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A Higher Harmonic Optimal Controller to Optimise Rotorcraft Aeromechanical Behaviour

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ABSTRACT

A Higher Harmonic Optimal Controller
to
Optimise Rotorcraft Aeromechanical Behaviour

by

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Three methods to optimise rotorcraft aeromechanical behaviour for those cases where the rotorcraft plant can be adequately represented by a linear model with a system matrix, were identified and implemented in a stand-alone code. For convenience, the helicopter vibration reduction problem was selected as the subject problem for this investigation. These methods determine the optimal control vector which minimises the vibration metric subject to constraints at discrete time points. These methods differ from the commonly used non-optimal constraint penalty methods such as those employed by conventional controllers (e.g., the Deterministic, Cautious, and Dual Controllers) in that the constraints are handled as actual constraints to an optimisation problem rather than as just additional terms in the performance index. The first method is to use a Non-linear Programming algorithm to solve the problem directly. The second method is to solve the full set of non-linear equations which define the necessary conditions for optimality. The third method is to solve each of the possible reduced sets of equations defining the necessary conditions for optimality when the constraints are pre-selected to be either active or inactive, and then to simply select the best solution. The effects of manoeuvres and aeroelasticity on the systems matrix are modelled by using a "proportional navigation" type pseudo-random pseudo-row-dependency scheme to define the systems matrix.

Cases run to date indicate that the first method of solution (i.e., the direct optimisation of the control vector subject to constraints, herein referred to as the "Optimal Controller") is reliable, robust, and, easiest to use. The algorithm employed for direct optimisation is particularly suitable to this problem since although it is designed to solve the general non-linear programming problem, it employs a successive quadratic programming method to solve this more general problem. Since this method initially estimates the Hessian and then updates it successively as the quadratic solutions are updated, and since the only difference between the problem of interest and the standard Quadratic Programming Problem is that the constraints are quadratic rather than linear, the Hessian is invariant to the optimisation process and is analytically known. Correspondingly, modifications to this successive quadratic programming method which will enhance its reliability, overall robustness, and speed, appear feasible for this particular problem of interest.

The second and third methods successfully solved systems of non-linear equations which define the necessary conditions for optimality. It doesn't appear to be practical to use these methods by themselves however, because there exist many (perhaps infinite) solutions to these equations and no real way to recognise the solution yielding the global minimum or to even guarantee convergence to this solution if it were known. The second method can however, be used to verify that the necessary conditions are satisfied when the first method is employed. The use of the second method for verification after the first method obtains a solution was made an option in the stand-alone code.

The Deterministic Controller was added to the stand-alone code to provide a convenient means of comparison. Options to directly optimise the weighting coefficients of the Deterministic Controller either in a specified ratio to one another (i.e., the "Conventional Controller" as referred to herein) or individually (i.e., the "Optimised Conventional Controller" as referred to herein) whilst satisfying the constraints were provided as a means to obtain a more representative and meaningful comparison of controllers and to help access the relative merit of solving the actual optimisation problem. Occasionally a mathematical conditioning problem occurs with the direct optimisation of the individual weighting coefficients of the Optimised Conventional Controller which causes a numerical overflow and subsequent error termination. This occurs when the optimisation process converges to a "solution" which has a harmonic phase angle with an $n\pi/2$ value. In this case the value of one of the associated harmonic coefficients approaches zero whilst the value of the associated weighting coefficient has no finite upper bound.

Cases run to date indicate that the performance (i.e., the reduction of the vibration metric) of the Optimal Controller was superior to that of both the Conventional and the Optimised Conventional Controllers. In accordance with theory, the Optimal Controller yielded a zero vibration metric for square non-singular T-Matrices (i.e., when the number of measurements equals the number of controls with no redundancy). As expected, the Optimised Conventional Controller was superior to the Conventional Controller in performance, but inferior to the Optimal Controller for all cases except a few degenerate cases where the performance of the three controllers was essentially equal. The performance gap was widest for square non-singular T-Matrices.

Nomenclature

A	diagonal $l.u.b.$ constraint limit matrix with dimension $(\frac{M}{2}\times\frac{M}{2})$. The diagonal elements are the A_n values in Eq (13).
A	coefficient of the scalar quadratic term in Relationship (34).
a	general vector with dimension $(l\times 1)$.
a	coefficient of the scalar quadratic term in Eq (72).
A_n	constraint limit for the n-th harmonic constraint equivalent to $(l.u.b.)^2_{Y_n}$ in Relationship (13).
A_1	equality constraint linear coefficient matrix with dimension $(M\times M)$ in Relationship (40).
A_2	inequality constraint linear coefficient matrix with dimension $(M\times M)$ in Relationship (41).
B	diagonal constraint weighting matrix in Relationship (31) with dimension $(M\times M)$. B is normally the identity matrix I_M.
B	coefficient of the scalar linear term in Relationship (34).
B	general matrix with dimension $(r\times s)$.
b	general vector with dimension $(\ell\times 1)$.
b	coefficient of the scalar linear term in Eq (72).
b_1	equality constraint value vector with dimension $(M\times 1)$ in Relationship (40).
b_2	inequality constraint value vector with dimension $(M\times 1)$ in Relationship (41).
C	proportionality constant $\in [0,1]$ for Eq (8) which defines the elements of the "flexible" part of the T-Matrix.
C	constant term in Relationship (34).
C	matrix of dimension $(M\times M)$ equal to $F + F^T$.
c	scalar constant term in Eq (72).
C_{ij}	(i,j)th element of C equal to $f_{ij} + f_{ji}$.
D	scalar $l.u.b.$ in Relationship (35).
F	matrix of dimension $(M\times M)$ equal to $T^T W_Z T$.
f	performance index value for the General Non-linear Programming Problem.

Nomenclature (Continued)

f(•)	performance index function for the General Non-linear Programming Problem.
f_{ij}	(i,j)th element of F.
$\mathcal{F}$	quadratic coefficient matrix of dimension $(M \times M)$ in the performance index of the general QPP (Relationship (39)).
G	vector of dimension $(M \times 1)$ equal to $\left[Z_0^T W_Z T \right]^T$.
$\mathcal{G}^T$	linear coefficient row vector of dimension $(1 \times M)$ in the performance index of the general QPP (Relationship (39)).
g.l.b.	greatest lower bound.
H	scalar equal to $Z_0^T W_Z T_0$.
$\mathcal{H}$	scalar in the performance index of the general QPP (Relationship (39)).
HHC	higher harmonic control.
HVRP	helicopter vibration reduction problem.
i	duty cycle number.
i	index for the elements of the θ-Vector.
I_M	identity matrix of dimension $(M \times M)$.
$\mathbb{I}_\psi$	pseudo-identity tensor of rank three and dimension $(M \times M \times \frac{M}{2})$.
J	performance index value.
J(•)	performance index function.
J_A	augmented performance index for the Conventional Controller problem.
J_{QPP}	performance index for the general QPP.
J_1	performance index for the scalar non-linear programming problem.
J_2	performance index for the scalar Conventional Controller problem.
$\mathcal{J}$	augmented performance index value for the Min/Max calculus problem.
$\mathcal{J}(\bullet,\bullet)$	augmented performance index function for the Min/Max calculus problem.
k	duty cycle number.
L	dimension of the "reference base set of sensors", the "base measurement dimension", the number of rows in the "core matrix".

Nomenclature (Continued)

l	general dimension used for vectors and matrices.
$l.h.s.$	left hand side.
$l.u.b.$	least upper bound.
M	number of control variables to be optimised, dimension of the θ-Vector (the control vector), number of columns in the T-Matrix.
N	total number of sensors, dimension of the Z-Vector (the measurement vector), number of rows in the T-Matrix.
n	harmonic number.
P	number of equality constraints for the general non-linear programming problem.
P	general symmetric matrix of dimension $(r \times r)$ or $(l \times l)$.
Q	number of inequality constraints for the general non-linear programming problem.
Q	upper left sub-matrix of the partitioned Jacobian with dimension $(\frac{3M}{2} \times M)$.
QPP	quadratic programming problem.
R	upper right sub-matrix of the partitioned Jacobian with dimension $(\frac{3M}{2} \times M)$.
R	common value of harmonic magnitude constraint limits equal to $(l.u.b.)_{Y_i}$ for $i = (n-1),\ n,\ (n+1)$.
R_{ij}	(i,j)th uniformly distributed pseudo-random number used to generate a T-Matrix.
R_k	right-sub-matrix of the partitioned Jacobian for the k-th reduced set of equations with dimension (9×3).
r	general dimension used for vectors and matrices.
$\boldsymbol{R}^{M}$	Euclidean M-Space.
$\boldsymbol{R}^{\frac{3M}{2}}$	Euclidean $\frac{3M}{2}$-Space.
$r.h.s.$	right hand side.
s	general dimension used for vectors and matrices.
S	lower left sub-matrix of the partitioned Jacobian with dimension $(\frac{M}{2} \times M)$.

Nomenclature (Continued)

T	quasi-static transfer matrix (the T-Matrix) as identified for the current duty cycle which is common to both Local and Global models, and which relates the Z-Vector to the θ-Vector with dimension $(N \times M)$.
T	general matrix with dimension $(N \times M)$.
T	lower right sub-matrix of the partitioned Jacobian with dimension $(\frac{M}{2} \times M)$.
T_G	quasi-static transfer matrix as identified for the current duty cycle for the Global model which relates the Z-Vector to the θ-Vector with dimension $(N \times M)$.
T_L	quasi-static transfer matrix as identified for the current duty cycle for the Local model which relates the Z-Vector to the θ-Vector with dimension $(N \times M)$.
T_{ij}	(i,j)th element of the T-Matrix.
t	time.
VCS	vibration control system.
W_z	diagonal weighting coefficient matrix of the quadratic term in Z of dimension $(N \times N)$.
W_z	general diagonal matrix of dimension $(N \times N)$ whose diagonal elements are all greater than zero.
W_i	i-th diagonal element of W_z.
$W_{\Delta\theta}$.	diagonal weighting coefficient matrix of the quadratic term in $\Delta\theta$ of dimension $(M \times M)$.
W_θ	diagonal weighting coefficient matrix of the quadratic term in θ of dimension $(M \times M)$.
W_θ	scalar weighting coefficient of the quadratic term in θ in Relationship (36) for the scalar Conventional Controller problem.
W_{θ_i}	i-th diagonal element of W_θ.
Y_n	amplitude of the n-th harmonic sinusoid.
y_n	value at time t of the n-th harmonic sinusoid.
Z	measurement vector (the Z-Vector) with dimension $(N \times 1)$.
Z(•)	measurement vector function with dimension $(N \times 1)$.

Nomenclature (Continued)

Z_0	actual measurement vector during the current duty cycle for the condition where no control is applied with dimension $(N \times 1)$.
Z_{0_i}	i-th element of Z_0.
Z_i	i-th element of Z.
Z_i	estimated measurement vector for the current duty cycle with dimension $(N \times 1)$.
Z_{i-1}	actual measurement vector for the previous duty cycle with dimension $(N \times 1)$.
α	slack variable vector with dimension $(\frac{M}{2} \times 1)$.
α_k	k-th element of α.
α_ψ	diagonal slack variable matrix with dimension $(\frac{M}{2} \times \frac{M}{2})$.
Γ	vector comprised of the α and λ vectors with dimension $(M \times 1)$ and equal to $[\alpha : \lambda]^T$.
Γ_k	vector with dimension (3×1) comprised of the k-th possible combination of elements from the α and λ vectors as specified by Relationship (123).
δ_{ij}	Kronecker Delta $= \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$.
$\Delta\theta$	change in the value of θ from that of the previous duty cycle with dimension $(M \times 1)$ and equal to $\theta_k - \theta_{k-1}$ for the k-th current duty cycle.
$\Delta\theta_k$	increment in the k-th element of θ.
ζ	general vector of dimension $(M \times 1)$.
η	general vector of dimension $(N \times 1)$.
η_0	general vector of dimension $(N \times 1)$.
θ	control vector (the θ-Vector) with dimension $(M \times 1)$.
θ	scalar control variable for the scalar non-linear programming and scalar Conventional Controller problems.
θ^*	solution value of θ.

Nomenclature (Continued)

θ_{c_n}	coefficient of the n-th harmonic Cosine term.
θ_i	i-th element of θ.
θ_i	control vector (the θ-Vector) defined for the current or i-th duty cycle with dimension $(M \times 1)$.
θ_{i-1}	control vector (the θ-Vector) defined for the previous or $(i-1)$th duty cycle with dimension $(M \times 1)$.
θ_{MAX}	*l.u.b.* constraint limit for θ in the scalar non-linear programming problem.
θ_{s_n}	coefficient of the n-th harmonic Sine term.
θ_ψ	constraint matrix form of θ with dimension $(M \times \frac{M}{2})$.
Θ	vector comprised of the θ and α vectors with dimension $(\frac{3M}{2} \times 1)$ and equal to $[\theta \vdots \alpha]^T$.
λ	adjoint vector comprised of $\frac{M}{2}$ Lagrangian multipliers with dimension $(\frac{M}{2} \times 1)$.
λ_k	k-th element of λ.
ξ	general vector of dimension $(M \times 1)$.
ξ	$(M \times 1)$ dimensional control vector counterpart of the θ-Vector for the quadratic programming sub-problem of the successive quadratic programming method used to solve the general non-linear programming problem.
Ξ	vector comprised of Equations (103) and (100) with dimension $(\frac{3M}{2} \times 1)$ and equal to [Equation(103) : Equation(100)]^T which equals $\left\{\left[\frac{\partial \mathcal{J}}{\partial \theta}\right]^T \vdots \; \phi_C(\Theta)\right\}^T$.
$\phi(\bullet)$	equality constraint vector function with dimension $(P \times 1)$.
$\phi(\bullet)$	equality constraint matrix function with dimension $(\frac{M}{2} \times \frac{M}{2})$.
$\phi_C(\bullet)$	equality constraint vector function with dimension $(\frac{M}{2} \times 1)$ formed by compression of the corresponding equality constraint matrix function $\phi(\bullet)$ into a column vector by the process described in Section 2.4.1.
ϕ_k	k-th element of $\phi_C(\bullet)$.

Nomenclature (Continued)

$\phi_{QPP}(\bullet)$	equality constraint vector function with dimension $(M \times 1)$ for a QPP.
Φ	vector comprised of the Θ and λ vectors with dimension $(2M \times 1)$ and equal to $[\Theta:\lambda]^T = [\theta:\alpha:\lambda]^T = [\theta:\Gamma]^T$.
Φ_k	vector comprised of the θ and Γ_k vectors with dimension (9×1) and equal to $[\theta:\Gamma_k]^T$.
$\psi(\bullet)$	inequality constraint vector function with dimension $(Q \times 1)$.
$\psi(\bullet)$	inequality constraint matrix function with dimension $(\frac{M}{2} \times \frac{M}{2})$.
$\psi_C(\bullet)$	inequality constraint vector function with dimension $(\frac{M}{2} \times 1)$ formed by compression of the corresponding inequality constraint matrix function $\psi(\bullet)$ into a column vector by the process described in Section 2.4.1.
$\psi_{ij}(\bullet)$	(i,j)th element of $\psi(\bullet)$.
$\psi_{QPP}(\bullet)$	inequality constraint vector function with dimension $(M \times 1)$ for a QPP.
Ψ	vector comprised of Equations (103), (100), and (104) with dimension $(2M \times 1)$ and equal to $[\Xi \vdots \text{Equation}(104)]^T$ which equals $\left\{\Xi \vdots \left[\frac{\partial \mathcal{J}}{\partial \alpha}\right]^T\right\}^T$.
0_M	zero vector of dimension $(M \times 1)$.
$0_{\frac{M}{2}}$	zero matrix of dimension $(\frac{M}{2} \times \frac{M}{2})$.
0_N	zero vector of dimension $(N \times 1)$.

Nomenclature (Continued)

Superscripts

T	matrix transposition
*	solution or unique solution to a problem.

Subscripts

C	denotes a column matrix (vector) formed from a diagonal matrix by shifting the diagonal elements to a single column.
G	global model.
i	duty cycle or element position number.
j	element position number.
k	duty cycle or element position number.
L	local model.
L	dimension of the “reference base set of sensors”, the “base measurement dimension”, the number of rows in the ‘core matrix”.
M	dimension of the control vector (the θ-Vector).
n	harmonic number.
N	dimension of the measurement vector (the Z-Vector).
QPP	quadratic programming problem.
0	value when no control is applied.
1FD	one-sided finite difference method.
2FD	two-sided finite difference method.

1.0 INTRODUCTION

Given the predicted growth in air transportation, the potential exists for significant market niches for rotary wing subsonic vehicles. Technological advances which optimise rotorcraft aeromechanical behaviour can contribute significantly to their commercial development, acceptance, and sales. An example of the optimisation of rotorcraft aeromechanical behaviour which is of interest is the Helicopter Vibration Reduction Problem (HVRP). Although the HVRP was selected as the subject problem for this investigation, it is *emphasised* that the analysis described herein is applicable to all those rotorcraft aeromechanical behaviour optimisation problems for which the relationship between the harmonic control vector and the measurement vector can be adequately described by a linear model.

The reduction of rotorcraft vibration and loads is an important means to extend the useful life of the vehicle and to improve its ride quality. Although vibration reduction can be accomplished by using passive dampers and/or tuned masses, active control has the potential to reduce vibration throughout a wider flight regime whilst requiring less additional weight to the aircraft than passive methods.

Active control is achieved using a closed-loop "feedback" controller (Figures 1 and 2 and Reference 1). Typically, the measurement vector is defined from the measurements obtained with vibration sensors (e.g., accelerometers), the relationship between the measurement vector and the control vector (i.e., the helicopter plant matrix) is determined (i.e., plant matrix identification), and then a new control vector which will hopefully reduce the vibration metric is defined and used to control the system (e.g., rotor blade pitch). It is this last mentioned function, that is the selection of the new control vector, that is the subject of this research. Commonly used conventional classical controller schemes (e.g., the Deterministic, Cautious, and Dual Controllers described in References 1 and 2) employ a non-optimal constraint penalty method to define the new control vector, whilst the Optimal Controller which is the primary subject of this document employs a non-linear programming optimisation technique that accounts for the constraints in accordance with optimisation theory.

2.0 TECHNICAL

The systems models used by the controllers described herein and their mathematical equivalency are presented first. Next, a heuristic "seat-of-the-pants" proportional navigation type technique used during controller performance verification to model aeroelastic effects for the Helicopter Vibration Reduction Problem (HVRP) is described. The equations used by both the conventional (Deterministic) and Optimal Controllers presented in Reference 1 are in general matrix/vector form. One of the purposes of this documentation is to present these equations in their expanded form which is amenable to computer coding. These equations together with their associated equations defining the necessary conditions for optimality and the required equations to solve the non-linear programming problem (i.e., the analytic gradient, the Hessian, and the Jacobian) are then presented. The pseudo-random pseudo-row-dependency scheme used to generate the systems matrix (i.e., plant matrix) and the initial measurement vector to heuristically model the aeroelastic effects is described next. Then the Stand-Alone Optimal Controller System is described, and finally the results of a numerical study comparing the Optimal Controller with optimised conventional controllers for various "proportions" of aeroelasticity is presented.

2.1 General Controller Definition

The general controller scheme (Figures 1 and 2) uses the measured state (i.e., the measurement vector) to define a new control vector which will hopefully reduce the vibration metric during the next duty cycle. To accomplish this, it is necessary to have first assumed a plant model (i.e., a mathematical model relating the estimated measurement vector to the control vector). A linear model with a Systems Matrix (also commonly referred to as the Plant Matrix or the T-Matrix) that estimates the measurement vector is assumed for all the controllers mentioned herein. It is necessary to identify or define this systems matrix (T-Matrix identification) before the new control vector can be determined because the new control vector is, in general, dependent on this matrix. This identification of the systems matrix is, in general, dependent on the actual measurement vector which defines the plant state of interest (i.e., not the estimated measurement vector defined by the assumed mathematical model) and is not a subject of this research.

2.1.1 Systems Models of Controlled Vibration Response

The mathematical model (i.e., the plant model) relating the estimated measurement vector to the control vector is assumed to be linear for all the controllers mentioned herein. The Local and Global model and their mathematical equivalence are presented first, then the sensitivity of the plant matrix elements for higher harmonic Fourier models and their representation by uniformly pseudo-random numbers are discussed, and lastly the modelling of aeroelastic effects by means of a Proportional Navigation type scheme to introduce some degree of linear dependency to the rows of a non-square matrix is described.

2.1.1.1. Local Model

The local model expresses the relationship of the estimated measurement vector during the current duty cycle to the difference in control between the current duty cycle and the previous one and the actual measurement vector for the previous duty cycle; specifically:

$$Z_i = T_L(\theta_i - \theta_{i-1}) + Z_{i-1} \tag{1}$$

where:

T_L	quasi-static local model T-Matrix as identified for the current duty cycle with dimension $(N \times M)$
Z_{i-1}	actual measurement vector for the previous duty cycle with dimension $(N \times 1)$
Z_i	estimated measurement vector for the current duty cycle with dimension $(N \times 1)$
θ_{i-1}	control vector (θ-vector) used during the previous duty cycle with dimension $(M \times 1)$
θ_i	control vector (θ-vector) defined for the current duty cycle with dimension $(M \times 1)$
i	duty cycle number
N	dimension of the Z-Vector, number of rows in the T-Matrix
M	dimension of the θ-vector, number of columns in the T-Matrix

2.1.1.2 Global Model

The global model expresses the relationship of the estimated measurement vector during the current duty cycle to the control in the current duty cycle and the actual measurement vector for the condition where no control is applied; specifically:

$$z_i = T_G \theta_i + z_0 \qquad (2)$$

where:

T_G	quasi-static global model T-Matrix as identified for the current duty cycle with dimension $(N \times M)$
z_0	actual measurement vector during the current duty cycle for the condition where no control is applied with dimension $(N \times 1)$
z_i	estimated measurement vector for the current duty cycle with dimension $(N \times 1)$
i	duty cycle number
N	dimension of the z-vector, number of rows in the T-Matrix
M	dimension of the θ-vector, number of columns in the T-Matrix

2.1.1.3 Mathematical Equivalence of the Local and Global Models

The expanded form of the controller algorithm equations presented herein were derived assuming the global linear model for the plant. It is noted however, that the global linear model can be made equivalent to the local linear model by the appropriate selection of the Z_0-vector in the global model. Equating the local and global expressions for Z_i defined by equations (1) and (2), respectively, yields

$$Z_i = T_L(\theta_i - \theta_{i-1}) + Z_{I-1} = T_G\,\theta_i + Z_0$$

If Z_0 is defined according to:

$$Z_0 = Z_{i-1} - T_L\,\theta_{i-1} \qquad (3)$$

then

$$T_L\,\theta_i = T_G\,\theta_i \;\Rightarrow\; \boxed{T_L \equiv T_G \equiv T} \qquad (4)$$

where T is the general T-Matrix equivalent to both T_L and T_G. Equation (3) then becomes

$$\boxed{Z_0 = Z_{i-1} - T\theta_{i-1}} \qquad (5)$$

Notice that equation (5) can be rearranged to

$$Z_{I-1} = T\theta_{i-1} + Z_0 \qquad (6)$$

which is just the global model applied to the previous duty cycle. A specified local model can be transformed to the global model assumed for the controller algorithm equations by simply using equation (5) to define Z_0 for the global model.

