves can be mapped onto $\overline{POQ}$ by a suitable relabeling of **X**, **Y**, **Z**. Symmetry may be further exploited to reduce the effort. An interchange of **X** and **Y** will map $\overline{OP}$ and $\overline{OQ}$ onto each other.

It turns out to be difficult to parameterize Y in terms of X and obtain the minima along the bounding curve by differentiation. We will bound the values of f along $\overline{OQ}$ by those on a simpler polygonal boundary below $\overline{OQ}$.

We begin at $Y = R - D$ and require

$$f(X,\, R - D) = X + (R - D) + Z(X,\, R - D) \ge 3R. \tag{10}$$

This leads to the inequalities

$$2X^2 - 2(2R + D)X + 2R^2 - D^2 + 2RD \le 0,\ R - \frac{\sqrt{3}-1}{2}D \le X \le R + \frac{\sqrt{3}+1}{2}D. \tag{11}$$

The largest value of X that needs to be considered is $\sqrt{R^2 + D^2 + RD}$. Extrema along this line segment, denoted by $\overline{QB}$ (see Fig. 2), must be at the endpoints;

$$f\left(\sqrt{R^2 + D^2 + RD},\ R - D\right) = R - D + 2\sqrt{R^2 + D^2 + RD} > 3R,$$
$$f\left(R - \left(\sqrt{3} - 1\right)D/2,\ R - D\right) = 3R. \tag{12}$$

The next portion of the bounding curve is drawn by leaving $X = R - \left(\sqrt{3} - 1\right)D/2$ and increasing Y to some point below $\overline{OQ}$. The location of this point may be found by solving for $\cos\theta$ in $X^2 = R^2 + D^2 - 2RD\cos\theta$. We find that

$$\cos\theta = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}}{4}\frac{D}{R},$$
$$\cos\left(\frac{2\pi}{3} - \theta\right) = \frac{\sqrt{3}}{2}\sqrt{1 - \left[\frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}D}{4R}\right]^2} - \frac{1}{2}\left[\frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}D}{4R}\right] \tag{13}$$
$$\le \frac{3^{3/4}}{2^{3/2}} - \frac{\sqrt{3}-1}{4} < \frac{2}{3}.$$

Using the law of cosines to write Y^2 in terms of R, D, and $\cos(2\pi/3 - \theta)$, we see that Y decreases as $\cos(2\pi/3 - \theta)$ increases. The smallest value of Y on $\overline{POQ}$ for $X = R - \left(\sqrt{3} - 1\right)D/2$ is bounded using (13);

$$Y \ge \sqrt{R^2 + D^2 - 4RD/3} > R - 2D/3. \tag{14}$$

We can proceed up to point C whose coordinates are $\left(R - \left(\sqrt{3} - 1\right)D/2,\ R - 2D/3\right)$.

As for the first part of the polygonal bounding curve, the extrema are at the endpoints. The minimum value of f along $\overline{BC}$ is $3R$ and occurs at B. The maximum is located at C and is given by

$$2R - \left[\left(\sqrt{3} - 1\right)/2 + 2/3\right]D + \sqrt{R^2 + \left(14/9 + \sqrt{3}/2\right)D^2 + \left(\sqrt{3} + 1/3\right)RD} > 3R. \tag{15}$$

Proceeding as before, we fix $Y = R - 2D/3$ and see how far to the left we can go and still have $f \ge 3R$. This time we obtain the inequalities

$$R - \frac{\sqrt{42}/2 - 1}{3}D \le X \le R + \frac{\sqrt{42}/2 + 1}{3}D. \tag{16}$$

It will not be necessary to proceed from C all the way to the value indicated in (16). We can stop at a point E which has the same X-coordinate as O. From the estimate

$$\left(R - \frac{\sqrt{42}/2 - 1}{3} D\right)^2 < (R - 0.7D)^2 < R^2 + D^2 - RD, \tag{17}$$

we see that $f(X, Y) > 3R$ at every point of this portion of the boundary.

Finally, we proceed vertically from E to O. The derivative of f with respect to Y is positive everywhere along this line segment. The smallest value of f here is at E.

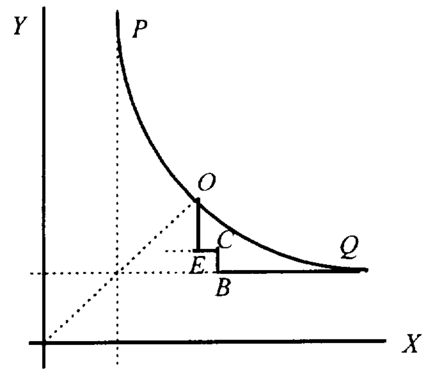

Fig. 2

Due to the relative positions of the actual and polygonal boundaries and the topography of the surface, the inequality is strict; $X + Y + Z > 3R$. When $R > 0$, the result for the case $R \leq D$ may be inferred by interchanging R and D in the preceding calculations and showing $X + Y + Z > 3D \geq 3R$. The case $R = 0$ is immediate.

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