2.1.1.4 Sensitivity of the T-Matrix Elements and Its Modelling Using Random Methods

The helicopter applications of interest typically require the definition of the harmonic content of the responses of interest to the harmonics of a specified control. The responses are measured by various sensors (e.g., accelerometers) mounted on the helicopter and its rotor system. A Fast Fourier Transform (FFT) is then performed on this response data to define the coefficients of the associated Fourier Series which describe these responses of interest. The measurement (response) vector is typically comprised of the coefficients of selected higher harmonic sinusoidal terms of these Fourier Series. The specified control vector is typically comprised of selected mono-frequency Sine/Cosine coefficient tuples, or equivalently mono-frequency amplitude/phase angle tuples, from those which describe the higher harmonic sinusoidal excitation of blade pitch. Once the measurement vector for the current duty cycle is defined, the T-Matrix can be identified by various methods (Reference 3). The elements of the T-Matrix can, however, appear to have random values when compared with one another for a particular duty cycle and to appear to vary significantly between duty cycles because of the sensitivity of the components of the measurement vector (i.e., the selected higher harmonic coefficients of the Fourier Series which describe the responses of interest) to minor variations in the wave-form of the sensor measurements which include the effects of process and measurement noise. This apparent randomness appears to have uniform distribution rather than a distribution with cluster points or modes.

This apparent randomness can be exploited to greatly simplify the T-Matrix definition process for controller feasibility and desirability analyses. The elements of the T-Matrix can be generated simply by using a uniform random distribution generator with a range of values appropriate for the problem rather than using a detailed rotor simulation for the purposes of the study reported herein. In general, the largest square sub-matrix of the T-Matrix will be non-singular (i.e., its rows and columns will be linearly independent) if its elements are generated in this manner. This fact provides a simple means to verify that the optimisation algorithm is working properly for a square T-Matrix when the constraints are not active. In this case, the minimised performance index should be identically equal to zero (see Section 2.3.2).

2.1.1.5 Modelling Aeroelastic Effects by Means of a "Proportional Navigation" Type Scheme to Introduce Some Degree of Linear Dependency to the Rows of a Non-Square T-Matrix

In general, the elements of the control vector should be at least partially independent of one another. Fortunately this requirement is relatively easy to satisfy by the appropriate selection of the control variables to be optimised (e.g., mono-frequency Sine/Cosine coefficient tuples or amplitude/phase angle tuples from the expression defining the control).

It is also desirable that the elements of the measurement vector be at least partially independent of one another to eliminate redundancy in the performance index and to reduce the computational burden resulting from a larger number of rows in the T-Matrix. For the HVRP, the independence of the elements of measurement vector depends to some extent on the elasticity of the aircraft. For example, if a helicopter had purely translational motion and if two sets of three orthogonal accelerometers were firmly fixed in different locations in its fuselage, one would expect that the acceleration vector determined from each set of accelerometers would be the same if the airframe was perfectly ridged and there were no measurement errors. On the other hand if the airframe was not perfectly ridged (e.g., elastic), one would expect that the acceleration vector determined from each set of accelerometers would differ somewhat. In the first case, the two sets of accelerometers would be fully dependent while in the second case the two sets of accelerometers would exhibit some degree of independence from each other.

The least number of possible independent measurements (i.e., the minimum number of measurement degrees-of-independence) occurs for an ideally perfect rigid aircraft whose motion is purely translational. For convenience, this least number of possible independent measurements for this "ideal aircraft" is referred to as the "base measurement dimension", and any reference set of sensors of "base measurement dimension" which spans the measurement degrees-of-independence space for this "ideal aircraft" is referred to as a "reference base set of sensors". Additional sensors mounted on this "ideal aircraft" will not provide any information in addition to that obtained from the "reference base set of sensors". Indeed, if any additional sensor is identically equivalent to, and has identically the same orientation as one of the reference base set sensors, the corresponding row of the T-Matrix for this redundant sensor would be identical to that corresponding to its associated reference base set sensor even though the two sensors might be mounted at different locations on this "ideal aircraft".

If the aircraft is flexible and "bends" under loads, sensors in addition to the "reference base set of sensors" will provide additional information, and the dimension of the measurement degrees-of-independence space will be greater than the "base measurement dimension". In general, the greater the flexibility of the aircraft, the greater the degree of independence additional measurements will have. This fact, together with the recognition that a perfectly rigid aircraft with pure translational motion (e.g., translational vibration) is the limiting case having the minimum number of independent measurements, suggests a simple means by which elastic and non-translational effects can be added to the simple random model defined in the previous section (i.e., Section 2.1.1.4). The degree of independence of the elastic and non-translational effects as measured by sensors in addition to the "reference base set of sensors" can be simulated by a "proportional navigation" type scheme in which the required additional rows of the T-Matrix are defined as a linear combination of the row corresponding to one of the "reference base set sensors" and a new randomly generated row.

The ideally perfect rigid aircraft whose motion is purely translational would be most efficiently modelled with a minimum size $(L \times M)$ T-Matrix where M is the minimum number of independent controls which comprise the Theta-Vector (i.e., the control vector) and L is the "base measurement dimension" (i.e., the least number of possible independent measurements for this "ideal aircraft"). For convenience this ideal minimum size T-Matrix is referred to as the "core matrix". Each row of this $(L \times M)$ "core matrix" corresponds to a specific sensor in the "reference base set of sensors". Each sensor in the "reference base set of sensors" has a corresponding row of this "core matrix", consequently this mapping is "one-to-one onto". Although it is usually the case that L is greater than or equal to M, it should be noted that the controller optimisation algorithm should be sufficiently robust to provide a usable non-catastrophic control update if sensors fail and L becomes less than M. An algorithm was devised to generate a synthetic T-Matrix which would simulate a flexible aircraft. This algorithm uses pseudo-random "uniformly distributed" numbers to define the rows of the more general (N×M) T-Matrix system according to:

1. Each row of the (L×M) "core matrix" (i.e., each row corresponding to a sensor belonging to the "reference base set of sensors") is generated by using a pseudo-random uniform distribution number generator to define each element in the row. This procedure is the same as that described in the previous section (i.e., Section 2.1.1.4). The general element T_{ij} of the "core matrix" is defined (see Figure 3):

$$\boxed{T_{ij} = R_{ij} \qquad \text{for } i \leq L \ \& \ j \leq M} \qquad (7)$$

where

$R_{ij} \in$ {uniformly distributed pseudo-random numbers}

i row number in the T-Matrix corresponding to the i-th sensor ∈ "reference base set of sensors"

j column number in the T-Matrix corresponding to the j-th element of the control vector

L dimension of the "reference base set of sensors", the "base measurement dimension", the number of rows in the "core matrix"

M number of control variables to be optimised, dimension of the θ-Vector (Control Vector), number of columns in the T-Matrix

2. Rows in addition to the first L rows of the "core matrix" define the "flexible" part of the T-Matrix and are defined by a "proportional navigation" type scheme. The general element T_{ij} of this "flexible" part of the T-Matrix is defined (see Figure 3):

$$\boxed{T_{ij} = C\,T_{\ell j} + (1-C)R_{ij} \quad \text{for } L < i \leq N \;\&\; j \leq M} \quad (8)$$

where

C is the proportionality constant $\in [0,1]$

ℓ defines the row number of the reference sensor of the "reference base set of sensors" which is used in the definition of the i-th row of the T-Matrix according to Equation (8)

N total number of sensors, number of rows in the T-Matrix, dimension of the Z-vector

Notes:

1. A least upper bound (l.u.b.) whose value is meaningful for the problem at issue is specified for the uniformly distributed pseudo-random numbers which are generated. The greatest lower bound (g.l.b.) equal to the negative of the l.u.b. is assumed for the pseudo-random numbers. The values of R_{ij} are scaled to be within these bounds.
2. The Z_0-vector is defined according to the same procedure that is used to define the T-Matrix elements.

2.1.2 General Controller

The Vibration Control System (VCS) described herein consists of a closed-loop controller (see Figures 1 and 2) which computes a new Higher Harmonic Control (HHC) vector to be used during the next duty cycle. This computation is designed to reduce vibration and is based on the latest measured state vector and the latest identified T-Matrix. The relationship between the measured state vector (i.e., the Z-Vector) and the control vector (i.e., the θ-Vector is referred to as the "helicopter plant function" and can be highly non-linear. The "helicopter plant function" is that part of the actual helicopter itself which relates the measured state vector to the control vector. It is assumed that the "helicopter plant function" for the controllers described herein can be approximated with sufficient accuracy between successive duty cycles by a linear relationship such as those described in Sections 2.1.1.1 and 2.1.1.2. It is emphasised that these linear approximations define an "estimated" state vector based on a computed control vector rather than an actual measured state vector, and that in general the two state vectors will not be equivalent. The computation of a new control vector is, in general, dependent on the T-Matrix part of this linear relationship and consequently the T-Matrix must be known before this computation can be accomplished. A T-Matrix identification process such as one of those described in Reference 3 is usually employed to define the T-Matrix. It is assumed that the T-Matrix is identifiable and is consequently known for the subject research described herein. Identification processes themselves are not the subject of this research. The emphasis of this research is the computation of the new control vector for the next duty cycle given the T-Matrix.

2.2 "Vibration Minimisation" Via the Conventional Controller

The Conventional Controller is simply a special case of the General Controller (see Section 2.1.2 and Figure 2) in which a specific computational procedure is assumed for the determination of the control vector to be used during the next duty cycle. The assumed computational scheme is similar to that of the Deterministic Controller which is described in References 1 and 2. The definition and disposal of constraints, the definition of the performance index and its non-optimal augmentation with constraints, the definition of the problem to be solved and its necessary conditions for optimality, and the specific computational procedure used to determine the new control vector are presented in Sections 2.2.1 and 2.2.2.

2.2.1 Conventional Controller Problem

The Conventional Controller is the "classic" controller which has been in use for many years. Although sub-optimal, this controller has worked well in many applications, particularly for steady state operating systems whose control approaches a steady state value as the desired operating conditions are achieved. Instead of solving the "real" problem which seeks to minimise the "real" performance index subject to "real" constraints imposed on the system and the control vector, the Conventional Controller solves a simpler problem which does not impose any "real" constraints during minimisation. This is accomplished by adjoining a specific form of the constraints with weighting coefficients to the "real" performance index to form an augmented performance index, and then by directly minimising this augmented performance index. This constraint form is designed to be positive for all conditions and to approach zero from this positive "side" as the system approaches its desired steady state value. This technique of adjoining constraints to the performance index is referred to as "Internal Limiting" (References 1 and 2). The idea is that by minimising this augmented performance index the constraint functions will be minimised simultaneously with the "real" performance index. The motivation behind this approach is simply that a known analytic solution exists to this problem, while solution of the real problem has historically been much more difficult. The Conventional Controllers described in References 1 and 2 also employ what is referred to as "External Limiting" which is an after-the-fact imposition of constraints with corresponding adjustment of the control after the minimisation of the augmented performance index has been accomplished. The Conventional Controller used during this research and described herein does not employ this External Limiting.

A convenient and simple form for the "real" part of the performance index which can provide an excellent measure of the helicopter vibration is a simple quadratic metric of the individual vibration measurements; specifically:

$$\boxed{J = Z^T W_Z Z} \qquad (9)$$

where:

J is the "real" performance index

W_Z is the diagonal weighting matrix for Z with dimension $(N \times N)$

Z is the measurement vector (Z-Vector) with dimension $(N \times 1)$

N is the dimension of the Z-Vector, number of rows in the T-Matrix

It is emphasised that this choice of a performance index is but one of many possibilities. This performance index is relatively simple, representative of the control objective, and amenable to the pertinent mathematical derivations.

The control vector for the helicopter applications of interest is typically comprised of selected mono-frequency Sine/Cosine coefficient tuples from the sinusoidal terms which describe the higher harmonic sinusoidal excitation of blade pitch. The actual hardware limitations require constraints having the form of a least upper bound ($l.u.b.$) imposed on the amplitude of each of the higher harmonic frequency sinusoids of interest (i.e., the mono-frequency Sine/Cosine coefficient tuples which form the control vector). For each frequency of interest, the corresponding sinusoid has the form:

$$y_m = \theta_{S_m} \operatorname{Sin}(n\Omega t) + \theta_{C_m} \operatorname{Cos}(n\Omega t) \qquad (10)$$

where

m is the harmonic number

t is the time

y_m is the pitch contribution of the n-th harmonic sinusoid

θ_{C_m} is the coefficient of the n-th harmonic Cosine term

θ_{S_m} is the coefficient of the n-th harmonic Sine term

The amplitude of the sinusoid Y_n is then expressed (see Reference 4):

$$Y_m = \sqrt{\theta_{S_m}^2 + \theta_{C_m}^2} \qquad (11)$$

from which the "real" (or "actual") $l.u.b.$ constraint can be written:

$$Y_m = \sqrt{\theta_{S_m}^2 + \theta_{C_m}^2} \leq (l.u.b.)_{Y_m} \qquad (12)$$

where

$(l.u.b.)_{Y_m}$ is the least upper bound constraint limit for Y_m.

For convenience, this constraint is expressed in the form

$$\boxed{\theta_{S_n}^2 + \theta_{C_n}^2 \leq (l.u.b.)_{Y_n}^2 \equiv A_n} \qquad (13)$$

where

A_n is the constraint limit for the n-th harmonic constraint.

Instead of attempting to solve the optimisation problem with the constraints expressed by relationship (13) treated as actual constraints to the problem, the Conventional Controller seeks to drive the harmonic coefficients in the control vector to zero by adding a weighted summation of their squares to the "real" performance index defined by equation (9) to form the augmented performance index to be minimised. This weighted summation is sometimes referred to as "internal limiting" and is expressed by

$$\sum_{i=1}^{M} W_{\theta_i} \theta_i^2 = \theta^T W_\theta \theta \quad \text{Internal Limiting Term} \qquad (14)$$

where

θ is the Control Vector (θ-Vector) with dimension $(M \times 1)$.

θ_i is the i-th element of the θ-Vector equivalent to an element from one of the $(\theta_{S_n}, \theta_{C_n})$ tuples which comprise the θ-Vector

W_{θ_i} is the weighting coefficient for the i-th element of the θ-Vector in the "internal limiting term", i-th diagonal element of the W_θ matrix.

W_θ is the diagonal weighting coefficient matrix in the "internal limiting term" with dimension $(M \times M)$.

i is the index for the elements of the θ-Vector.

M is the dimension of the θ-Vector, number of columns in the T-Matrix.

The augmented performance index J_A is

$$\boxed{J_A = Z^T W_Z Z + \theta^T W_\theta \theta} \qquad (15)$$

The Conventional Controllers described in References 1 and 2 also include a $\Delta\theta$ term of the form

$$\sum_{i=1}^{M} W_{\Delta\theta_i} (\Delta\theta_i)^2 = \Delta\theta^T W_{\Delta\theta} \Delta\theta$$

where

$\Delta\theta = \theta_k - \theta_{k-1}$ with dimension $(M \times 1)$

$W_{\Delta\theta}$ is the diagonal weighting coefficient matrix with dimension $(M \times M)$.

k is the duty cycle number.

This $\Delta\theta$ term is not included in the analysis reported herein for simplicity, and because its inclusion will not enhance the performance of the Conventional Controller.

The Conventional Controller problem in its simple form is

$$\boxed{\begin{array}{ll} \underset{\theta \in \mathbb{R}^M}{\text{Minimise:}} & J_A = Z^T W_Z Z + \theta^T W_\theta \theta \\ \\ \text{Subject to:} & \underline{\text{NO}} \text{ Constraints per se} \end{array}} \qquad (16)$$

where

$\mathbb{R}^M$ denotes Euclidean M-Space.

2.2.2 Necessary Conditions for "Optimality'

The necessary conditions for "optimality" for the Conventional Controller are readily derived by differentiating an appropriate equation defining J_A with respect to θ, setting the resulting derivative equal to zero, and then solving for θ in accordence with matrix Max/Min calculus. First the global model defining Z as a function of θ (i.e, Equation (2)) is substituted into Equation (16) yielding

$$J_A = [T\theta + Z_o]^T W_Z [T\theta + Z_o] + \theta^T W_\theta \theta \qquad (17)$$

where the duty cycle index has been omitted for convenience and

T has dimension $(N \times M)$
W_Z has dimension $(N \times N)$
W_θ has dimension $(M \times M)$
Z_o has dimension $(N \times 1)$
θ has dimension $(M \times 1)$

Expansion of Equation (17) together with manipulation of the various terms yields

$$J_A = \theta^T T^T W_Z T\theta + \theta^T T^T W_Z Z_o + Z_o^T W_Z T\theta + Z_o^T W_Z Z_o + \theta^T W_\theta \theta \qquad (18)$$

Noting that

$$\theta^T T^T W_z Z_o = \theta^T T^T [Z_o^T W_z^T]^T = [T\theta]^T [Z_o^T W_z^T]^T$$

$$= \left([Z_o^T W_z^T][T\theta]\right)^T = [Z_o^T W_z^T T\theta]^T \qquad (19)$$

Since W_z is diagonal, $W_z^T = W_z$

and since $Z_o^T W_z^T T\theta$ is scalar,

$$[Z_o^T W_z^T T\theta]^T = Z_o^T W_z^T T\theta$$

and so Equation (19) can be written as

$$\theta^T T^T W_z Z_o = Z_o^T W_z^T T\theta \qquad (20)$$

Substitution of Equation (20) into Equation (18) yields

$$J_A = \theta^T T^T W_z T\theta + 2Z_o^T W_z T\theta + Z_o^T W_z Z_o + \theta^T W_\theta \theta \qquad (21)$$

Let

$$\boxed{\begin{array}{c} F = T^T W_z T \\ \hline G^T = Z_o^T W_z T \\ \hline H = Z_o^T W_z Z_o \end{array}} \quad \begin{array}{ll} \text{with dimension } (M \times M) & (22) \\ \text{with dimension } (1 \times M) & (23) \\ \text{a scalar (i.e., with dimension } (1 \times 1)) & (24) \end{array}$$

Then

$$\boxed{J_A = \theta^T F\theta + 2G^T\theta + H + \theta^T W_\theta \theta} \qquad (25)$$

Differentiating Equation (25) with respect to θ and setting resulting equation equal to zero yields

$$\frac{\partial J_A}{\partial \theta} = 2\theta^T [F + W_\theta] + 2G^T = 0 \qquad (26)$$

from which

$$\theta^T = -G^T [F + W_\theta]^{-1}$$

or equivalently (noting that F and W_j are symmetric)

$$\theta = -[F + W_\theta]^{-1} G \qquad (27)$$

Noting that

$$G = [G^T]^T = [Z_0^T W_z T]^T = [W_z T]^T Z_0 = T^T W_z^T Z_0$$

Since W_z is diagonal, $W_z^T = W_z$ and consequently

$$G = T^T W_z Z_0 \qquad (28)$$

Substituting Equations (22) and (28) into Equation (27) yields

$$\boxed{\theta = -[T^T W_z T + W_\theta]^{-1} T^T W_z Z_0} \qquad (29)$$

which agrees with that presented in References 1 and 2.

Selection of the control vector (i.e., the θ-Vector) utilising Equation (29) satisfies the necessary conditions for "optimality" for the Conventional Controller as per matrix Max/Min calculus.

The solutions to three variations of the problem posed by Relationship (16) were provided in the "stand-alone" code developed within this study; these are:

1. Specify W_θ by input and solve Relationship (16) explicitly by using Equation (29). This is NOT computationally a numerical optimisation problem since the control vector (i.e., the θ-Vector) is explicitly determined.

2. Specify W_θ and an initial scalar scaling coefficient C_{W_θ} for W_θ by input, and then optimise C_{W_θ} to yield the minimum J_A solving Relationship (16) explicitly at each optimisation step by using Equation (29). This is a one dimensional optimisation problem. The controller which solves this problem is referred to herein as the "<u>Conventional Controller</u>."

3. Specify the initial W_θ by input and then optimise each of its diagonal elements to yield the minimum J_A by solving Relationship (16) explicitly at each optimisation step by using Equation (29). This is an M-dimensional optimisation problem. The controller which solves this problem is referred to herein as the "<u>Optimised Conventional Controller</u>."

2.3 Vibration Minimisation as a Non-Linear Programming Problem

The Optimal Controller is simply a special case of the General Controller (see Section 2.1.2 and Figure 2) in which a specific computational procedure is assumed for the determination of the control vector to be used during the next duty cycle. The Helicopter Vibration Reduction Problem (HVRP) was posed as a non-linear programming problem and a successive quadratic programming method developed by Schittkowski, Stoer, and Gill et all (References 7 and 9 through 15) was employed to solve it. The definition of constraints, performance index, the problem to be solved, and the specific computational procedure used to determine the new control vector are presented in Sections 2.3.1 and 2.3.2.

2.3.1 General Non-Linear Programming Problem

The general non-linear programming problem can be expressed in the form

$$\left.\begin{array}{ll} \underset{\theta \in \mathbb{R}^M}{\text{Minimise}} & J = f[z(\theta)] \\ \text{Subject to:} & \phi(\theta) = 0 \\ & \psi(\theta) \geq 0 \end{array}\right\} \quad (30)$$

where

$\mathbb{R}^M$ denotes Euclidean M-Space.

$z(\bullet)$ is the measurement vector with dimension $(N \times 1)$.

θ is the control vector with dimension $(M \times 1)$.

$f[\bullet]$ is the scalar performance index with dimension (1×1).

$\phi(\bullet)$ is the equality constraint vector with dimension $(p \times 1)$.

$\psi(\bullet)$ is the inequality constraint vector with dimension $(Q \times 1)$.

p is the number of equality constraints.

Q is the number of inequality constraints.

The helicopter vibration reduction problem was described somewhat in the text prior to the definition the Conventional Controller Problem in Section 2.2.1. The performance index is that defined by Equation (9) and the control vector (i.e., the Theta-Vector) is comprised of mono-frequency Sine/Cosine coefficient tuples which typically describe the blade pitch sinusoids as per Equation (10). Since the control vector is comprised of tuples, its dimension is always even. It is assumed that there is an amplitude constraint of the form defined by Equation (13) for each mono-frequency tuple in the control vector, consequently the number of constraints is identically half the dimension of the control vector. The helicopter vibration reduction problem can be expressed as a non-linear programming problem in the form

$$\left.\begin{array}{ll} \underset{\theta \in \mathbb{R}^M}{\text{Minimise}} & J = [z(\theta)]^T W_z [z(\theta)] \\ \text{Subject to:} & \theta_\psi^T B \theta_\psi \leq A \end{array}\right\} \quad (31)$$

where

$Z(\theta) = T\theta + Z_0$ with dimension $(N \times 1)$

W_Z is a diagonal weighting coefficient matrix for $Z(\theta)$ with dimension $(N \times N)$

θ is the control vector with dimension $(M \times 1)$ where M is even

θ_ψ is the constraint matrix form of θ with dimension $(M \times \frac{M}{2})$. θ_ψ has the form

$$\theta_\psi = \begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 \\ \theta_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \theta_3 & 0 & & 0 & 0 \\ 0 & \theta_4 & 0 & & 0 & 0 \\ 0 & 0 & \theta_5 & & 0 & 0 \\ 0 & 0 & \theta_6 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \theta_{M-3} & 0 \\ \vdots & \vdots & \vdots & & \theta_{M-2} & 0 \\ \cdot & \cdots & \cdots & \cdots & 0 & \theta_{M-1} \\ \cdot & \cdots & \cdots & \cdots & 0 & \theta_M \end{bmatrix} \qquad (32)$$

A is the diagonal l.u.b. constraint limit matrix with dimension $(\frac{M}{2} \times \frac{M}{2})$. The diagonal elements are the A_m values in Equation (13).

B is the diagonal constraint weighting matrix with dimension $(M \times M)$. B is normally the I_M identy matrix.

In accordance with the development of Equation (25) (i.e., Equations (17) through (24)), the helicopter vibration reduction problem defined by Relations (31) can be expressed

$$\boxed{\begin{array}{l} \underset{\theta \in \mathbb{R}^M}{\text{Minimise}} \; J = \begin{cases} [z(\theta)]^T W_z [z(\theta)] \\ \qquad \text{or equivalently} \\ \theta^T T^T W_z T \theta + 2 Z_0^T W_z T \theta + Z_0^T W_z Z_0 \\ \qquad \text{or equivalently} \\ \theta^T F \theta + 2 G^T \theta + H \end{cases} \\ \\ \text{Subject to} \quad \theta_\psi^T B \theta_\psi \leq A \end{array}} \qquad (33)$$

where the various vectors and matrices are defined as previously (i.e., in accordance with Relations (17) through (32)).

Relations (33) are most likely the simplest reasonably accurate expression of the "real" helicopter vibration problem consistent with the assumed linear plant models described in Sections 2.1.1.1 and 2.1.1.2. The controller which solves this problem is referred to herein as the "Optimal Controller."

The question arises as to why bother to try to solve the more complicated non-linear programming problem defined by Relationship (33) when the Conventional Controller problem defined by Relationship (16) has produced what has appeared to be satisfactory results for many years. A graphical comparison of a scalar example of both methods provides insight into why the solution to the more complicated non-linear programming problem would be preferable.

Relationship (33) for a scalar Optimal Controller problem (i.e., a scalar helicopter vibration reduction problem posed as a scalar non-linear programming problem) is

$$\underset{\theta \in \mathbb{R}^M}{\text{Minimise:}} \quad J_1 = A\theta^2 + B\theta + C \tag{34}$$

$$\text{Subject to:} \quad \theta^2 \leq D \not\Rightarrow \theta \leq +\sqrt{D} \equiv \theta_{MAX} \tag{35}$$

where:

A, B, C, and D are scalar constants

$A > 0$

$D > 0$

J_1 is the "real" performance index

θ is the scalar control variable

θ_{MAX} is the l.u.b. constraint limit for θ

$M = 1$

Relationship (16) for a scalar Conventional Controller problem is:

$$\underset{\theta \in \mathbb{R}^M}{\text{Minimise}} \quad J_2 = \begin{cases} J_1 + W_\theta \theta^2 \\ A\theta^2 + B\theta + C + W_\theta \theta^2 \end{cases} \qquad (36)$$

Subject to: <u>NO</u> constraints per se

where:

$W_\theta \theta^2$ is the "internal limiting" attempt to constrain θ

J_2 is the augmented performance index

W_θ is the scalar weighting coefficient for the "internal limiting" term

The graphical representation of these expressions for J_1, J_2, and $W_\theta \theta^2$ is shown in Figure 4. Letting the superscript asterisk denote the value at solution of either the Optimal or Conventional Controller problem as appropriate, it can clearly be seen that:

1. The solution performance index J_1^* to the Optimal Controller problem is less than the solution performance index J_2^* to the Conventional Controller problem.

2. The solution control variable θ_1^* to the Optimal Controller problem is different than the solution control variable θ_2^* to the Conventional Controller Problem.

3. The solution control variable θ_1^* to the Optimal Controller problem can be on the constraint limit θ_{MAX} but <u>never</u> above it while the solution control variable θ_2^* to the Conventional Controller can be below or above this limit θ_{MAX} but only coincidentally on it.

4. The above three observations indicates that there will in general be discrepancies in the solution performance index and control. The value of J_2 can never be less than that of J_1 and it is possible, depending on the value of A, B, C, and W_θ, that θ_2^* could be greater than θ_{MAX}. In practice, the discrepancy in performance and control can be significantly large.

2.3.2 Solution

The Helicopter Vibration Reduction Problem (HVRP) defined by Relationship (33) expressed in Standard Form is

$$\boxed{\begin{array}{ll} \underset{\theta \in \mathbb{R}}{\text{Minimise}} & J = \theta^T F \theta + 2G^T \theta + H \\ & \\ \text{Subject to:} & \psi(\theta) = A - \theta_\psi^T B \theta_\psi \geq 0_{\frac{M}{2}} \end{array}} \qquad \begin{array}{r} (37) \\ \\ (38) \end{array}$$

where

F, G^T, and H are defined by Equations (22), (23), and (24), respectively

θ_ψ is defined by Equation (32)

A and B are defined according to their usage in Equation (31)

$0_{\frac{M}{2}}$ is the zero matrix with dimension $(\frac{M}{2} \times \frac{M}{2})$

$\psi(\bullet)$ is the Constraint Function Matrix with dimension $(\frac{M}{2} \times \frac{M}{2})$

It is useful and convenient at this point to present five simple, but important, theorems dealing with matters of symmetry and positive definiteness of pertinent matrices. It is assumed that all matrices and vectors as used in these theorems are conformable and that their elements are real numbers.

Theorem #1

If given that

W_Z is a diagonal matrix of dimension $(N \times N)$ whose diagonal elements are all greater then zero

ξ is any vector $\in \mathbb{R}^M$

η and η_o are vectors $\in \mathbb{R}^N$

T is a matrix of dimension $(N \times M)$

0_N is the zero vector of dimension $(N \times 1)$

0_M is the zero vector of dimension $(M \times 1)$

Then

A. $\eta^T W_Z \eta > 0 \quad \forall \eta \in \mathbb{R}^N \,|\, \eta \neq 0_N$

or equivalently W_Z is Positive Definite.

B. $\xi^T [T^T W_Z T] \xi > 0 \quad \forall \xi \in \mathbb{R}^M \,|\, \xi \neq 0_M$

or equivalently $[T^T W_Z T]$ is Positive Definite.

Proof:

Part A

$$\eta^T W_Z \eta = [\eta_1 \; \eta_2 \; \eta_3 \cdots \eta_N] \begin{bmatrix} W_1 & 0 & 0 & \cdots & 0 \\ 0 & W_2 & 0 & \cdots & 0 \\ 0 & 0 & W_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W_N \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_N \end{bmatrix}$$

$$\eta^T W_z \eta = W_1 \eta_1^2 + W_2 \eta_2^2 + W_3 \eta_3^2 + \dots\dots + W_N \eta_N^2$$

since $W_k > 0 \quad \forall k \in \{1, 2, 3, \dots\dots N\}$

$$\eta^T W_z \eta > 0 \quad \forall \eta \in \mathbb{R}^N | \eta \neq 0_N$$

or equivalently W_z is Positive Definite

Part B

Consider $\quad \zeta = \eta - \eta_0$

then $\quad \zeta \in \mathbb{R}^N$

and by Part A $\quad \zeta^T W_z \zeta > 0 \quad \forall \zeta \in \mathbb{R}^N | \zeta \neq 0_N$

If $\quad \eta = T\xi + \eta_0 \quad \forall \xi \in \mathbb{R}^M$

then $\quad \zeta = \eta - \eta_0 = T\xi$

and $\quad \zeta^T W_z \zeta = [T\xi]^T W_z [T\xi] = [\xi^T T^T] W_z [T\xi]$

$$\zeta^T W_z \zeta = \xi^T [T^T W_z T] \xi > 0 \quad \forall \xi \in \mathbb{R}^M | \xi \neq 0_M$$

or equivalently $[T^T W_z T]$ is Positive Definite Q.E.D.

Theorem #2

If given that

P is a symmetric matrix of dimension $(r \times r)$

B is a matrix of dimension $(r \times s)$

Then

$B^T P B$ is symmetric

Proof

Since P is symmetric, $P = P^T$

Then

$$B^T P B = B^T P^T B = [PB]^T B = \left(B^T [PB]\right)^T = [B^T P B]^T$$

$\therefore$ $B^T P B$ is symmetric. Q.E.D.

Theorem #3

If given that

W_z is a diagonal matrix of dimension $(N \times N)$ whose diagonal elements are all greater than zero

T is a matrix of dimension $(N \times M)$

F is a matrix of dimension $(M \times M)$, $F = T^T W_z T$

C is a matrix of dimension $(M \times M)$, $C = F + F^T$

Then C is Positive Definite, symmetric, and equals $2F$

Proof

F is Positive Definite by Theorem #1

W_z is symmetric since W_z is diagonal

F is symmetric by Theorem #2, thus $F = F^T$

and

$$C = F + F^T = F + F = 2F$$

$\therefore$ C is symmetric and $C = 2F$

Since F is Positive Definite

$$\zeta^T F \zeta > 0 \quad \forall \zeta \in \mathbb{R}^M \mid \zeta \neq 0_M$$

$$\zeta^T C \zeta = \zeta^T [2F] \zeta = 2\zeta^T F \zeta > 0$$

$$\forall \zeta \in \mathbb{R}^M \mid \zeta \neq 0_M$$

$\therefore$ C is Positive Definite Q.E.D.

Theorem 4

If given that

P is symmetric with dimension $(\ell \times \ell)$

a and b are vectors with dimension $(\ell \times 1)$

Then

$$a^T P b = b^T P a$$

Proof

Since P is symmetric, $P = P^T$

and

$$a^T P b = [a^T P] b = [a^T P^T] b = [Pa]^T b$$

$$a^T P b = \left(b^T [Pa]\right)^T = [b^T P a]^T$$

Since $b^T P a$ is a scalar, $b^T P a = [b^T P a]^T$

$\therefore$ $a^T P b = b^T P a$ Q.E.D.

Theorem 5

If addressing the problem posed by Relationship (33) and if given that

$$Z(\theta) = T\theta + Z_0 \quad (\text{Equation } (2))$$

T is square and non-singular

and

No constraints are active, that is

$$\theta_\psi^T B \theta_\psi < A \quad \forall \theta \in \text{Solution Space}$$

Then

$$\exists \text{ a unique } \theta^* \neq 0_M \ni \begin{cases} J = 0 \text{ for } \theta = \theta^* \\ J > 0 \ \forall \theta \neq \theta^* \text{ and } \theta \neq 0_M \end{cases}$$

Proof:

By Cramer's Theorem,

$$\exists \text{ a unique } \theta^* \neq 0_M \ni Z(\theta^*) = T\theta^* + Z_0 = 0_N$$

Then $J = [Z(\theta^*)]^T W_Z [Z(\theta^*)] = 0_N^T W_Z 0_N = 0$

By Theorem 1

$$J > 0 \quad \forall Z \in \mathbb{R}^N \mid Z \neq 0_N$$

or equivalently

$$J > 0 \quad \forall \theta \in \mathbb{R}^M \mid \theta \neq \theta^*$$

Theorem 5 provides a convenient means to test for error in the algorithms and code designed to solve the HVRP posed by Relationship (33), or equivalently by Relationships (37) and (38). If T is non-singular and square, and if the constraints are not active, then the algorithm should yield a minimum value of J equal to zero.

Investigation of various methods to solve the HVRP as a non-linear programming problem led to the selection of the highly successful successive quadratic programming method developed by Schittkowski, Stoer, and Gill et al (References 9 through 15) which solves the general non-linear programming problem by solving a sequence of related quadratic programming sub-problems. One advantage of this method is that quadratic programming problems can be solved efficiently. A very important property of quadratic programming problems is that if the quadratic coefficient matrix in the performance index is positive definite, the problem has a unique solution which is, of course, the global solution. This means that the sequence of solutions to the quadratic programming sub-problems will converge to the global solution of the general problem in the limit. This method is computationally implemented in the Stand-Alone Optimal Controller System (Section 2.7) by selection of the IMSL system (Reference 7) of codes (i.e., the IMSL main driver routines DNCONF and DNCONG and their subroutines) which were designed to solve quite general non-linear programming problems using this method.

The general Quadratic Programming Problem (QPP) is:

$$\boxed{\begin{array}{lll} \underset{\theta \in \mathbb{R}^M}{\text{Minimise}} & J_{QPP} = \theta^T \mathcal{F} \theta + 2\mathcal{G}^T \theta + \mathcal{H} & \\ \text{Subject to} & \underset{QPP}{\phi}(\theta) = b_1 - A_1 \theta = 0_M & \\ & \underset{QPP}{\psi}(\theta) = b_2 - A_2 \theta \geq 0_M & \end{array}} \quad \begin{array}{r} (39) \\ (40) \\ (41) \end{array}$$

where

J_{QPP}	is the scalar QPP performance index
$\mathcal{F}$	is the quadratic coefficient matrix with dimension $(M \times M)$
$\mathcal{G}^T$	is the linear coefficient row vector with dimension $(1 \times M)$
$\mathcal{H}$	is a scalar
A_1	is the equality constraint linear coefficient matrix with dimension $(M \times M)$
A_2	is the inequality constraint linear coefficient matrix with dimension $(M \times M)$
b_1	is the equality constraint value vector with dimension $(M \times 1)$
b_2	is the inequality constraint value vector with dimension $(M \times 1)$
0_M	is the zero vector with dimension $(M \times 1)$

θ is the control vector with dimension $(M \times 1)$

$\phi_{QPP}(\cdot)$ is the equality constraint function vector with dimension $(M \times 1)$

$\psi_{QPP}(\cdot)$ is the inequality constraint function vector with dimension $(M \times 1)$

The quadratic programming sub-problems used by the successive quadratic programming method are formulated by using a quadratic approximation of the performance index and linear approximations of the constraint functions at each step of the iteration process. These approximations are obtained by simple replacement of the f, ϕ, and ψ functions in Relationship (30) with their appropriately truncated matrix Taylor Series expansions, where if the Hessian of $f(\cdot)$ is not positive definite, the algorithm adjusts it so that it is. The sub-problem is:

$$\boxed{\begin{array}{lll}
\underset{\xi \in \mathbb{R}^M}{\text{Minimise}} & \frac{1}{2}\xi^T\left(\overline{\left.\frac{\partial^2 f}{\partial \theta^2}\right|}_{\theta_k}\right)\xi + \left(\left.\frac{\partial f}{\partial \theta}\right|_{\theta_k}\right)\xi & (42) \\[2ex]
\text{Subject to} & \left(\left.\frac{\partial \phi}{\partial \theta}\right|_{\theta_k}\right)\xi + \phi(\theta_k) = 0 & (43) \\[2ex]
 & \left(\left.\frac{\partial \psi}{\partial \theta}\right|_{\theta_k}\right)\xi + \psi(\theta_k) \geq 0 & (44)
\end{array}}$$

where

$\overline{\left(\frac{\partial^2 f}{\partial \theta^2}\bigg|_{\theta_k}\right)}$ is the positive definite approximation of the Hessian of $f(\bullet)$ evaluated at θ_k with dimension $(M \times M)$

$\left(\frac{\partial f}{\partial \theta}\bigg|_{\theta_k}\right)$ is the gradient of $f(\bullet)$ evaluated at θ_k with dimension $(1 \times M)$

$\left(\frac{\partial \phi}{\partial \theta}\bigg|_{\theta_k}\right)$ is the gradient of $\phi(\bullet)$ evaluated at θ_k with dimension $(P \times M)$

$\left(\frac{\partial \psi}{\partial \theta}\bigg|_{\theta_k}\right)$ is the gradient of $\psi(\bullet)$ evaluated at θ_k with dimension $(Q \times M)$

ξ is the control vector counterpart of θ for the quadratic programming sub-problem with dimension $(M \times 1)$

If optimality is not achieved at the completion of an iterative step, the Hessian is updated (References 10 and 11) and a new step is attempted.

It is once again noted and emphasised that if the quadratic coefficient matrix in the performance index (i.e., $\mathcal{H}$ in Relationship (39) and the Hessian $\left(\frac{\partial^2 f}{\partial \theta^2}\bigg|_{\theta_k}\right)$ in Relationship (42)) is <u>Positive Definite</u>, then there is a <u>unique</u> solution (i.e., θ^* for Relationships (39), (40), and (41); and ξ^* for Relationships

(42), (43), and (44)) to the QPP which is, of course, the global solution.

The attraction of a method based on solving quadratic programming sub-problems is the close resemblance that the HVRP has to the QPP. Comparison of the HVRP defined by Relationships (37) and (38) with the QPP defined by Relationships (39), (40), and (41) shows that:

1. The performance index J of the HVRP has identically the same quadratic form as J_{QPP} of the QPP.
2. The HVRP has no equality constraints, or equivalently b_1 is identically the zero vector and A_1 is identically the zero matrix in Relationships (40) and (43) for the quadratic programming sub-problem solved at each iteration step.
3. The only difference between the HVRP and a QPP is the inequality constraint function which is quadratic for the HVRP (Relationship (38)) and linear for the QPP (Relationship (41)).
4. Since J and $\psi(\cdot)$ of the HVRP (Relationship (37) and (38)) are quadratics and known, their gradients are known, linear, and easy to evaluate. It is noted that the gradient of a quadratic evaluated by two-sided finite differences is exact analytically (see Sub-Section 2.5.3.1) and hence use of gradients so evaluated need not degrade gradient accuracy.

5. Since J and $\psi(\bullet)$ of the HVRP (Relationships (37) and (38)) are quadratics and known, their Hessians are invariant and known. Accordingly, the Hessians need not be updated during the iterative process in future codes dedicated to the HVRP.

6. Since the quadratic coefficient matrix F of the HVRP is positive definite, its Hessian is also positive definite* and each step of the iteration process yields the unique solution to the quadratic programming sub-problem and the Hessian need not be adjusted to be positive definite. This unique solution is the global solution and the iteration process consequently converges to the global solution in the limit.

Experience to date indicates that the selected successive quadratic programming method works very well indeed; it is reliable, robust, reasonably efficient, and appears to actually converge to the global solution of the HVRP.

* It is shown in Section 2.5.4 that the Hessian of f, $\frac{\partial^2 f}{\partial \theta^2}$, equals $F+F^T$ which by Theorem 3 equals 2F and is positive definite

2.4 Vibration Minimisation as a Constrained Calculus Min/Max Problem

Variations of the Optimal Controller which determine the control vector that satisfies the necessary conditions for optimality for the HVRP defined by Relationships (37) and (38) are conceptually possible. It is emphasised however, that satisfaction of necessary conditions for optimality does not guarantee identification of the global solution. Optimal Controllers of this type (i.e., controllers which attempt to solve the HVRP by determining a control vector which satisfies the necessary conditions for optimality) are simply special cases of the General Controller (see Section 2.1.2 and Figure 2) in which a specific computational procedure is assumed for the determination of the control vector to be used during the next duty cycle. The control vector solution to the non-linear equations which define the necessary conditions for optimality is determined using the Levenburg-Marquardt algorithm (References 6, 16, 17, 18). The definition of calculus Min/Max HVRP problem and the necessary conditions for optimality are presented in Sections 2.4.1 and 2.4.2, the method of solution of the full set of non-linear equations which define the necessary conditions for optimality is described in Section 2.4.3., and the method of solution by selecting the best of solutions of each of the possible reduced sets of equations which define the necessary conditions for optimality when the constraints are preselected to be either active or inactive is described in Section 2.4.4.

2.4.1 Constrained Calculus Min/Max Problem

The HVRP Standard Form defined by Relationships (37) and (38) is transformed to Standard Min/Max Calculus Form by introduction of a function of a slack variable vector α to the inequality constraint function $\psi(\cdot)$ which transforms this function to an equality constraint function $\phi(\cdot)$. This is accomplished by defining $\phi(\cdot)$ to be

$$\phi(\Theta) \equiv \psi(\theta) - \alpha_\psi^T \alpha_\psi \equiv 0 \qquad (45)$$

with dimension $\left(\frac{M}{2} \times \frac{M}{2}\right)$

where

$\psi(\cdot)$ is the inequality constraint function matrix defined by Relationship (38) with dimension $\left(\frac{M}{2} \times \frac{M}{2}\right)$

α is the slack variable vector with dimension $\left(\frac{M}{2} \times 1\right)$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\frac{M}{2}} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta \\ \hline \alpha \end{bmatrix} \quad \text{with dimension } \left(\frac{3M}{2} \times 1\right)$$

α_ψ is the diagonal slack variable matrix with dimension $(\frac{M}{2} \times \frac{M}{2})$. The diagonal elements are the individual slack variables α_k, $k = 1, 2, \ldots \frac{M}{2}$ each one of which is associated with an inequality constraint $\psi_k(\cdot)$, $k = 1, 2, \ldots \frac{M}{2}$.

$$\alpha_\psi = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & 0 & \\ 0 & & \ddots & \\ & & & \alpha_{\frac{M}{2}} \end{bmatrix}$$

Substituting Relationship (38) into Equation (45) yields the desired result; specifically

$$\phi(\Theta) = A - \theta_\psi^T B \theta_\psi - \alpha_\psi^T \alpha_\psi = 0 \qquad (46)$$

Notice that for each row k, $k = 1, 2, \ldots \frac{M}{2}$, of $\phi(\Theta)$ if the constraint is active (i.e., $[\theta_\psi^T B \theta_\psi]_k = A_k$), then $\alpha_k = 0$. If the constraint is not active (i.e., $[\theta_\psi^T B \theta_\psi < A_k$), then $\alpha_k \neq 0$.

Because of the way θ_ψ, A, B, and α_ψ are defined, $\phi(\Theta)$ is diagonal. Without any loss in generality, this diagonal matrix $\phi(\Theta)$ as defined by Equation (46) can be compressed into a column matrix (vector) $\phi_c(\Theta)^*$ comprised of just the

* The subscript C denotes a column matrix formed from a diagonal matrix by this type of compression.

the diagonal elements of $\Phi(\Theta)$ by eliminating the off-diagonal zero elements. The Standard Min/Max Calculus Form of the HVRP is

$$\boxed{\begin{array}{ll} \underset{\Theta \in \mathbb{R}^{\frac{3M}{2}}}{\text{Minimise}} & J = \theta^T F \theta + 2G^T \theta + H \\ \\ \text{Subject to} & \phi_c(\Theta) = \left[A - \theta_\psi^T B \theta_\psi - \alpha_\psi^T \alpha_\psi\right]_c = 0 \end{array}} \qquad \begin{array}{r} (47) \\ \\ \\ (48) \end{array}$$

2.4.2 Necessary Conditions for Optimality

The necessary conditions for optimality can be readily derived using standard Min/Max Calculus. First, form the augmented performance index $\mathcal{J}[J(\theta), \phi_c(\Theta), \lambda]$ by adjoining the equality constraint vector $\phi_c(\Theta)$ to the actual performance index $J(\theta)$ using an adjoint vector λ of dimension $(\frac{M}{2} \times 1)$ whose elements are the $\frac{M}{2}$ Lagrangian multipliers each one of which corresponds to a constraint.

$$\mathcal{J}[J(\theta), \phi_c(\Theta), \lambda] = \mathcal{J}[\Theta, \lambda] = J(\theta) + \lambda^T \phi_c(\Theta) \qquad (49)$$

where

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{\frac{M}{2}} \end{bmatrix} \quad \text{and} \quad \phi_c(\cdot) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{\frac{M}{2}} \end{bmatrix} \quad \text{each with dimension } \left(\frac{M}{2} \times 1\right)$$

The necessary conditions for optimality are then obtained by satisfaction of the $\frac{M}{2}$ constraint equations and the $\frac{3M}{2}$ equations obtained by taking the gradient with respect to Θ of the augmented performance index $\mathcal{J}[\Theta, \lambda]$. These equations are

$$\boxed{\begin{aligned} \mathrm{Grad}_{\Theta}\, \mathcal{J}[\Theta, \lambda] = \left[\frac{\partial \mathcal{J}}{\partial \Theta}\right]^T &= 0 \quad \text{with dimension } \left(\frac{3M}{2} \times 1\right) \qquad (50) \\ \phi_c(\Theta) &= 0 \quad \text{with dimension } \left(\frac{M}{2} \times 1\right) \qquad (51) \end{aligned}}$$

It is noted that since these equations are in general non-linear, more than one solution for Θ and λ can exist. These additional "solutions" do not necessarily correspond to relative minima, let alone the global minimum, of the HVRP.

For the HVRP, the Grad_α part of Grad_Θ of $\mathcal{J}[\Theta,\lambda]$ leads to "Switching Equations".

$$\text{Grad}_\alpha \mathcal{J}[\Theta,\lambda] = \begin{bmatrix} \frac{\partial \mathcal{J}}{\partial \alpha_1} \\ \frac{\partial \mathcal{J}}{\partial \alpha_2} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial \alpha_{\frac{M}{2}}} \end{bmatrix} = -2 \begin{bmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_{\frac{M}{2}} \alpha_{\frac{M}{2}} \end{bmatrix} = 0$$

from which

$$\boxed{\lambda_k \alpha_k = 0 \quad \text{for } k = 1, 2, \ldots \tfrac{M}{2}} \qquad (52)$$

Clearly either $\lambda_k = 0$ or $\alpha_k = 0$. If the k-th constraint is active, $\alpha_k = 0$ and in general $\lambda_k \neq 0$. If the constraint is not active, $\alpha_k \neq 0$ and $\lambda_k = 0$. In the event that both $\alpha_k = 0$ and $\lambda_k = 0$, there would be more "surviving" simultaneous equations from Relationships (50) and (51) than unknowns from Θ and λ and accordingly the problem would be over-determined and a solution might not exist. These cases where $\lambda_k = \alpha_k = 0$ are not of interest. Consequently, only cases where either $\lambda_k = 0$ or $\alpha_k = 0$, but not both, are considered.

2.4.3 Solution of the Full Set of Equations Which Define the Necessary Conditions for Optimality

Investigation of various methods to solve the non-linear simultaneous system of algebraic equations (i.e., Equations (50) and (51)) that define the necessary conditions for optimality for Θ and λ led to the selection of the IMSL system (Reference 6) of codes (i.e., the IMSL main driver routines DNEQNF and DNEQNJ and their subroutines) which were designed to solve a quite general non-linear simultaneous system of algebraic equations by using the Levenberg-Marquardt version of the algorithm employed by MINPACK routines HYBRD and HYBRD1 (References 16 and 17). These MINPACK routines are modifications of M. J. D. Powell's Hybrid method (Reference 18) which is in itself a variation of Newton's Method.

Experience to date indicates that these routines work reasonably well for the HVRP in that they do reliably converge to a "solution". Unfortunately, in general there exist many (perhaps infinite) combinations of Θ and λ which satisfy these equations which define the necessary conditions for optimality. Indeed, not only do solutions exist which correspond to relative extrema not the global minimum, but solutions can also exist which do not correspond to any extremum at all. The solution that the algorithm converges to is quite sensitive to the initial (starting) estimate of Θ and λ. There doesn't appear to be any practical way to select an initial estimate for Θ and λ which will guarantee convergence to the desired global minimum unless, of course, the solution is already known and is used as the starting estimate. This method can, however, be used to verify that a previously determined solution obtained by solving the non-linear programming problem (Section 2.3.2) is indeed a solution. Accordingly, options were provided in the stand-alone program (i.e., options defined by IOPT = 11, 12, 13, and 14) which provide this verification to the solution obtained by solving the non-linear programming problem.

2.4.4 Solution of a Reduced Set of Equations Which Define the Necessary Conditions for Optimality for Each Possible Combination of Active/Inactive Constraints

Recognising that satisfaction of the full set equations (i.e., Equations (50) and (51)) that define the necessary conditions for optimality do not necessarily yield the global minimum, a more reliable method to obtain the global minimum was sought which was likewise based on the satisfaction of necessary conditions. The approach which was developed selects the "best" (i.e., that which yields the minimum) of the solutions of each possible set of reduced equations obtained when the constraints are preselected to be either active or inactive. Pre-selection of the activity of a constraint eliminates the corresponding switching equation (see Section 2.4.2) from the set of equations to be solved. Since there is a switching equation for each constraint, the number of switching equations is M/2 for the HVRP. Correspondingly, it is possible to reduce the dimension of the system of equations defining the necessary conditions for optimality from 2M for the full set of equations to 3M/2 for the reduced set by pre-selection of the activity for all the constraints. Since the number of arithmetic operations are proportional to the square of the dimension, an approximate reduction of 9/16 in computation time for solution of a reduced set of equations from that of a full set of equations is obtained. Unfortunately this method requires solution of several sets of equations, the amount being dependent on the M, and generally results in a net increase in computation time. But notwithstanding this increase in computation time, the solutions (accomplished using the same IMSL routines specified in Section 2.4.3) of lower dimension systems are more reliable in general.

It was pointed out in Section 2.4.1 with reference to Equations (45), (46), and (48) that

if the k-th constraint is active, $\alpha_k = 0$
if the k-th constraint is inactive, $\alpha_k \neq 0$

It was pointed out in Section 2.4.2 with reference to the switching equations, Equation (52), that

if the k-th constraint is active, $\alpha_k = 0$ and $\lambda_k \neq 0$
if the k-th constraint is inactive, $\alpha_k \neq 0$ and $\lambda_k = 0$
the problem is over-determined and not of interest
if $\alpha_k = \lambda_k = 0$

Cases where M=6 (e.g., where the control vector is comprised of the coefficients for the n-1, n, and n+1 harmonic terms for blade pitch) were investigated. The appropriate options were provided in the stand-alone

Code (i.e., IOPT = 5, 6, 7 and 8) for cases where M=6. The three constraint equations which are the scalar form of Equation (48) corresponding to M=6 are

$$A_1 - \theta_1^2 - \theta_2^2 - \alpha_1^2 = 0 \qquad \text{Constraint 1}$$

$$A_2 - \theta_3^2 - \theta_4^2 - \alpha_2^2 = 0 \qquad \text{Constraint 2}$$

$$A_3 - \theta_5^2 - \theta_6^2 - \alpha_3^2 = 0 \qquad \text{Constraint 3}$$

The eight constraint activity possibilities corresponding to M = 6 are

Possibility	Parameters Equal to Zero	Active Constraints
1	α_1 α_2 α_3	1, 2, 3
2	λ_1 α_2 α_3	2,3
3	α_1 λ_2 α_3	1,3
4	λ_1 λ_2 α_3	3
5	α_1 α_2 λ_3	1,2
6	λ_1 α_2 λ_3	2
7	α_1 λ_2 λ_3	1
8	λ_1 λ_2 λ_3	None

Experience to date indicates that, as in the case of solution to the full set of equations, these routines work reasonably well for the HVRP in that they do reliably converge to a "solution" for each of the reduced sets of equations. But likewise, there also exists many (perhaps infinite) combinations of Θ and λ which satisfy these reduced sets of equations and not all of them correspond to any extrema. Similarly, there doesn't appear to be any practical way to select an initial estimate for Θ and λ which will guarantee convergence to the desired global minimum. Since the computation times were greater for this technique than for solution of the full set of equations, and since there did not appear to be any useful application of this technique such as providing verification that the necessary conditions were satisfied, further use of this method is not recommended.

2.5 Expanded Equations Required for Solution of the Non-Linear Programming Problem and the Necessary Conditions for Optimality

The equations presented in the previous sections were, for the most part, in matrix/vector form and a bit more general than convenient for coding purposes. The corresponding expanded and detailed forms of these equations and those equations required for the computational processes described herein are presented subsequently. Specifically, the expanded constraint and performance index equations are presented first, followed by the analytic gradient and Hessian, and finally the necessary conditions for optimality and the analytic Jacobian. The proof of the interesting and useful fact that the numerically derived gradient of a quadratic function obtained by the conventional two-sided finite difference method is mathematically equivalent to the analytically derived gradient of the same function is presented right after the analytic gradient equations.

2.5.1 Constraints

The matrix form of the constraints, $\psi(\Theta)$, for the HVRP defined by Relationship (38) is presented here again for convenience, specifically

$$\psi(\theta) = A - \theta_\psi^T B \theta_\psi \geq 0_{\frac{M}{2}} \qquad (38)$$

where A, B, and θ_ψ are defined by Relationships (31) and (32).

Since the θ_k of θ_ψ are actually harmonic sine and cosine coefficients of the sinusoidal excitation of blade pitch (see Section 2.2.1, Equations (10) through (13)), the B matrix is identically the identity matrix I_M and accordingly the matrix form of the constraints becomes

$$\psi(\theta) = A - \theta_\psi^T I_M \theta_\psi = A - \theta_\psi^T \theta_\psi \geq 0_{\frac{M}{2}} \qquad (53)$$

Because of the way A and θ_ψ are defined, the off-diagonal elements of Relationship (53) are identically zero, and accordingly Relationship (53) can be expressed in element form as

$$\psi_{ij}(\theta) = \delta_{ij}\left[A - \theta_\psi^T \theta_\psi\right]_{ij} \geq 0 \quad \text{(not summed)} \qquad (54)$$
$$\text{for } i, j \in \left\{1, 2, \cdots, \tfrac{M}{2}\right\}$$

where δ_{ij} is the Kronecker Delta

The constraint relations of interest are the diagonal elements $\psi_{ii}(\theta)$ of $\psi(\theta)$ and are simply

$$\psi_{ii}(\theta) = [A - \theta_\psi^T \theta_\psi]_{ii} \geq 0 \quad (\text{not summed}) \qquad (55)$$

$$\text{for } i \in \{1, 2, \ldots, \tfrac{M}{2}\}$$

or equivalently

$$\boxed{\left.\begin{aligned} \psi_{11}(\theta) &= A_1 - \theta_1^2 - \theta_2^2 \geq 0 \\ \psi_{22}(\theta) &= A_2 - \theta_3^2 - \theta_4^2 \geq 0 \\ &\vdots \\ \psi_{\frac{k}{2}\frac{k}{2}}(\theta) &= A_{\frac{k}{2}} - \theta_{k-1}^2 - \theta_k^2 \geq 0 \\ &\vdots \\ \psi_{\frac{M}{2}\frac{M}{2}}(\theta) &= A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 \geq 0 \end{aligned}\right\} \text{for } k \in \{1, 2, \ldots, M\}} \qquad (56)$$

2.5.2 Performance Index

The matrix form of the equation defining the performance index J for the HVRP is defined by Equation (37) with associated coefficients being defined by Equations (22), (23), and (24). These equations are presented here again for convenience; specifically:

$$J = \theta^T F \theta + 2G^T\theta + H \qquad (37)$$

where

$$F = T^T W_z T \qquad (22)$$

$$G^T = Z_0^T W_z T \qquad (23)$$

$$H = Z_0^T W_z Z_0 \qquad (24)$$

In practice, the elements f_{ij} of F are generated by using a matrix product algorithm in the stand-alone code to evaluate Equation (22) to yield

$$F = T^T W_z T = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1M} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2M} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{M1} & f_{M2} & f_{M3} & \cdots & f_{MM} \end{bmatrix} \qquad (57)$$

The product $\theta^T F \theta$ evaluated manually yields

$$\left.\begin{array}{rl}\theta^T F \theta = f_{11}\theta_1^2 & + [f_{12}+f_{21}]\theta_1\theta_2 + [f_{13}+f_{31}]\theta_1\theta_3 + [f_{14}+f_{41}]\theta_1\theta_4 + \cdots\cdots + [f_{1M}+f_{M1}]\theta_1\theta_M \\ & + f_{22}\theta_2^2 + [f_{23}+f_{32}]\theta_2\theta_3 + [f_{24}+f_{42}]\theta_2\theta_4 + \cdots\cdots + [f_{2M}+f_{M2}]\theta_2\theta_M \\ & \quad + f_{33}\theta_3^2 + [f_{34}+f_{43}]\theta_3\theta_4 + \cdots\cdots + [f_{3M}+f_{M3}]\theta_3\theta_M \\ & \qquad \ddots \qquad \vdots \\ & \qquad\qquad + [f_{(M-1)M}+f_{M(M-1)}]\theta_{M-1}\theta_M \\ & \qquad\qquad\qquad + f_{MM}\theta_M^2 \end{array}\right\} \quad (58)$$

It is convenient to define the $(M \times M)$ dimensional C matrix as

$$C = F + F^T \qquad (59)$$

Then the elements of C are

$$C_{ij} = f_{ij} + f_{ji} \quad \text{for } i,j \in \{1, 2, \ldots M\} \qquad (60)$$

and $\theta^T F \theta$ can be written as

$$\left.\begin{array}{rl}\theta^T F \theta = \frac{1}{2}C_{11}\theta_1^2 & + C_{12}\theta_1\theta_2 + C_{13}\theta_1\theta_3 + C_{14}\theta_1\theta_4 + \cdots\cdots + C_{1M}\theta_1\theta_M \\ & + \frac{1}{2}C_{22}\theta_2^2 + C_{23}\theta_2\theta_3 + C_{24}\theta_2\theta_4 + \cdots\cdots + C_{2M}\theta_2\theta_M \\ & \quad + \frac{1}{2}C_{33}\theta_3^2 + C_{34}\theta_3\theta_4 + \cdots\cdots + C_{3M}\theta_3\theta_M \\ & \qquad \ddots \\ & \qquad\qquad + C_{(M-1)M}\theta_{(M-1)}\theta_M \\ & \qquad\qquad\qquad + \frac{1}{2}C_{MM}\theta_M^2 \end{array}\right\}. \quad (61)$$

Next, $Z_0^T W_Z$ of Equations (23) and (24) is evaluated manually.

$$Z_0^T W_Z = [Z_{0_1} \; Z_{0_2} \; Z_{0_3} \; \cdots \; Z_{0_N}] \begin{bmatrix} W_1 & 0 & 0 & \cdots & 0 \\ 0 & W_2 & 0 & & \vdots \\ 0 & 0 & W_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & W_N \end{bmatrix}$$

$$Z_0^T W_Z = [W_1 Z_{0_1} \;\; W_2 Z_{0_2} \;\; W_3 Z_{0_3} \; \cdots \; W_N Z_{0_N}] \qquad (62)$$

The elements g_i of G^T are determined by manually taking the product specified in Equation (23).

$$G^T = (Z_0^T W_Z) T = [g_1 \;\; g_2 \;\; g_3 \; \cdots \; g_M] \qquad (63)$$

where

$$g_k = W_1 Z_{0_1} T_{1k} + W_2 Z_{0_2} T_{2k} + W_3 Z_{0_3} T_{3k} + \cdots + W_N Z_{0_N} T_{Nk} \quad \text{for } k = 1, 2, \cdots M \qquad (64)$$

Then $G^T \theta$ is

$$G^T \theta = g_1 \theta_1 + g_2 \theta_2 + g_3 \theta_3 + \cdots + g_M \theta_M \qquad (65)$$

Finally, H is evaluated by manually taking the product specified in Equation (24)

$$H = (Z_o^T W) Z_o = W_1 Z_{o_1}^2 + W_2 Z_{o_2}^2 + W_3 Z_{o_3}^2 + \cdots\cdot + W_N Z_{o_N}^2 \qquad (66)$$

The performance index can now be expressed completely with scalar operations by evaluating Equation (37) using Equations (61), (65), and (66); specifically

$$\boxed{\begin{array}{rl} J = & \frac{1}{2} C_{11}\theta_1^2 + C_{12}\theta_1\theta_2 + C_{13}\theta_1\theta_3 + C_{14}\theta_1\theta_4 + \cdots\cdots + C_{1M}\theta_1\theta_M \\ & \quad + \frac{1}{2} C_{22}\theta_2^2 + C_{23}\theta_2\theta_3 + C_{24}\theta_2\theta_4 + \cdots\cdots + C_{2M}\theta_2\theta_M \\ & \qquad + \frac{1}{2} C_{33}\theta_3^2 + C_{34}\theta_3\theta_4 + \cdots\cdots + C_{3M}\theta_3\theta_M \\ & \qquad\qquad \ddots \qquad \ddots \qquad \vdots \\ & \qquad\qquad\qquad + C_{(M-1)M}\theta_{(M-1)}\theta_M \\ & \qquad\qquad\qquad + \frac{1}{2} C_{MM}\theta_M^2 \\ & + 2(g_1\theta_1 + g_2\theta_2 + g_3\theta_3 + \cdots\cdots + g_M\theta_M) \\ & + W_1 Z_{o_1}^2 + W_2 Z_{o_2}^2 + W_3 Z_{o_3}^2 + \cdots\cdots + W_N Z_{o_N}^2 \end{array}} \qquad (67)$$

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2.5.3 Analytic Gradient

The analytic gradient with respect to θ of the performance index J defined by Equation (37) is easily evaluated directly from this matrix equation and is

$$\boxed{\mathrm{Grad}_\theta\, J = \left[\frac{\partial J}{\partial \theta}\right]^T = 2F\theta + 2G^T = C\theta + 2G^T} \tag{68}$$

where by Theorem 2, F is symmetric; and by Theorem 3, C is also symmetric and

$$2F = F + F^T = C = C^T$$

This gradient, $\mathrm{Grad}_\theta\, J$, can likewise be evaluated directly from scalar Equation (67) in element form and is

$$\boxed{\mathrm{Grad}_\theta\, J = \begin{bmatrix} C_{11}\theta_1 + C_{12}\theta_2 + C_{13}\theta_3 + \cdots + C_{1M}\theta_M + 2g_1 \\ C_{12}\theta_1 + C_{22}\theta_2 + C_{23}\theta_3 + \cdots + C_{2M}\theta_M + 2g_2 \\ C_{13}\theta_1 + C_{23}\theta_2 + C_{33}\theta_3 + \cdots + C_{3M}\theta_M + 2g_3 \\ \vdots \\ C_{1M}\theta_1 + C_{2M}\theta_2 + C_{3M}\theta_3 + \cdots + C_{MM}\theta_M + 2g_M \end{bmatrix}} \tag{69}$$

which is identically equivalent to Equation (68).

The analytic gradient with respect to θ of the constraint function $\psi(\theta)$ defined by Equation (53) (i.e., Equation (38) with $B \equiv I_M$) is

$$\boxed{\mathrm{Grad}_\theta\left[\psi_c^T(\theta)\right] = \left[\frac{\partial \psi_c}{\partial \theta}\right]^T = -2\theta_\psi} \qquad (70)$$

where the subscript C denotes a column matrix formed from a diagonal matrix by shifting the diagonal elements to a single column.

This result can likewise be directly obtained from the scalar system defined by Equation (56). $\mathrm{Grad}_\theta[\psi_C^T(\theta)]$ in element form is

$$\boxed{\mathrm{Grad}_\theta\left[\psi_C^T(\theta)\right] = -2\begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 \\ \theta_2 & 0 & 0 & & \vdots & \vdots \\ 0 & \theta_3 & 0 & & \vdots & \vdots \\ 0 & \theta_4 & 0 & & \vdots & \vdots \\ 0 & 0 & \theta_5 & & \vdots & \vdots \\ 0 & 0 & \theta_6 & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \theta_{M-3} & 0 \\ 0 & \cdots & \cdots & \cdots & \theta_{M-2} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \theta_{M-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \theta_M \end{bmatrix}} \qquad (71)$$

which has dimension $\left(M \times \frac{M}{2}\right)$

2.5.3.1. Mathematical Equivalence of the Analytic Gradient and the Gradient by Two-Sided Finite Differences of a Quadratic Function

It is an interesting and useful fact that the gradient of a general quadratic function derived numerically using the standard two-sided finite difference method is mathematically equivalent to the analytically derived gradient of the same function. This fact means that there is NO analytic error (there is, of course, a computational error due to the finite computer word size) when numerically evaluating the gradient of a general quadratic function using the two-sided finite difference method. This result can be of great significance when attempting to numerically evaluate the gradient as part of an optimisation process. Inaccurate gradients are a frequent cause of failure to converge to the solution or other problems in such processes, and correspondingly great care should be taken to assure sufficient accuracy of the gradient.

Note first that a general scalar function of a vector will appear to be a scalar function of a scalar variable when a scalar partial derivative of the function with respect to one of the elements of the vector is evaluated. In the case of a general function $J(\theta)$ of a $(M \times 1)$ vector θ, $J(\theta)$ will appear to be $J(\theta_k)$ when $\left(\frac{\partial J}{\partial \theta_k}\right)$ is evaluated. If $J(\theta)$ is the general quadratic function defined by Equation (67), $J(\theta)$ will be comprised of only terms containing a coefficient times one of θ_k^2, or θ_k^1, or $\theta_k^0 \equiv 1$. $J(\theta)$ has, under these circumstances, the form

$$J(\theta) = J(\theta_k) = a\theta_k^2 + b\theta_k + c \quad \forall k \in \{1,2,\cdots M\} \quad (72)$$

for the purpose of evaluating $\left(\frac{\partial J}{\partial \theta_k}\right)$ where a, b, and c are scalar "constants" for this particular circumstance. Each one of a, b, and c is actually a function of the other θ_j for all $j \in \{1, 2, \cdots M \mid j \neq k\}$ but which are assumed to be constant when evaluating the partial derivative with respect to θ_k.

Since k is arbitrarily chosen from $\{1, 2, \cdots M\}$, Equation (72) is true for each of the M elements of θ when evaluating the scalar partial derivative with respect to that element. Consequently, all the partial derivatives required for the gradient are of a scalar quadratic function of the form defined by Equation (72).

To illustrate the possible presence of an analytic error in the numerical evaluation of the gradient, consider first the frequently used standard numerical one-sided finite difference method. Evaluation of $\left(\frac{\partial J}{\partial \theta_k}\bigg|_{\theta_k}\right)$ by this method is specified by

$$\left(\frac{\partial J}{\partial \theta_k}\bigg|_{\theta_k}\right)_{1FD} \equiv \left(\frac{dJ}{d\theta_k}\bigg|_{\theta_k}\right)_{1FD} \equiv \frac{J(\theta_k + \Delta\theta_k) - J(\theta_k)}{\Delta\theta_k} \quad (73)$$

where

$\Delta\theta_k$ is an increment in θ_k

subscript 1FD denotes numerical evaluation of the derivative by the one-sided finite difference method

Evaluation of $J(\theta_k + \Delta\theta_k)$ by Taylor Series Expansion yields

$$J(\theta_k + \Delta\theta_k) = J(\theta_k) + \left(\frac{dJ}{d\theta_k}\bigg|_{\theta_k}\right)(\Delta\theta_k) + \frac{1}{2!}\left(\frac{d^2J}{d\theta_k}\bigg|_{\theta_k}\right)(\Delta\theta_k)^2 + \frac{1}{3!}\left(\frac{d^3J}{d\theta_k}\bigg|_{\theta_k}\right)(\Delta\theta_k)^3 + \cdot\cdot\cdot\cdot + R_n \qquad (74)$$

where

$|\Delta\theta_k| << 1$

R_n is the n-th order residual

Substitution of Equation (74) into Equation (73) with simplification yields

$$\left(\frac{\partial J}{\partial\theta_k}\bigg|_{\theta_k}\right)_{1FD} = \left(\frac{dJ}{d\theta_k}\bigg|_{\theta_k}\right)_{1FD} = \left(\frac{dJ}{d\theta_k}\bigg|_{\theta_k}\right) + \frac{1}{2!}\left(\frac{d^2J}{d\theta_k^2}\bigg|_{\theta_k}\right)(\Delta\theta_k) + \frac{1}{3!}\left(\frac{d^3J}{d\theta_k^3}\bigg|_{\theta_k}\right)(\Delta\theta_k)^2 + \cdot\cdot\cdot\cdot + \frac{R_n}{(\Delta\theta_k)} \qquad (75)$$

Since the mathematically correct value of the derivative is the analytically derived value of $\left(\frac{dJ}{d\theta_k}\Big|_{\theta_k}\right)$, the partial derivative evaluated by the one-sided finite difference method, $\left(\frac{\partial J}{\partial\theta_k}\Big|_{\theta_k}\right)_{1FD}$ has an analytic error of

$$(\text{Analytic Error})_{1FD} = \frac{1}{2!}\left(\frac{d^2J}{d\theta_k^2}\bigg|_{\theta_k}\right)(\Delta\theta_k) + \frac{1}{3!}\left(\frac{d^3J}{d\theta_k^3}\bigg|_{\theta_k}\right)(\Delta\theta_k)^2 + \cdot\cdot\cdot\cdot + \frac{R_n}{(\Delta\theta_k)} \qquad (76)$$

Which for $|\Delta\theta_k| << 1$

$$(\text{Analytic Error})_{1\,FD} \approx \frac{1}{2}\left(\frac{d^2J}{d\theta_k^2}\bigg|_{\theta_k}\right)(\Delta\theta_k) \qquad (77)$$

The practical computational significance of Relationship (76) is that there exists an optimal value of $(\Delta\theta_k)$ which optimally trades-off computer computation errors against the analytic error defined by Relationship (76) to yield the minimum net error (i.e., the sum of the analytic error and the computer computation errors). This optimal value of $(\Delta\theta_k)$ is dependent on the computer word size (or equivalently, the number of digits available for each number) used in the computations. In general, larger values of $(\Delta\theta_k)$ result in larger analytic errors, but smaller computer computation errors, and vice versa.

For the $J(\theta_k)$ quadratic function defined by Equation (72), the analytically derived derivatives appearing in the right hand side (r.h.s.) of Equations (75), (76), and (77) are:

$$\left(\frac{dJ}{d\theta_k}\bigg|_{\theta_k}\right) = 2a\,\theta_k + b \qquad (78)$$

$$\left(\frac{d^2J}{d\theta_k^2}\bigg|_{\theta_k}\right) = 2a \qquad (79)$$

$$\left(\frac{d^3 J}{d\theta_k^3}\bigg|_{\theta_k}\right) = \left(\frac{d^4 J}{d\theta_k^4}\bigg|_{\theta_k}\right) = \cdot \cdot \cdot \cdot = \left(\frac{d^{m-1} J}{d\theta_k^{m-1}}\bigg|_{\theta_k}\right) = R_m \equiv 0 \qquad (80)$$

Substitution of Equations (78), (79), and (80) into Equation (75) yields

$$\left(\frac{\partial J}{\partial\theta_k}\bigg|_{\theta_k}\right) = 2a\theta_k + b + a(\Delta\theta_k) \qquad (81)$$

with the analytic error from Equation (76) or (77) being

$$(\text{Analytic Error})_{1FD} = a(\Delta\theta_k) \qquad (82)$$

The same result (i.e., Equations (81) and (82)) could have been obtained more easily by considering the quadratic functions $J(\theta_k)$ and $J(\theta_k+\Delta\theta_k)$ defined by Equation (72) first, and then by directly substituting them into the definition of the gradient by the one-sided finite difference method, Equation (73). Specifically

$$\left(\frac{\partial J}{\partial\theta_k}\bigg|_{\theta_k}\right) = \frac{[a(\theta_k+\Delta\theta_k)^2 + b(\theta_k+\Delta\theta_k) + c] - [a(\theta_k)^2 + b(\theta_k) + c]}{(\Delta\theta_k)} \qquad (83)$$

which when simplified yields Equations (81) and (82).

In a manner similar to that employed for the evaluation of the gradient by the one-sided finite difference method, evaluation of $\left(\frac{\partial J}{\partial\theta_k}\bigg|_{\theta_k}\right)$ by the two-sided finite difference method is specified by

$$\left(\frac{\partial J}{\partial \theta_K} \bigg|_{\theta_K} \right)_{2FD} \equiv \left(\frac{dJ}{d\theta_K} \bigg|_{\theta_K} \right)_{2FD} \equiv \frac{J(\theta_K + \Delta\theta_K) - J(\theta_K - \Delta\theta_K)}{2(\Delta\theta_K)} \qquad (84)$$

where

$\Delta\theta_K$ is both the increment and the decrement in θ_c (i.e., the <u>same</u> value is used for both)

subscript 2FD denotes numerical evaluation of the derivative by the two-sided finite difference method

The Taylor Series Expansion of $J(\theta_K + \Delta\theta_K)$ is defined by Equation (74).
Evaluation of $J(\theta_K - \Delta\theta_K)$ by Taylor Series Expansion yields

$$J(\theta_K - \Delta\theta_K) = J(\theta_K) - \left(\frac{dJ}{d\theta_K} \bigg|_{\theta_K} \right)(\Delta\theta_K) + \frac{1}{2!}\left(\frac{d^2J}{d\theta_K^2} \bigg|_{\theta_K} \right)(\Delta\theta_K)^2 - \frac{1}{3!}\left(\frac{d^3J}{d\theta_K^3} \bigg|_{\theta_K} \right)(\Delta\theta_K)^3 + \cdot\cdot\cdot\cdot + R_n \qquad (85)$$

where

$|\Delta\theta_K| << 1$

R_n is the n-th order residual

Substitution of Equations (74) and (85) into Equation (84) with simplification yields

$$\left(\frac{\partial J}{\partial \theta_K} \bigg|_{\theta_K} \right)_{2FD} = \left(\frac{dJ}{d\theta_K} \bigg|_{\theta_K} \right) + \frac{1}{3!}\left(\frac{d^3J}{d\theta_K^3} \bigg|_{\theta_K} \right)(\Delta\theta_K)^2 + \frac{1}{5!}\left(\frac{d^5J}{d\theta_K^5} \bigg|_{\theta_K} \right)(\Delta\theta_K)^4 + \cdot\cdot\cdot\cdot + \frac{R_n}{(\Delta\theta_K)} \qquad (86)$$

Notice that the terms containing odd powers of $(\Delta\theta_k)$ have cancelled out and do not appear in Equation (86). Since the mathematically correct value of the derivative is the analytically derived value of $\left(\frac{dJ}{d\theta_k}\Big|_{\theta_k}\right)$, the partial derivative evaluated by the two-sided finite difference method, $\left(\frac{\partial J}{\partial\theta_k}\Big|_{\theta_k}\right)$, has an analytic error of

$$(\text{Analytic Error})_{2FD} \;=\; \frac{1}{3!}\left(\frac{d^3J}{d\theta_k^3}\Big|_{\theta_k}\right)(\Delta\theta_k)^2 \;+\; \frac{1}{5!}\left(\frac{d^5J}{d\theta_k^5}\Big|_{\theta_k}\right)(\Delta\theta_k)^4 + \cdots\cdot + \frac{R_n}{(\Delta\theta_k)} \qquad (87)$$

which for $|\Delta\theta_k| << 1$

$$(\text{Analytic Error})_{2FD} \;\approx\; \frac{1}{6}\left(\frac{\partial^3J}{\partial\theta_k^3}\Big|_{\theta_k}\right)(\Delta\theta_k)^2 \qquad (88)$$

Comparison of Equation (87) with Equation (76) reveals that the lowest order term in the analytic error expression for the two-sided finite difference method is one order higher than that of the one-sided finite difference method. Since $\Delta\theta_k << 1$, at least one order of magnitude improvement in accuracy in numerical gradient evaluation can be obtained by using the two-sided finite difference method instead of the one-sided finite difference method. As in the case of the one-sided finite difference method, an optimal $(\Delta\theta_k)$ exists which will minimise the net error (i.e., the sum of the analytic error and the computer computation error) however the net error is much sensitive to this value.

If the $J(\theta_k)$ function is the general quadratic function defined by Equation (72),

$\left(\frac{\partial J}{\partial \theta_k}\Big|_{\theta_k}\right)_{2FD}$ and $(\text{Analytic Error})_{2FD}$ are easily evaluated by substituting Equations (78), (79), and (80) into Equations (87) and (88) to yield

$$\boxed{\begin{array}{l} \left(\frac{\partial J}{\partial \theta_k}\Big|_{\theta_k}\right)_{2FD} = 2a\theta_k + b \\ \text{and} \\ (\text{Analytic Error})_{2FD} = 0 \\ \text{for } J(\theta_k) = a\theta_k^2 + b\theta_k + c \end{array}} \qquad \begin{array}{r} (89) \\ \\ (90) \\ (72) \end{array}$$

The same result (i.e, Equations (89) and (90) could have been obtained more easily by considering the quadratic functions $J(\theta_k+\Delta\theta_k)$ and $J(\theta_k-\Delta\theta_k)$ defined by Equation (72) first, and then by directly substituting them into the definition of the gradient by the two-sided finite difference method. Specifically

$$\left(\frac{\partial J}{\partial \theta_k}\Big|_{\theta_k}\right) = \frac{[a(\theta_k+\Delta\theta_k)^2 + b(\theta_k+\Delta\theta_k) + c] - [a(\theta_k-\Delta\theta_k)^2 + b(\theta_k-\Delta\theta_k) + c]}{2(\Delta\theta_k)} \qquad (91)$$

which when simplified yields Equations (89) and (90)

Equations (89) and (90) show that the partial derivative of a general quadratic function evaluated numerically using the two-sided finite difference method is identically the same as that derived analytically and hence is analytically exact.

Since the analytic error is identically zero (Equation (90)) for the general quadratic function (Equation (72)), there is no optimal $\Delta\theta_k$ which will minimise the net error (i.e., the sum of the analytic error and the computer computation errors). In this case, a good general rule is to make $\Delta\theta_k$ large enough to safely avoid unacceptable computer computation errors. Typically, values of $\Delta\theta_k \in [10^{-7}, 10^{-5}]\,\theta_k$ will be satisfactory.

2.5.4 Analytic Hessian

The analytic Hessian with respect to θ of the performance index J defined by Equation (37) is easily evaluated directly from the matrix equation which defines $\text{Grad}_\theta J$, Equation (68), and is

$$\boxed{\text{Hessian}_\theta J = \frac{\partial}{\partial\theta}\left[\text{Grad}_\theta J\right] = \frac{\partial^2 J}{\partial\theta^2} = C} \qquad (92)$$

This Hessian can likewise be evaluated directly from the scalar equation for $\text{Grad}_\theta J$, Equation (69) in element form and is

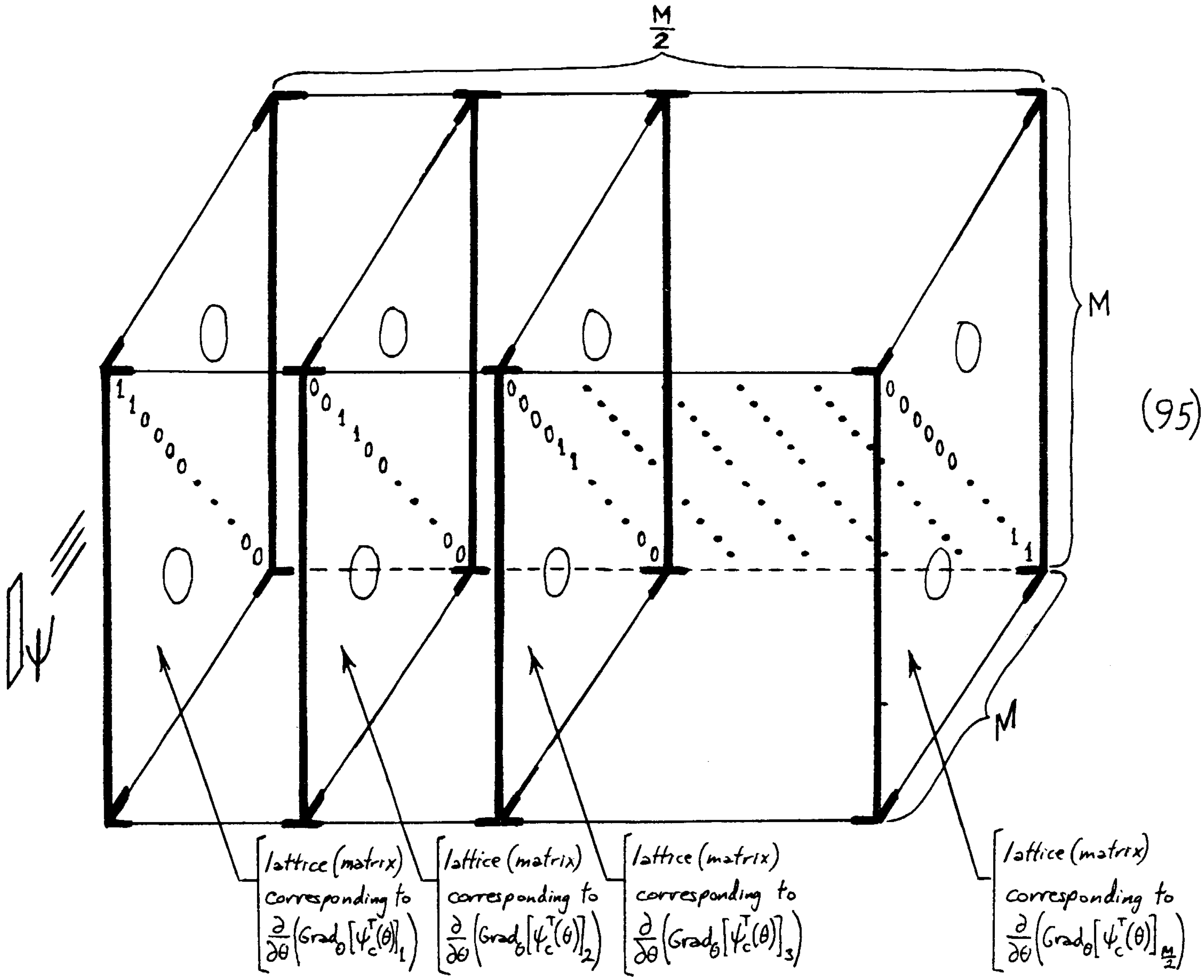
$$\boxed{\text{Hessian}_\theta J = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1M} \\ C_{12} & C_{22} & C_{23} & \cdots & C_{2M} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{3M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1M} & C_{2M} & C_{3M} & \cdots & C_{MM} \end{bmatrix}} \qquad (93)$$

The analytic Hessian with respect to θ of the constraint function $\psi(\theta)$ defined by Equation (53) (i.e., Equation (38) with $B \equiv I_M$) is

$$\boxed{\text{Hessian}_\theta\left[\psi_c(\theta)\right] = \frac{\partial}{\partial\theta}\left(\text{Grad}_\theta\left[\psi_c^T(\theta)\right]\right) = \frac{\partial^2\psi_c}{\partial\theta^2} = -2\,\mathbb{I}_\psi} \qquad (94)$$

where the subscript C denotes a column matrix formed from a diagonal matrix by shifting the diagonal elements to a single column, and $\mathbb{I}_\psi$ is the constraint form pseudo-identity tensor of rank three and dimension $(M \times M \times \frac{M}{2})$.

$\square_\psi$ is



(95)

This result can likewise be directly obtained from the element form of $\text{Grad}_\theta[\psi_c^T(\theta)]$ defined by Equation (71). It is convenient, when using this notation, to evaluate the Hessian for each element k, $k = 1, 2, \ldots, \frac{M}{2}$, of $\psi_c(\theta)$, or equivalently for each column k, $k = 1, 2, \ldots, \frac{M}{2}$ of $\text{Grad}_\theta[\psi_c^T(\theta)]$. This produces a $(M \times M)$ lattice (matrix) of the rank three tensor defined by Equation (94) which corresponds to the k-th element of $\psi_c(\theta)$, or equivalently to the k-th column of $\text{Grad}_\theta[\psi_c^T(\theta)]$.

For $k = 1$

$$\text{Hessian}_\theta[\psi_{c_1}(\theta)] = \frac{\partial}{\partial\theta}\left(\text{Grad}_\theta[\psi_c^T(\theta)]\right)_1 = \frac{\partial^2 \psi_{c_1}}{\partial\theta^2}$$

$$\text{Hessian}_\theta[\psi_{c_1}(\theta)] = -2\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & & \cdot \\ 0 & 0 & 0 & 0 & & \cdot \\ 0 & 0 & 0 & 0 & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdots & \cdot & 0 \end{bmatrix} \qquad (96)$$

For $1 < k < \frac{M}{2}$

$$\mathrm{Hessian}_\theta\left[\psi_{c_k}(\theta)\right] = \frac{\partial}{\partial\theta}\left(\mathrm{Grad}_\theta\left[\psi_c^T(\theta)\right]\right)_k = \frac{\partial^2\psi_{c_k}}{\partial\theta^2}$$

$$\mathrm{Hessian}_\theta\left[\psi_{c_k}(\theta)\right] = -2\left[\begin{array}{ccccccccccc}
 & & & & & \overset{\substack{(2k-1)\text{-th}\\ \text{Column}}}{\downarrow} & \overset{\substack{2k\text{-th}\\ \text{Column}}}{\downarrow} & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 1 & \vdots & \vdots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \vdots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & \vdots & \vdots \\
\vdots & \vdots & \vdots & & & & & & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{array}\right] \quad (97)$$

For $k = \frac{M}{2}$

$$\text{Hessian}_{\theta}\left[\psi_{c_{\frac{M}{2}}}(\theta)\right] = \frac{\partial}{\partial\theta}\left(\text{Grad}_{\theta}\left[\psi_{c}^{T}(\theta)\right]\right)_{\frac{M}{2}} = \frac{\partial^2\psi_{c_{\frac{M}{2}}}}{\partial\theta^2}$$

$$\text{Hessian}_{\theta}\left[\psi_{c_{\frac{M}{2}}}(\theta)\right] = -2\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \quad (98)$$

2.5.5 Equations Which Define the Necessary Conditions for Optimality

The matrix form of the equations which define the necessary conditions for optimality are defined by Equations (50) and (51) and are presented here again for convenience.

$$\boxed{\begin{aligned} \text{Grad}_{\underset{\sim}{\Theta}}\,\mathcal{J}[\underset{\sim}{\Theta},\lambda] = \left[\frac{\partial \mathcal{J}}{\partial \underset{\sim}{\Theta}}\right]^{T} &= 0 \quad \text{with dimension } \left(\tfrac{3M}{2} \times 1\right) \qquad (50) \\ \phi_c(\underset{\sim}{\Theta}) &= 0 \quad \text{with dimension } \left(\tfrac{M}{2} \times 1\right) \qquad (51) \end{aligned}}$$

where

$$\mathcal{J}[\underset{\sim}{\Theta},\lambda] = J(\theta) + \lambda^{T}\phi_c(\underset{\sim}{\Theta}) \qquad (49)$$

$$\left.\begin{array}{l} \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \lambda_{\frac{M}{2}} \end{bmatrix}, \quad \phi_c(\cdot) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \phi_{\frac{M}{2}} \end{bmatrix} \text{ each with dimension } \left(\tfrac{M}{2} \times 1\right) \\ \text{and} \\ \underset{\sim}{\Theta} = \begin{bmatrix} \theta \\ --- \\ \alpha \end{bmatrix} \text{ with dimension } \left(\tfrac{3M}{2} \times 1\right) \end{array}\right\} \qquad (99)$$

Noting the scalar form of ψ as shown by Equation (56) and the definitions of α, $\underset{\sim}{\Theta}$, and α_ϕ which complement Equation (45), Equations (48) and (51) can be expressed in scalar form as

$$\boxed{\phi_c(\underset{\sim}{\Theta}) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{\frac{M}{2}} \end{bmatrix} = \begin{bmatrix} A_1 - \theta_1^2 - \theta_2^2 - \alpha_1^2 \\ A_2 - \theta_3^2 - \theta_4^2 - \alpha_2^2 \\ A_3 - \theta_5^2 - \theta_6^2 - \alpha_3^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 - \alpha_{\frac{M}{2}}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \underline{\text{Constraint}} \; \underline{\text{Equations}}} \qquad (100)$$

$\mathfrak{f}[\underset{\sim}{\Theta},\lambda]$ as defined by Equation (49) can then easily be expressed in scalar form using Equation (67) to define $J(\theta)$, Equation (99) to define λ, and Equation (100) to define $\phi_c(\underset{\sim}{\Theta})$.

$$\mathfrak{f}[\underset{\sim}{\Theta},\lambda] = \left[J(\theta)\right]_{Eq(67)} + \left(\lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3 + \cdot\cdot\cdot\cdot + \lambda_{\frac{M}{2}}\phi_{\frac{M}{2}}\right) \qquad (101)$$

where

$\left[J(\theta)\right]_{Eq(67)}$ is the scalar form of $J(\theta)$ as expressed by Equation (67).

Referring to Equations (67), (99), and (100), $\mathfrak{f}[\underset{\sim}{\Theta},\lambda]$ can be expressed as

$$\left.\begin{aligned}
J[\underset{\sim}{\Theta},\lambda] = {} & \frac{1}{2}C_{11}\theta_1^2 + C_{12}\theta_1\theta_2 + C_{13}\theta_1\theta_3 + C_{14}\theta_1\theta_4 + \cdots\cdots + C_{1M}\theta_1\theta_M \\
& + \frac{1}{2}C_{22}\theta_2^2 + C_{23}\theta_2\theta_3 + C_{24}\theta_2\theta_4 + \cdots\cdots + C_{2M}\theta_2\theta_M \\
& + \frac{1}{2}C_{33}\theta_3^2 + C_{34}\theta_3\theta_4 + \cdots\cdots + C_{3M}\theta_3\theta_M \\
& \qquad \ddots \qquad \ddots \qquad \vdots \\
& + C_{(M-1)M}\theta_{(M-1)}\theta_M \\
& + \frac{1}{2}C_{MM}\theta_M^2 \\
& + 2(g_1\theta_1 + g_2\theta_2 + g_3\theta_3 + \cdots\cdots + g_M\theta_M) \\
& + W_1 Z_{o_1}^2 + W_2 Z_{o_2}^2 + W_3 Z_{o_3}^2 + \cdots\cdots + W_N Z_{o_N}^2 \\
& + \lambda_1(A_1 - \theta_1^2 - \theta_2^2 - \alpha_1^2) \\
& + \lambda_2(A_2 - \theta_3^2 - \theta_4^2 - \alpha_2^2) \\
& + \lambda_3(A_3 - \theta_5^2 - \theta_6^2 - \alpha_3^2) \\
& \qquad \vdots \\
& + \lambda_{\frac{M}{2}}(A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 - \alpha_{\frac{M}{2}}^2)
\end{aligned}\right\} \quad (102)$$

The equations resulting from evaluation of Equation (50) using Equation (102) and Equation (100) comprise the full set of scalar equations which define the necessary conditions for optimality. Specifically

$$\left.\begin{aligned}
\frac{\partial f}{\partial \theta_1} &= C_{11}\theta_1 + C_{12}\theta_2 + C_{13}\theta_3 + \cdots\cdots + C_{1M}\theta_M + 2g_1 - 2\lambda_1\theta_1 = 0 \\
\frac{\partial f}{\partial \theta_2} &= C_{12}\theta_1 + C_{22}\theta_2 + C_{23}\theta_3 + \cdots\cdots + C_{2M}\theta_M + 2g_2 - 2\lambda_1\theta_2 = 0 \\
\frac{\partial f}{\partial \theta_3} &= C_{13}\theta_1 + C_{23}\theta_2 + C_{33}\theta_3 + \cdots\cdots + C_{3M}\theta_M + 2g_3 - 2\lambda_2\theta_3 = 0 \\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{\partial f}{\partial \theta_M} &= C_{1M}\theta_1 + C_{2M}\theta_2 + C_{3M}\theta_3 + \cdots\cdots + C_{MM}\theta_M + 2g_M - 2\lambda_{\frac{M}{2}}\theta_M = 0
\end{aligned}\right\} \quad (103)$$

Necessary Conditions for Optimality

$$\left.\begin{aligned}
\frac{\partial f}{\partial \alpha_1} &= -2\lambda_1\alpha_1 = 0 \\
\frac{\partial f}{\partial \alpha_2} &= -2\lambda_2\alpha_2 = 0 \\
\frac{\partial f}{\partial \alpha_3} &= -2\lambda_3\alpha_3 = 0 \\
&\quad\vdots \\
\frac{\partial f}{\partial \alpha_{\frac{M}{2}}} &= -2\lambda_{\frac{M}{2}}\alpha_{\frac{M}{2}} = 0
\end{aligned}\right\} \qquad
\left.\begin{aligned}
\lambda_1\alpha_1 &= 0 \\
\lambda_2\alpha_2 &= 0 \\
\lambda_3\alpha_3 &= 0 \\
&\vdots \\
\lambda_{\frac{M}{2}}\alpha_{\frac{M}{2}} &= 0
\end{aligned}\right\} \quad \underline{\text{Switching Equations}} \qquad (104)$$

where the constraint equations are defined by Equation (100).

2.5.6 Analytic Jacobian

The solutions described in Sections 2.4.3 and 2.4.4 of the full and reduced sets of equations which define the necessary conditions for optimality (i.e., Equations (100), (103), and (104)) require evaluation of the Jacobian of these equations during the solution process. Although the ordering of these equations is arbitrary, the order for this analysis was selected to be Equation (103) first, then Equation (100), and finally Equation (104). It is convenient at this point to define four vectors; the (MX1) dimensional vector Γ comprised of some of the unknowns (i.e., the slack variables and the Lagrangian multipliers), the (2M×1) dimensional vector Φ comprised of all the unknowns (i.e, the control vector, the slack variables, and the Lagrangian multipliers), the $(\frac{3M}{2} \times 1)$ dimensional vector Ξ comprised of ordered Equations (103) and (100), and the (2M×1) dimensional vector Ψ of the ordered set of all the equations defining the necessary conditions for optimality (i.e., Equations (103), (100), and (104)). Specifically, define

$$\Gamma = \begin{bmatrix} \alpha \\ --- \\ \lambda \end{bmatrix} = [\alpha_1\ \alpha_2\ \alpha_3 \cdot\cdot\cdot\cdot\cdot \alpha_{\frac{M}{2}} \vdots\ \lambda_1\ \lambda_2\ \lambda_3 \cdot\cdot\cdot\cdot\cdot \lambda_{\frac{M}{2}}]^T \quad \text{with dimension } (M\times1) \tag{105}$$

$$\Phi = \begin{bmatrix} \Theta \\ --- \\ \lambda \end{bmatrix} = \begin{bmatrix} \theta \\ --- \\ \alpha \\ --- \\ \lambda \end{bmatrix} = \begin{bmatrix} \theta \\ --- \\ \Gamma \end{bmatrix} = [\theta_1\ \theta_2\ \theta_3 \cdot\cdot\cdot\cdot\cdot \theta_M \vdots\ \alpha_1\ \alpha_2\ \alpha_3 \cdot\cdot\cdot\cdot\cdot \alpha_{\frac{M}{2}} \vdots\ \lambda_1\ \lambda_2\ \lambda_3 \cdot\cdot\cdot\cdot\cdot \lambda_{\frac{M}{2}}]^T \text{ with dimension } (2M\times1) \tag{106}$$

Noting that

Equation (103) in matrix form is $\left[\dfrac{\partial \mathcal{L}}{\partial \theta}\right]^T = 0$ with dimension $(M \times 1)$ (107)

Equation (100) in matrix form is $\phi_c(\underset{\sim}{\Theta}) = 0$ with dimension $\left(\frac{M}{2} \times 1\right)$ (108)

and

Equation (104) in matrix form is $\left[\dfrac{\partial \mathcal{L}}{\partial \alpha}\right]^T = 0$ with dimension $\left(\frac{M}{2} \times 1\right)$ (109)

Define

$$\underset{\sim}{\Xi} = \left[\begin{array}{c} \text{Equation (103)} \\ \hdashline \text{Equation (100)} \end{array}\right] = \left[\begin{array}{c} \left[\dfrac{\partial \mathcal{L}}{\partial \theta}\right]^T \\ \hdashline \phi_c(\underset{\sim}{\Theta}) \end{array}\right] = \left[\begin{array}{c:c} \dfrac{\partial \mathcal{L}}{\partial \theta_1} \; \dfrac{\partial \mathcal{L}}{\partial \theta_2} \; \dfrac{\partial \mathcal{L}}{\partial \theta_3} \cdots\cdots \dfrac{\partial \mathcal{L}}{\partial \theta_M} & \phi_1 \; \phi_2 \; \phi_3 \cdots\cdots \phi_{\frac{M}{2}} \end{array}\right]^T \text{ with dimension } \left(\tfrac{3M}{2} \times 1\right) \tag{110}$$

and

$$\Psi = \left[\begin{array}{c} \text{Equation (103)} \\ \hdashline \text{Equation (100)} \\ \hdashline \text{Equation (104)} \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{\Xi} \\ \hdashline \left[\dfrac{\partial \mathcal{L}}{\partial \alpha}\right]^T \end{array}\right] = \left[\begin{array}{c:c:c} \dfrac{\partial \mathcal{L}}{\partial \theta_1} \; \dfrac{\partial \mathcal{L}}{\partial \theta_2} \; \dfrac{\partial \mathcal{L}}{\partial \theta_3} \cdots\cdots \dfrac{\partial \mathcal{L}}{\partial \theta_M} & \phi_1 \; \phi_2 \; \phi_3 \cdots\cdots \phi_{\frac{M}{2}} & \dfrac{\partial \mathcal{L}}{\partial \alpha_1} \; \dfrac{\partial \mathcal{L}}{\partial \alpha_2} \; \dfrac{\partial \mathcal{L}}{\partial \alpha_3} \cdots\cdots \dfrac{\partial \mathcal{L}}{\partial \alpha_{\frac{M}{2}}} \end{array}\right]^T \text{ with dimension } (2M \times 1) \tag{111}$$

Then for the full set of equations which define the necessary conditions for optimality

$$\boxed{\text{JACOBIAN} = \left[\dfrac{\partial \Psi}{\partial \Phi}\right] \qquad \text{with dimension } (2M \times 2M)} \tag{112}$$

It is convenient to partition the Jacobian according to

$$\boxed{\left[\frac{\partial \Psi}{\partial \Phi}\right] = \left[\begin{array}{c:c} Q & R \\ \hdashline S & T \end{array}\right]} \qquad \underline{\text{JACOBIAN for}} \qquad (113)$$

<u>Full Set of Equations</u>

where

$$Q = \left[\frac{\partial \Xi}{\partial \theta}\right] \quad \text{with dimension} \left(\frac{3M}{2} \times M\right) \qquad (114)$$

$$R = \left[\frac{\partial \Xi}{\partial \Gamma}\right] \quad \text{with dimension} \left(\frac{3M}{2} \times M\right) \qquad (115)$$

$$S = \frac{\partial}{\partial \theta}\left[\frac{\partial \gamma}{\partial \alpha}\right]^T \quad \text{with dimension} \left(\frac{M}{2} \times M\right) \qquad (116)$$

$$T = \frac{\partial}{\partial \Gamma}\left[\frac{\partial \gamma}{\partial \alpha}\right]^T \quad \text{with dimension} \left(\frac{M}{2} \times M\right) \qquad (117)$$

The Jacobian can be expressed in element form from Equations (103), (100), and (104) by using Equations (114), (115), (116), and (117) to evaluate Q, R, S, and T, respectively; specifically

$$Q = \underbrace{\left[\begin{array}{cccccccc}
(C_{11}-2\lambda_1) & C_{12} & C_{13} & C_{14} & \cdots & \cdots & & C_{1M} \\
C_{12} & (C_{22}-2\lambda_1) & C_{23} & C_{24} & \cdots & \cdots & & C_{2M} \\
C_{13} & C_{23} & (C_{33}-2\lambda_2) & C_{34} & \cdots & \cdots & & C_{3M} \\
C_{14} & C_{24} & C_{34} & (C_{44}-2\lambda_2) & \cdots & \cdots & & C_{4M} \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots \\
C_{1M} & C_{2M} & C_{3M} & C_{4M} & \cdots & \cdots & & (C_{MM}-2\lambda_{\frac{M}{2}}) \\
\hdashline
-2\theta_1 & -2\theta_2 & 0 & 0 & \cdots & \cdots & \cdot & 0 \\
0 & 0 & -2\theta_3 & -2\theta_4 & & & \cdot & 0 \\
0 & 0 & 0 & 0 & & & \cdot & 0 \\
0 & 0 & 0 & 0 & & & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & -2\theta_{(M-1)} & -2\theta_M
\end{array}\right]}_{M}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} M \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} \frac{M}{2}
\end{array}
\text{ with dimension } \left(\frac{3M}{2} \times M\right) \quad (118)$$

$$
R = \left[\begin{array}{ccccc|ccccc}
0 & 0 & 0 & \cdots & 0 & -2\theta_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & -2\theta_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -2\theta_3 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -2\theta_4 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -2\theta_5 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -2\theta_6 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -2\theta_{(M-1)} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -2\theta_M \\
\hline
-2\alpha_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & -2\alpha_2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & -2\alpha_3 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -2\alpha_{\frac{M}{2}} & 0 & 0 & 0 & \cdots & 0 \\
\multicolumn{5}{c}{\underbrace{\hspace{6cm}}_{\frac{M}{2}}} & \multicolumn{5}{c}{\underbrace{\hspace{6cm}}_{\frac{M}{2}}}
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\0\end{array}}\right\} M \\
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} \frac{M}{2}
\end{array}
\quad \begin{array}{l}\text{with}\\ \text{dimension}\\ \left(\frac{3M}{2} \times M\right)\end{array}
\qquad (119)
$$

$$S = \left.\left[\begin{array}{c:c} \underbrace{\begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}}_{\frac{M}{2}} & \underbrace{\begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}}_{\frac{M}{2}} \end{array}\right]\right\} \frac{M}{2} \text{ with dimension } \left(\frac{M}{2} \times M\right) \qquad (120)$$

$$T = \left.\left[\begin{array}{c:c} \underbrace{\begin{matrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{\frac{M}{2}} \end{matrix}}_{\frac{M}{2}} & \underbrace{\begin{matrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{\frac{M}{2}} \end{matrix}}_{\frac{M}{2}} \end{array}\right]\right\} \frac{M}{2} \text{ with dimension } \left(\frac{M}{2} \times M\right) \qquad (121)$$

It was assumed that $M=6$ and that the constraint activity was known a priori for the reduced sets of equations defining necessary conditions for optimality as described in Section 2.4.4. Correspondingly, the number of unknowns and equations to be solved was reduced from 12 to 9 by appropriately selecting three of $\{\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3\}$ to be zero for each constraint activity possibility and by eliminating the switching equations (Equation (104)) altogether from those to be solved.

Define

$$\Phi_k = \left[\begin{array}{c} \theta \\ \hdashline \Gamma_k \end{array}\right] \quad \text{for } k \in \{1, 2, \ldots . 8\} \text{ with dimension } (9 \times 1) \qquad (122)$$

where Γ_k is defined according to

Possibility k	Parameters Equal to Zero	Γ_k with dimension (3×1)
1	$\alpha_1\ \alpha_2\ \alpha_3$	$\Gamma_1 = [\lambda_1\ \lambda_2\ \lambda_3]^T$
2	$\lambda_1\ \alpha_2\ \alpha_3$	$\Gamma_2 = [\alpha_1\ \lambda_2\ \lambda_3]^T$
3	$\alpha_1\ \lambda_2\ \alpha_3$	$\Gamma_3 = [\alpha_2\ \lambda_1\ \lambda_3]^T$
4	$\lambda_1\ \lambda_2\ \alpha_3$	$\Gamma_4 = [\alpha_1\ \alpha_2\ \lambda_3]^T$
5	$\alpha_1\ \alpha_2\ \lambda_3$	$\Gamma_5 = [\alpha_3\ \lambda_1\ \lambda_2]^T$
6	$\lambda_1\ \alpha_2\ \lambda_3$	$\Gamma_6 = [\alpha_1\ \alpha_3\ \lambda_2]^T$
7	$\alpha_1\ \lambda_2\ \lambda_3$	$\Gamma_7 = [\alpha_2\ \alpha_3\ \lambda_1]^T$
8	$\lambda_1\ \lambda_2\ \lambda_3$	$\Gamma_8 = [\alpha_1\ \alpha_2\ \alpha_3]^T$

(123)

Then

$$\left.\begin{aligned}
\Phi_1 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \lambda_1\ \lambda_2\ \lambda_3\right]^T \\
\Phi_2 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_1\ \lambda_2\ \lambda_3\right]^T \\
\Phi_3 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_2\ \lambda_1\ \lambda_3\right]^T \\
\Phi_4 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_1\ \alpha_2\ \lambda_3\right]^T \\
\Phi_5 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_3\ \lambda_1\ \lambda_2\right]^T \\
\Phi_6 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_1\ \alpha_3\ \lambda_2\right]^T \\
\Phi_7 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_2\ \alpha_3\ \lambda_1\right]^T \\
\Phi_8 &= \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6 \,\vdots\, \alpha_1\ \alpha_2\ \alpha_3\right]^T
\end{aligned}\right\}\ \begin{array}{l}\text{with}\\ \text{dimension}\\ (9\times 1)\end{array} \qquad (124)$$

The equations to be solved are specified by the Ξ vector defined by Equation (110) and correspondingly, the Jacobian is

$$\boxed{\left(\text{JACOBIAN}\right)_k = \left[\frac{\partial \Xi}{\partial \Phi_k}\right] \quad \begin{array}{l}\text{for } k \in \{1, 2, \cdots\cdots 8\}\\ \text{with dimension } (9\times 9)\end{array}} \qquad (125)$$

It is convenient to partition the Jacobian according to

JACOBIAN for Reduced Sets of Equations

$$\boxed{\left[\frac{\partial \Xi}{\partial \Phi_k}\right] = [Q \,\vdots\, R_k] \text{ for } k \in \{1,2\cdots\cdot 8\}} \qquad (126)$$

where Q is defined as previously by Equations (114) and (118) for $M = 6$ and is

$$Q = \left[\frac{\partial \Xi}{\partial \theta}\right] = \underbrace{\left[\begin{array}{cccccc} (C_{11}-2\lambda_1) & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & (C_{22}-2\lambda_1) & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & (C_{33}-2\lambda_2) & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & (C_{44}-2\lambda_2) & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & (C_{55}-2\lambda_3) & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & (C_{66}-2\lambda_3) \\ \hline -2\theta_1 & -2\theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\theta_3 & -2\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\theta_5 & -2\theta_6 \end{array}\right]}_{6} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}6 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}3 \,. \end{array} \text{ with dimension } (9\times 6) \qquad (127)$$

and

$$R_k = \left[\frac{\partial \Xi}{\partial \Gamma_k}\right] \quad \text{for } k \in \{1, 2, \cdots\cdot 8\} \quad \text{with dimension } (9\times 3) \qquad (128)$$

If $k = 1$, constraints 1, 2, and 3 are active, $\Gamma_1 = [\lambda_1 \lambda_2 \lambda_3]^T$, and

$$R_1 = \left[\frac{\partial \Xi}{\partial \Gamma_1}\right] = \underbrace{\left[\begin{array}{ccc} -2\theta_1 & 0 & 0 \\ -2\theta_2 & 0 & 0 \\ 0 & -2\theta_3 & 0 \\ 0 & -2\theta_4 & 0 \\ 0 & 0 & -2\theta_5 \\ 0 & 0 & -2\theta_6 \\ \hdashline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}}\right\} 6 \\ \text{with dimension} (9 \times 3) \\ \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}}\right\} 3 \end{array} \tag{129}$$

If $k = 2$, constraints 2 and 3 are active, $\Gamma_2 = [\alpha_1 \ \lambda_2 \ \lambda_3]^T$, and

$$R_2 = \left[\frac{\partial \Xi}{\partial \Gamma_2}\right] = \underbrace{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2\theta_3 & 0 \\ 0 & -2\theta_4 & 0 \\ 0 & 0 & -2\theta_5 \\ 0 & 0 & -2\theta_6 \\ \hdashline -2\alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}}\right\} 6 \\ \text{with dimension} (9 \times 3) \\ \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}}\right\} 3 \end{array} \tag{130}$$

If $k=3$, constraints 1 and 3 are active, $\Gamma_3 = [\alpha_2 \ \lambda_1 \ \lambda_3]^T$, and

$$R_3 = \left[\frac{\partial \Xi}{\partial \Gamma_3}\right] = \underbrace{\left[\begin{array}{ccc} 0 & -2\theta_1 & 0 \\ 0 & -2\theta_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\theta_5 \\ 0 & 0 & -2\theta_6 \\ \hdashline 0 & 0 & 0 \\ -2\alpha_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 6 \\ \text{with dimension } (9\text{X}3) \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3 \end{array} \tag{131}$$

If $k=4$, constraint 3 is active, $\Gamma_4 = [\alpha_1 \ \alpha_2 \ \lambda_3]^T$, and

$$R_4 = \left[\frac{\partial \Xi}{\partial \Gamma_4}\right] = \underbrace{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\theta_5 \\ 0 & 0 & -2\theta_6 \\ \hdashline -2\alpha_1 & 0 & 0 \\ 0 & -2\alpha_2 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 6 \\ \text{with dimension } (9\text{X}3) \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3 \end{array} \tag{132}$$

If $k=5$, constraints 1 and 2 are active, $\Gamma_5 = [\alpha_3 \; \lambda_1 \; \lambda_2]^T$, and

$$R_5 = \left[\frac{\partial \vec{\Xi}}{\partial \Gamma_5}\right] = \underbrace{\left[\begin{array}{ccc} 0 & -2\theta_1 & 0 \\ 0 & -2\theta_2 & 0 \\ 0 & 0 & -2\theta_3 \\ 0 & 0 & -2\theta_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2\alpha_3 & 0 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\} 6 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} 3 \end{array} \tag{133}$$

with dimension (9x3)

If $k=6$, constraint 2 is active, $\Gamma_6 = [\alpha_1 \; \alpha_3 \; \lambda_2]^T$, and

$$R_6 = \left[\frac{\partial \vec{\Xi}}{\partial \Gamma_6}\right] = \underbrace{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\theta_3 \\ 0 & 0 & -2\theta_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline -2\alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2\alpha_3 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\} 6 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} 3 \end{array} \tag{134}$$

with dimension (9x3)

If $k=7$, constraint 1 is active, $\Gamma_7 = [\alpha_2\ \alpha_3\ \lambda_1]^T$, and

$$R_7 = \left[\frac{\partial \Xi}{\partial \Gamma_7}\right] = \underbrace{\left[\begin{array}{ccc} 0 & 0 & -2\theta_1 \\ 0 & 0 & -2\theta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ -2\alpha_2 & 0 & 0 \\ 0 & -2\alpha_3 & 0 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\}6 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}3 \end{array} \qquad (135)$$

with dimension (9x3)

If $k=8$, No constraints are active, $\Gamma_8 = [\alpha_1\ \alpha_2\ \alpha_3]^T$, and

$$R_8 = \left[\frac{\partial \Xi}{\partial \Gamma_8}\right] = \underbrace{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline -2\alpha_1 & 0 & 0 \\ 0 & -2\alpha_2 & 0 \\ 0 & 0 & -2\alpha_3 \end{array}\right]}_{3} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\}6 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}3 \end{array} \qquad (136)$$

with dimension (9x3)

2.6 The T-Matrix Generation System

The stand-alone code URAND was written to define the elements of T and Z_0 (see Subsections 2.1.1.2 and 2.1.1.3). Pseudo-random numbers were generated from a uniform distribution (Reference 8) and employed in this code to define these elements in accordance with the random concepts described in Subsections 2.1.1.4 and 2.1.1.5. A listing of definitions of the principal parameters in URAND including all the input parameters and a brief description of the principal routines comprising URAND is presented in Appendix A. Listings of the VAX/VMS Command File used to execute URAND and the URAND FORTRAN routines themselves are presented in Appendix B. Input to, and output from, a sample case is presented in Appendix C.

2.7 The Stand-Alone Optimal Controller System

The stand-alone code OPTIM was written as part of this research to provide a means to experiment with, to compare, and to evaluate the controller schemes described herein. Numerous controller options (see definition of IOPT in Appendix D) were provided to specify the optimal controllers defined according to the theory described in Sections 2.3 and 2.4 and the pseudo-optimal controllers defined according to the theory described in Section 2.2 and variations thereof. Several IMSL MATH/LIBRARY routines (see Appendix D and References 5, 6, and 7) were employed in this code to accomplish the required computations and to provide verification of analytic gradients and Jacobians. A listing of definitions of the principal parameters in OPTIM including all the input parameters and a brief description of the principal routines comprising OPTIM is presented in Appendix D. Listings of the VAX/VMS Command File used to execute OPTIM and the OPTIM FORTRAN routines themselves are presented in Appendix E. Input to, and output from, a sample case is presented in Appendix F.

Early on in the development of this code during the initial verification phases before the T-Matrix Generation System was developed, T-Matrices were obtained from actual CAMRAD/JA (Reference 19) closed loop HHC simulations of the BO-105 and S-76 rotor systems. HHC problems of various dimensions (i.e., various T-Matrix dimensions) were examined. A typical and illustrative example of the behaviour of the Optimal Controller as described in Section 2.3, was a 6-vector HHC applied to the four bladed S-76 rotor system with a 6-vector measurement. The Control Vector for this case is:

$$\theta = \begin{bmatrix} \theta_{c_3} \\ \theta_{s_3} \\ \theta_{c_4} \\ \theta_{s_4} \\ \theta_{c_5} \\ \theta_{s_5} \end{bmatrix} \quad \underline{\text{CONTROL VECTOR}} \; \underline{\text{ROTATING FRAME}} \; \underline{\text{BLADE PITCH}} \qquad (137)$$

The Measurement Vector is

$$Z = \begin{bmatrix}\text{Drag Force}\\ \text{Side Force}\\ \text{Thrust}\\ \text{Roll Moment}\\ \text{Pitch Moment}\\ \text{Torque}\end{bmatrix} \quad \begin{array}{l}\underline{\text{MEASUREMENT VECTOR}}\\ \underline{\text{NON-ROTATING FRAME}}\\ \underline{\text{4 per Rev Hub Loads}}\end{array} \qquad (138)$$

Applying the same magnitude constraint limit to each of the pitch control harmonics, that is setting

$$(l.u.b)_{Y_3} = (l.u.b)_{Y_4} = (l.u.b)_{Y_5} = R \qquad (139)$$

in Equation (12) or (13), the optimal values of θ_{C_n} and θ_{S_n} for $n = 3, 4,$ and 5 and for parametric values of R were determined using the Stand-Alone Optimal Controller System during its verification process. The results, which are presented in Figures 5, 6, and 7, exhibit typical behaviour; specifically

0. From a performance index (PINDX) starting value ($PINDX_0$) of 32.94, the optimisation process yields a performance index value $PINDX < PINDX_0$ in all cases.

1. For R large enough (i.e., $R > 2.0463$), none of the constraints are active (i.e., $Y_n > (\ell.u.b.)_{Y_n}$ for $n = 3, 4, 5$) and the resulting performance index is zero as required by Theorem 5 in Section 2.3.2.

2. Decreasing R to 2.0463 and below activates the 4th harmonic constraint (see Figure 6) and causes the minimum performance index (PINDX) to be greater than zero (see Figure 5).

3. Further reduction of R to 1.3448 and below activates the 5th harmonic constraint (see Figure 6) and causes the minimum performance index (PINDX) to increase at a somewhat higher rate (see Figure 5).

4. Further reduction of R to 0.7697 and below activates the 3rd harmonic constraint (see Figure 6) and causes the minimum performance index (PINDX) to increase at a higher rate yet (see Figure 5) until R reaches zero at which point the performance index (PINDX) has the value that it would have without the application of any HHC at all ($PINDX \equiv PINDX_{00} = 24.59$).

From Figure 5, it is noted that a very restrictive constraint $\ell.u.b.$ of 0.8 degree applied to all three HHC harmonics under consideration still yielded a 90% reduction in performance index from that obtained with no HHC at all. From Figure 6, it is noted that the decreasing order of the optimal unconstrained values of the harmonic coefficients of interest are the 4th harmonic, the 5th harmonic, and the 3rd harmonic. Correspondingly, the constraints become active in that order as the common l.u.b. value (i.e., R) is decreased. From Figure 7, it is noted that as the common l.u.b. value (i.e., R) is decreased, the phase angle of the three harmonics of interest appear to cluster about the value of -135 degrees.

During the verification process, cases were run to test the reliability and robustness of the Optimal Controller System. Numerous T-Matrix dimensions up to (24 x 12) were considered. A partial list of these cases together with their constraint activity and resulting performance index value (J) for each controller investigated (i.e., 1.) the Optimal Controller, 2.) the Conventional Controller, and 3.) the Optimised Conventional Controller) is presented in Table 1. The cases with T-Matrix dimensions of the form (N x 6) for N = 6, 7, 8, • • • 24 are described subsequently in Section 2.8.

One of the important indicators of algorithm reliability and robustness is what happens if a sufficient number of the measurement sensors are "lost" causing the number of remaining sensors (N) to be less than the number of available controls (M) (i.e., $N < M$ for $T(N, M)$). From Table 1, it can be seen that all three controllers still seek a solution and that the Optimal Controller yields a performance index less than or equal to that of the other two controllers. The important point is that with the exception of the degenerate case where $N = 0$, all three controllers yielded a result. The case where $N = 0$ can easily be handled with simple logic.

2.8 Numerical Study of Aeroelastic Effects by Introducing Some Degree of Linear Dependency to the Rows of a Non-square T-Matrix by means of the "Proportional Navigation" Scheme

In accordance with the "Proportional Navigation" type scheme to model aeroelastic effects as described in Section 2.1.1.5, a numerical study of aeroelastic effects was accomplished for a family of T-Matrices with dimensions (Nx6) where N=6,7,8, ••• 24 and for proportionality constants (C in Equation (8) and input parameter RATIO to the T-Matrix Generation System) from 100% to 75%. An important class of cases are those for which C ∈ [95%, 100%] because of both the actual physical characteristics of the aircraft in question (a generous allowance of up to 5% higher harmonic bending in the aircraft) and the statistical nature of the "Proportional Navigation" type scheme when a uniform distribution is assumed. A rapid decrease in minimum performance index computed by the Optimal Controller (defined in Section 2.3.1) for cases where no constraints are active typically occurs near C=95%. Accordingly, the minimum performance index J computed by the Optimal Controller, the Conventional Controller (defined in Section 2.2.2), and the Optimised Conventional Controller (defined in Section 2.2.2) for T-Matrix dimensions* of (N x 6) where N= 6, 7, 8, ••• 24 was

---

* The reference T-Matrix dimension from which T-Matrices of smaller dimensions were derived is (24x6). The Row Duplication Pointer-Vector (input parameter IROW(•) to the T-Matrix Generation System used to specify row duplication in accordance with Equation (8)) is

IROW = 1,1,1,1,1,1, | 1,2,3,4,5,6, | 1,2,3,4,5,6, | 1,2,3,4,5,6, computed for RATIO values of 95% and 100% and the results are presented in Figures 8 and 9, respectively. Note that the lines connecting the data points in these figures define an envelope, not a continuous function. The Optimal Controller outperformed the other two controllers for all dimensions considered and, as expected, the Optimised Conventional Controller yielded performance indices between that of the Optimal Controller and that of the Conventional Controller. It is noted however, that the Optimised Conventional Controller is an ideal case which would be difficult to realise in practice. Although the Optimised Conventional Controller has the same dimension as that of the Optimal Controller, it is poorly conditioned mathematically and in general has more than one solution in the neighbourhood of the global solution. Correspondingly, it is the least reliable of the three controllers and generally requires significantly more computer time to process.

The minimum performance index J computed by the Optimal Controller, the Conventional Controller, and the Optimised Conventional Controller was computed for RATIO values $\in [75\%, 100\%]$ and T-Matrix dimensions* of $(N \times 6)$ where $N = 6, 7, 8, \cdots 12$. The results are presented in Figures 10 through 16, respectively. As in the case of the data presented in Figures 8 and 9, the Optimal Controller outperformed the other controllers. In accordance with the requirement of Theorem 5 in Section 2.3.2 and the row duplication

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* See previous footnote.

specifications defined by IROW(•)*, the minimum performance index computed by the Optimal Controller is identically zero for N = 6 regardless of the RATIO value since the T-Matrix is square and non-singular in that case, and also when RATIO has a value of 100% for cases where N > 6 since no new information is contained in the T-Matrix rows after the sixth. The aforementioned rapid decrease in the minimum performance index for the optimal controller with no active constraints can clearly be seen in Figures 12 through 16.

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* See previous footnote.

2.9 Addition of a Least Upper Bound Constraint on the Magnitude of the Sum of Amplitudes of All Harmonics Being Considered

An additional constraint was formulated and implemented as an option to the Stand-Alone Optimal Controller System. This constraint has the form of a least upper bound (l.u.b) on the magnitude of the sum of the amplitudes of all harmonics being considered. Specifically, Relationship (13) becomes

$$
\begin{aligned}
\theta_1^2 + \theta_2^2 &\le (l.u.b)^2_{Y_1} &&\equiv A_1 \\
\theta_3^2 + \theta_4^2 &\le (l.u.b)^2_{Y_2} &&\equiv A_2 \\
\theta_5^2 + \theta_6^2 &\le (l.u.b)^2_{Y_3} &&\equiv A_3 \\
&\vdots \\
\theta_{M-3}^2 + \theta_{M-2}^2 &\le (l.u.b)^2_{Y_{\frac{M-2}{2}}} &&\equiv A_{\frac{M-2}{2}} \\
\theta_{M-1}^2 + \theta_M^2 &\le (l.u.b)^2_{Y_{\frac{M}{2}}} &&\equiv A_{\frac{M}{2}} \\
\sqrt{\theta_1^2+\theta_2^2} + \sqrt{\theta_3^2+\theta_4^2} + \sqrt{\theta_5^2+\theta_6^2} + \cdots + \sqrt{\theta_{M-1}^2+\theta_M^2} &\le (l.u.b)_{Y_{\frac{M+2}{2}}} &&\equiv A_{\frac{M+2}{2}}
\end{aligned}
\qquad (13\text{-}A)
$$

where A_k, $k = 1, 2, 3, \cdots \frac{M-2}{2}, \frac{M}{2}, \frac{M+1}{2}$ is the constraint limit for the k-th constraint.

The constraint matrix form of θ, θ_ψ, defined by Equation (32) becomes

$$\theta_\psi = \underbrace{\left.\begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 & (\theta_1^2+\theta_2^2)^{1/4} \\ \theta_2 & 0 & 0 & & 0 & 0 & 0 \\ 0 & \theta_3 & 0 & & 0 & 0 & (\theta_3^2+\theta_4^2)^{1/4} \\ 0 & \theta_4 & 0 & & 0 & 0 & 0 \\ 0 & 0 & \theta_5 & & 0 & 0 & (\theta_5^2+\theta_6^2)^{1/4} \\ 0 & 0 & \theta_6 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \theta_{M-3} & 0 & (\theta_{M-3}^2+\theta_{M-2}^2)^{1/4} \\ \cdot & \cdot & \cdot & & \theta_{M-2} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 & \theta_{M-1} & (\theta_{M-1}^2+\theta_M^2)^{1/4} \\ \cdot & \cdot & \cdot & \cdots & 0 & \theta_M & 0 \end{bmatrix}\right\} M}_{\frac{M}{2}+1 \;=\; \frac{M+2}{2}} \qquad \text{Dimension } \left(M \times \frac{M+2}{2}\right) \qquad (32\text{-A})$$

A is the diagonal l.u.b. constraint limit matrix with dimension $\left(\frac{M+2}{2} \times \frac{M+2}{2}\right)$.
The diagonal elements are the A_n values in Equation (13-A).

Correspondingly, the constraint relations of interest, Relations (56), become

$$\boxed{\begin{array}{l} \left.\begin{array}{lll} \psi_{11}(\theta) & = & A_1 - \theta_1^2 - \theta_2^2 \geq 0 \\ \psi_{22}(\theta) & = & A_2 - \theta_3^2 - \theta_4^2 \geq 0 \\ \psi_{33}(\theta) & = & A_3 - \theta_5^2 - \theta_6^2 \geq 0 \\ & & \quad\vdots \\ \psi_{\frac{k}{2}\frac{k}{2}}(\theta) & = & A_{\frac{k}{2}} - \theta_{k-1}^2 - \theta_k^2 \geq 0 \\ & & \quad\vdots \\ \psi_{\frac{M}{2}\frac{M}{2}}(\theta) & = & A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 \geq 0 \end{array}\right\} \quad \text{for } k \in \{1,2,3,\cdots M\} \\ \psi_{\frac{M+2}{2}\frac{M+2}{2}}(\theta) = A_{\frac{M+2}{2}} - \sqrt{\theta_1^2+\theta_2^2} - \sqrt{\theta_3^2+\theta_4^2} - \sqrt{\theta_5^2+\theta_6^2} - \cdots\cdots - \sqrt{\theta_{M-1}^2+\theta_M^2} \geq 0 \end{array}} \qquad (56\text{-}A)$$

The basic performance index J and its analytic gradient $\text{Grad}_\theta J$, Equations (67), (68), and (69), remain unchanged.

The analytic gradient of the constraint function $\text{Grad}_\theta[\psi_c^T(\theta)]$, Equation (71), becomes

$$\text{Grad}_\theta[\psi_c^T(\theta)] = -2 \begin{bmatrix} \theta_1 & 0 & 0 & \cdots\cdots & 0 & 0 & \frac{1}{2}\theta_1(\theta_1^2+\theta_2^2)^{-1/2} \\ \theta_2 & 0 & 0 & & 0 & 0 & \frac{1}{2}\theta_2(\theta_1^2+\theta_2^2)^{-1/2} \\ 0 & \theta_3 & 0 & & 0 & 0 & \frac{1}{2}\theta_3(\theta_3^2+\theta_4^2)^{-1/2} \\ 0 & \theta_4 & 0 & & 0 & 0 & \frac{1}{2}\theta_4(\theta_3^2+\theta_4^2)^{-1/2} \\ 0 & 0 & \theta_5 & & 0 & 0 & \frac{1}{2}\theta_5(\theta_5^2+\theta_6^2)^{-1/2} \\ 0 & 0 & \theta_6 & \cdot & 0 & 0 & \frac{1}{2}\theta_6(\theta_5^2+\theta_6^2)^{-1/2} \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \theta_{M-3} & 0 & \frac{1}{2}\theta_{M-3}(\theta_{M-3}^2+\theta_{M-2}^2)^{-1/2} \\ \cdot & \cdot & \cdot & & \theta_{M-2} & 0 & \frac{1}{2}\theta_{M-2}(\theta_{M-3}^2+\theta_{M-2}^2)^{-1/2} \\ \cdot & \cdot & \cdot & \cdots\cdots & 0 & \theta_{M-1} & \frac{1}{2}\theta_{M-1}(\theta_{M-1}^2+\theta_M^2)^{-1/2} \\ \cdot & \cdot & \cdot & \cdots\cdots & 0 & \theta_M & \frac{1}{2}\theta_M(\theta_{M-1}^2+\theta_M^2)^{-1/2} \end{bmatrix} \qquad (71\text{-}A)$$

which has dimension $\left(M \times \frac{M+2}{2}\right)$

The equations which define the necessary conditions for optimality, Equations (50), (51), (49), and (99) through (104), become

$$\boxed{\begin{aligned} \text{Grad}_{\tilde{\Theta}} f[\tilde{\Theta},\lambda] = \left[\frac{\partial f}{\partial \tilde{\Theta}}\right]^T &= 0 \quad \text{with dimension } \left(\frac{3M+2}{2} \times 1\right) \\ \phi_c(\tilde{\Theta}) &= 0 \quad \text{with dimension } \left(\frac{M+2}{2} \times 1\right) \end{aligned}} \quad \begin{aligned} &(50\text{-}A) \\ &(51\text{-}A) \end{aligned}$$

where

$$f[\tilde{\Theta},\lambda] = J(\theta) + \lambda^T \phi_c(\tilde{\Theta}) \qquad (49\text{-}A)$$

$$\left.\begin{array}{l} \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \lambda_{\frac{M}{2}} \\ \lambda_{\frac{M+2}{2}} \end{bmatrix} \qquad \phi_c(\bullet) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \phi_{\frac{M}{2}} \\ \phi_{\frac{M+2}{2}} \end{bmatrix} \begin{array}{l} \text{each with} \\ \text{dimension } \left(\frac{M+2}{2} \times 1\right) \end{array} \\ \\ \text{and} \\ \\ \qquad \tilde{\Theta} = \begin{bmatrix} \theta \\ --- \\ \alpha \end{bmatrix} \quad \text{with dimension } \left(\frac{3M+2}{2} \times 1\right) \end{array}\right\} \quad (99\text{-}A)$$

$$\boxed{\phi_c(\underset{\sim}{\Theta}) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{\frac{M}{2}} \\ \phi_{\frac{M+2}{2}} \end{bmatrix} = \begin{bmatrix} A_1 - \theta_1^2 - \theta_2^2 - \alpha_1^2 \\ A_2 - \theta_3^2 - \theta_4^2 - \alpha_2^2 \\ A_3 - \theta_5^2 - \theta_6^2 - \alpha_3^2 \\ \vdots \\ A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 - \alpha_{\frac{M}{2}}^2 \\ A_{\frac{M+2}{2}} - \sqrt{\theta_1^2 + \theta_2^2} - \sqrt{\theta_3^2 - \theta_4^2} - \sqrt{\theta_5^2 - \theta_6^2} - \cdots\cdots - \sqrt{\theta_{M-1}^2 + \theta_M^2} - \alpha_{\frac{M+2}{2}}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} \quad \underline{\underline{\text{Constraint}}} \ \underline{\underline{\text{Equations}}}} \tag{100-A}$$

$$J[\underset{\sim}{\Theta}, \lambda] = [J(\theta)]_{Eq(67)} + \left(\lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3 + \cdots\cdots + \lambda_{\frac{M}{2}}\phi_{\frac{M}{2}} + \lambda_{\frac{M+2}{2}}\phi_{\frac{M+2}{2}}\right) \tag{101-A}$$

$$
J[\theta,\lambda] = \left.\begin{aligned}
&\tfrac{1}{2}C_{11}\theta_1^2 + C_{12}\theta_1\theta_2 + C_{13}\theta_1\theta_3 + C_{14}\theta_1\theta_4 + \cdots\cdots + C_{1M}\theta_1\theta_M \\
&\qquad + \tfrac{1}{2}C_{22}\,\theta_2^2 + C_{23}\,\theta_2\theta_3 + C_{24}\theta_2\theta_4 + \cdots\cdots + C_{2M}\theta_2\theta_M \\
&\qquad\qquad + \tfrac{1}{2}C_{33}\,\theta_3 + C_{34}\theta_3\theta_4 + \cdots\cdots + C_{3M}\theta_3\theta_M \\
&\qquad\qquad\qquad \ddots \qquad\qquad\qquad \vdots \\
&\qquad\qquad\qquad\qquad + C_{(M-1)M}\,\theta_{M-1}\theta_M \\
&\qquad\qquad\qquad\qquad + \tfrac{1}{2}C_{MM}\,\theta_M^2 \\
&+ 2(g_1\theta_1 + g_2\theta_2 + g_3\theta_3 + \cdots\cdots + g_M\theta_M) \\
&+ W_1 Z_{0_1}^2 + W_2 Z_{0_2}^2 + W_3 Z_{0_3}^2 + \cdots\cdots + W_N Z_{0_N}^2 \\
&+ \lambda_1(A_1 - \theta_1^2 - \theta_2^2 - \alpha_1^2) \\
&+ \lambda_2(A_2 - \theta_3^2 - \theta_4^2 - \alpha_2^2) \\
&+ \lambda_3(A_3 - \theta_5^2 - \theta_6^2 - \alpha_3^2) \\
&\qquad\quad \vdots \\
&+ \lambda_{\frac{M}{2}}(A_{\frac{M}{2}} - \theta_{M-1}^2 - \theta_M^2 - \alpha_{\frac{M}{2}}^2) \\
&+ \lambda_{\frac{M+2}{2}}\left(A_{\frac{M+2}{2}} - \sqrt{\theta_1^2+\theta_2^2} - \sqrt{\theta_3^2+\theta_4^2} - \sqrt{\theta_5^2+\theta_6^2} - \cdots\cdots - \sqrt{\theta_{M-1}^2+\theta_M^2} - \alpha_{\frac{M+2}{2}}^2\right)
\end{aligned}\right\} \qquad (102\text{-}A)
$$

$$\left.\begin{aligned}
\frac{\partial f}{\partial \theta_1} &= C_{11}\theta_1 + C_{12}\theta_2 + C_{13}\theta_3 + \cdots\cdots + C_{1M}\theta_M + 2g_1 - 2\lambda_1\theta_1 - \lambda_{\frac{M+2}{2}}\theta_1(\theta_1^2+\theta_2^2)^{-1/2} = 0\\
\frac{\partial f}{\partial \theta_2} &= C_{12}\theta_1 + C_{22}\theta_2 + C_{23}\theta_3 + \cdots\cdots + C_{2M}\theta_M + 2g_2 - 2\lambda_1\theta_2 - \lambda_{\frac{M+2}{2}}\theta_2(\theta_1^2+\theta_2^2)^{-1/2} = 0\\
\frac{\partial f}{\partial \theta_3} &= C_{13}\theta_1 + C_{23}\theta_2 + C_{33}\theta_3 + \cdots\cdots + C_{3M}\theta_M + 2g_3 - 2\lambda_2\theta_3 - \lambda_{\frac{M+2}{2}}\theta_3(\theta_3^2+\theta_4^2)^{-1/2} = 0\\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\frac{\partial f}{\partial \theta_M} &= C_{1M}\theta_1 + C_{2M}\theta_2 + C_{3M}\theta_3 + \cdots\cdots + C_{MM}\theta_M + 2g_M - 2\lambda_{\frac{M}{2}}\theta_M - \lambda_{\frac{M+2}{2}}\theta_M(\theta_{M-1}^2+\theta_M^2)^{-1/2} = 0
\end{aligned}\right\} \quad (103\text{-A})$$

$$\left.\begin{aligned}
\frac{\partial f}{\partial \alpha_1} &= -2\lambda_1\alpha_1 = 0\\
\frac{\partial f}{\partial \alpha_2} &= -2\lambda_2\alpha_2 = 0\\
\frac{\partial f}{\partial \alpha_3} &= -2\lambda_3\alpha_3 = 0\\
&\quad\vdots\\
\frac{\partial f}{\partial \alpha_{\frac{M}{2}}} &= -2\lambda_{\frac{M}{2}}\alpha_{\frac{M}{2}} = 0\\
\frac{\partial f}{\partial \alpha_{\frac{M+2}{2}}} &= -2\lambda_{\frac{M+2}{2}}\alpha_{\frac{M+2}{2}} = 0
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
\lambda_1\alpha_1 &= 0\\
\lambda_2\alpha_2 &= 0\\
\lambda_3\alpha_3 &= 0\\
\\
\lambda_{\frac{M}{2}}\alpha_{\frac{M}{2}} &= 0\\
\lambda_{\frac{M+2}{2}}\alpha_{\frac{M+2}{2}} &= 0
\end{aligned}\right\}
\quad \underline{\text{Switching Equations}} \qquad (104\text{-A})$$

where the constraint equations are defined by Equation (100-A).

<u>Necessary Conditions for Optimality</u>

Other equations are similarly changed.

It is emphasised that this augmented system of constraints is not a simple quadratic system of constraints, but rather a significantly non-linear system. Correspondingly, the simplifications and computational efficiencies obtained with the strictly quadratic system of constraints will in general not exist for this non-linear system. Specifically

1. The gradient of the constraint function system $\psi_c^T(\theta)$ is not linear.
2. The Hessian of the constraint function system $\psi_c^T(\theta)$ is not invariant.
3. The HVRP differs significantly from the quadratic programming problem and correspondingly, there is an increased possibility of difficulty in convergence.

A "production version" of the Stand-Alone Optimal Controller System was developed by adding this additional constraint as an option, and by removing those options providing solutions to the equations defining necessary conditions for optimality and those used to verify analytic gradients and Jacobians. Corresponding listings of definitions of the principal parameters in URAND including all the input parameters and a brief description of the principal routines comprising URAND is presented in Appendix G. Likewise, corresponding listings of the VAX/VMS Command File used to execute URAND and the URAND FORTRAN routines themselves are presented in Appendix H. Input to, and output from a sample case run by this production version is presented in Appendix I. Corresponding listings of definitions to the principal parameters in OPTIM including all the input parameters and a brief description of the principal routines comprising OPTIM is presented in Appendix J. Likewise, corresponding listings of the VAX/VMS Command File used to execute OPTIM and the OPTIM FORTRAN routines themselves are presented in Appendix K. Input to, and output from a sample case run by this production version is presented in Appendix L.

3.0 CONCLUSIONS

Three optimisation methods to solve the helicopter vibration control problem were identified and implemented. These methods attempt to determine the optimal control vector which minimises the vibration metric subject to constraints at discrete time points. These methods differ from the commonly used non-optimal constraint penalty methods such as those employed by conventional controllers (e.g., the Deterministic, Cautious, and Dual Controllers) in that the constraints are handled as actual constraints to an optimisation problem rather than as just additional terms in the performance index. The first method is to use a Non-linear Programming algorithm to solve the problem directly. The second method is to solve the full set of non-linear equations which define the necessary conditions for optimality. The third method is to solve each of the possible reduced sets of equations which define the necessary conditions for optimality when the constraints are preselected to be either active or inactive, and then to select the best solution.

Cases run to date indicate that the first method of solution (i.e., the direct optimisation of the control vector subject to constraints herein referred to as the "Optimal Controller") is reliable, robust, and easiest to use. The algorithm employed for direct optimisation is particularly suitable to this problem since although it is designed to solve the general non-linear programming problem, it employs a successive quadratic programming method to solve this more general problem. Since this method initially estimates the Hessian of the performance index and constraint functions and then updates it successively as the quadratic programming solutions are updated, and since the only difference between the problem of interest and the standard Quadratic Programming Problem is that the constraints are quadratic rather than linear, the Hessian is invariant to the optimisation process and is analytically known. Correspondingly, modifications to this successive quadratic programming method which will enhance its reliability, overall robustness, and speed, appear feasible for this particular problem of interest.

The second and third methods (i.e., solution of a full set or solution of reduced sets of non-linear equations which define the necessary conditions for optimality) successfully solved systems of non-linear equations defining the necessary conditions for optimality. It doesn't appear to be practical to use these methods by themselves, however, because there exist many (perhaps infinite) solutions to these equations and no real way to recognise the solution yielding the global minimum or to even guarantee convergence to this solution if it were known. The second method can, however, be used to verify that the necessary conditions are satisfied when the first method is employed. The use of the second method for verification after the first method obtains a solution was made an option in the stand-alone program.

The conventional controller was investigated to provide a convenient means of comparison. In addition, options to directly optimise the weighting coefficients of the conventional controller either in a specified ratio to one another (i.e., the "Conventional Controller" as referred to herein) or individually (i.e., the "Optimised Conventional Controller" as referred to herein) while satisfying the constraints was provided as a means to obtain a more representative and meaningful comparison of controllers and to help access the relative merit of solving the actual optimisation problem. Occasionally a mathematical conditioning problem occurs with the direct optimisation of the individual weighting coefficients of the conventional controller which causes a numerical overflow and subsequent error termination. This occurs when the "solution" has "zero" harmonic components whose associated weighting coefficients approach infinity as the optimisation process converges to the solution weighting coefficient vector.

Cases run to date indicate that the performance (i.e., the reduction of the vibration metric) of the Optimal Controller was superior to that of both the Conventional and the Optimised Conventional Controllers. In accordance with theory, the Optimal Controller yielded a zero vibration metric when no constraints are active for square non-singular T-Matrices (i.e., when the number of measurements equals the number of controls with no redundancy) or when the proportionality constant (C in Equation (8) and input parameter RATIO to the T-Matrix Generation System) is 100 per cent. As expected, the Optimised Conventional Controller was superior to the Conventional Controller in performance, but inferior to the Optimal Controller for all cases except a few degenerate cases where the performance of the three controllers was essentially equal. The performance gap was widest for those cases where the Optimal Controller yielded a zero vibration metric in accordance with theory.

4.0 REFERENCES

1. Leyland, Jane A.: Comparison of Three Controllers Applied to Helicopter Vibration, NASA Technical Memorandum 102192, May 1992

2. Davis, Mark W.: Refinement and Evaluation of Helicopter Real-Time Self-Adaptive Active Vibration Controller Algorithms, NASA CR-3821, 1984

3. Jacklin, Stephen A., "System Identification Techniques for Helicopter Higher Harmonic Control", 42nd Annual Forum of the American Helicopter Society, Washington, DC., 2-4 June 1986

4. Leyland, Jane A.: Kinematics and Constraints Associated with Swashplate Blade Pitch Control, NASA Technical Memorandum 102265, March 1993

5. IMSL MATH/LIBRARY User's Manual, FORTRAN Subroutines for Mathematical Applications, Version 1.1, Volume 1, MALB-USM-UNBND-EN8908-1.1, August 1989

6. IMSL MATH/LIBRARY User's Manual, FORTRAN Subroutines for Mathematical Applications, Version 1.1, Volume 2, MALB-USM-UNBND-EN8901-1.1, January 1989

7. IMSL MATH/LIBRARY User's Manual, FORTRAN Subroutines for Mathematical Applications, Version 1.1, Volume 3, MALB-USM-UNBND-EN8901-1.1, January 1989

8. IMSL STAT/LIBRARY User's Manual, FORTRAN Subroutines for Statistical Analysis, Version 1.1, STLB-USM-PERFECT-EN8901-1.1, January 1989

9. Gill, P. E., W. Murray, M. A. Saunders, and M. H. Wright, "Model Building and Practical Aspects of Non-Linear Programming", in Computational Mathematical Programming, (edited by K. Schittkowski), NATO ASI Series, 15, Springer-Verlag, Berlin, Germany, 1985

10. Powell, M. J. D., "A Fast Algorithm for Non-linearly Constrained Optimisation Calculations", in Numerical Analysis Proceedings, Dundee 1977

11. Powell, M. J. D., "A Fast Algorithm for Non-linearly Constrained Optimisation Calculations", in Lecture Notes in Economics and Mathematics, 630, Springer-Verlag, Berlin, Germany, pp 144-157, 1978

12. Schittkowski, K., "Non-linear Programming Codes", in Lecture Notes in Economics and Mathematics, 183, Springer-Verlag, Berlin, Germany, 1980

13. Schittkowski, K., "On the Convergence of a Sequential Quadratic Programming Method with an Augmented Lagrangian Line Search Function", in Mathematik Operationsforschung und Statistik, Serie Optimization, 14, pp 197-216, 1983

14. Schittkowski, K., "NLPQL: A FORTRAN Subroutine Solving Constrained Non-linear Programming Problems", (edited by Clyde L. Monna), in Annals of Operations Research, 5, pp 485-500, 1986

15. Stoer, J., "Principles of Sequential Quadratic Programming Methods for Solving Non-linear Problems", in Computational Mathematical Programming, (Edited by K. Schittkowski), NATO ASI Series, 15, Springer-Verlag, Berlin, Germany, 1985

16. More, Jorge J., Garbow, Burton S., and Hillstrom, Kenneth E., "Documentation for MINPACK Subroutine HYBRD1", in User Guide for MINPACK-1, Argonne National Laboratory Report ANL-80-74, Argonne, Illinois, pp 51-56, March 1980

17. More, Jorge J., Garbow, Burton S., and Hillstrom, Kenneth E., "Documentation for MINPACK Subroutine HYBRD", in User Guide for MINPACK-1, Argonne National Laboratory Report ANL-80-74, Argonne, Illinois, pp 57-64, March 1980

18 Powell, M. J. D., "A Hybrid Method for Non-linear Equations", in Numerical Methods for Non-linear Algebraic Equations, P. Rabinowitz, editor, Gordon and Breach, 1970

19 Johnson, Wayne, "CAMRAD/JA, A Comprehensive Analytical Model of Rotorcraft Aerodynamics and Dynamics", Johnson Aeronautics, Palo Alto, CA, 1988

T-Matrix Dimension	Constraint l.μ.b.⊥ (Degrees)	Constraint Activity	Resulting Performance Index		
			Optimal Controller	Conventional Controller	Optimised Conventional Controller
6×6	2.000	1	6.010	11.589	7.712
6×6	120.000	none	0	11.589	7.712
5×6	2.000	2	1.456	9.918	6.464
5×6	120.000	none	0	9.918	6.464
4×6	2.000	none	0	1.865	0.075
3×6	2.000	none	0	2.354	0.731
2×6	2.000	none	0	0.014	0
1×6	2.000	none	0	0	0
0×6	2.000	none	No Change	Abort	Abort
24×12*	80.000	none	0	31.559	5.478
20×12*	80.000	none	0	28.373	4.686
24×2**	80.000	none	0	6.099	0

Notes:

1. The proportionality constant C in Equation (8), input as RATIO to program URAND for T-Matrix generation, is 1.000 for full 100% row duplication.

2. The reference T-Matrix dimension, from which T-Matrices of smaller dimensions listed in this table were derived, is (24×6) unless otherwise noted. The Row Duplication Pointer-Vector IROW(•) input to program URAND for T-Matrix generation is

   IROW = 1, 1, 1, 1, 1, 1, ¦ 1, 2, 3, 4, 5, 6, ¦ 1, 2, 3, 4, 5, 6, ¦ 1, 2, 3, 4, 5, 6,

* The reference T-Matrix has dimension (24×12).

** The reference T-Matrix has dimension (24×2).

Table 1, Partial List of Cases Run During Verification

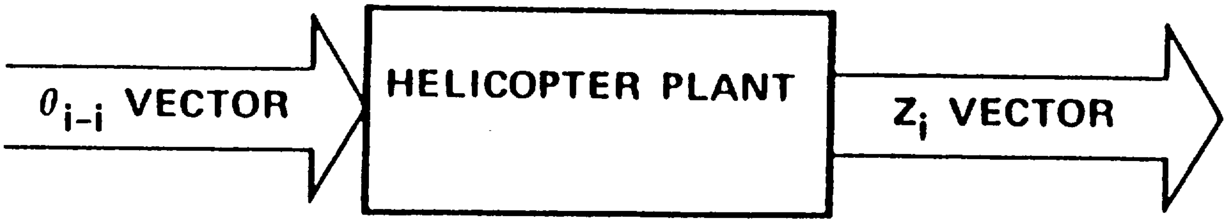


Figure 1. Controlled vibration response.

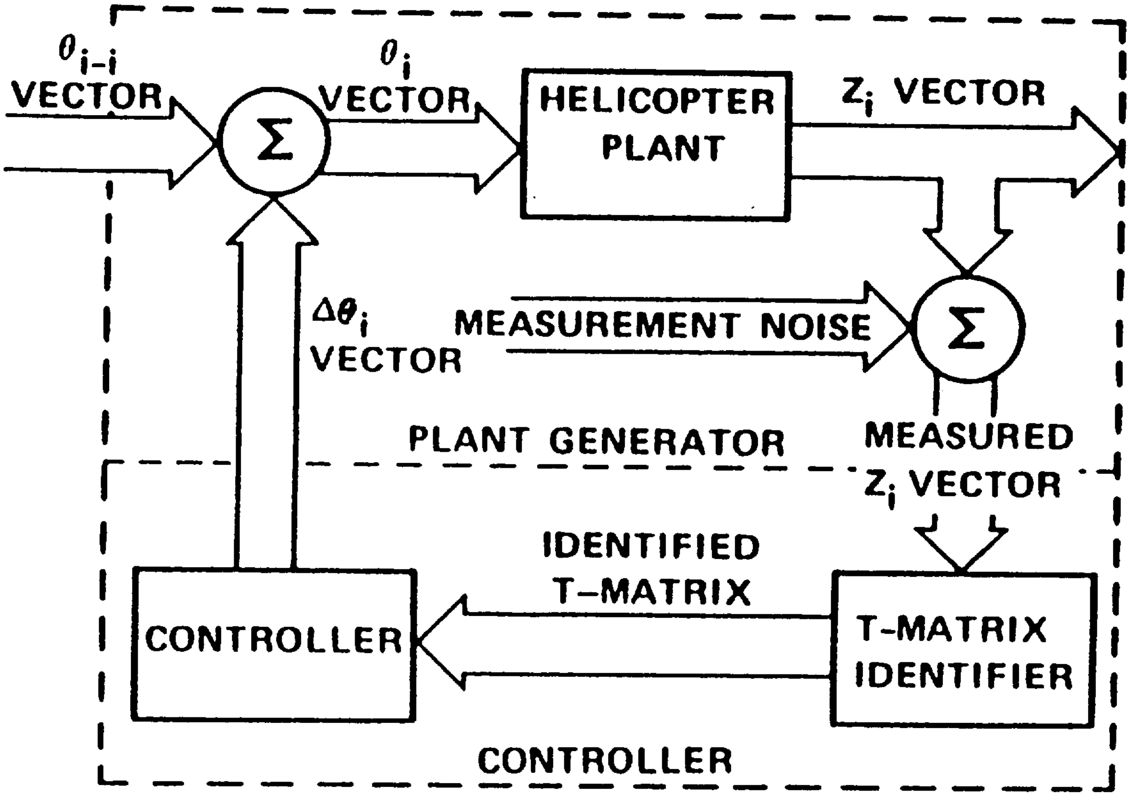
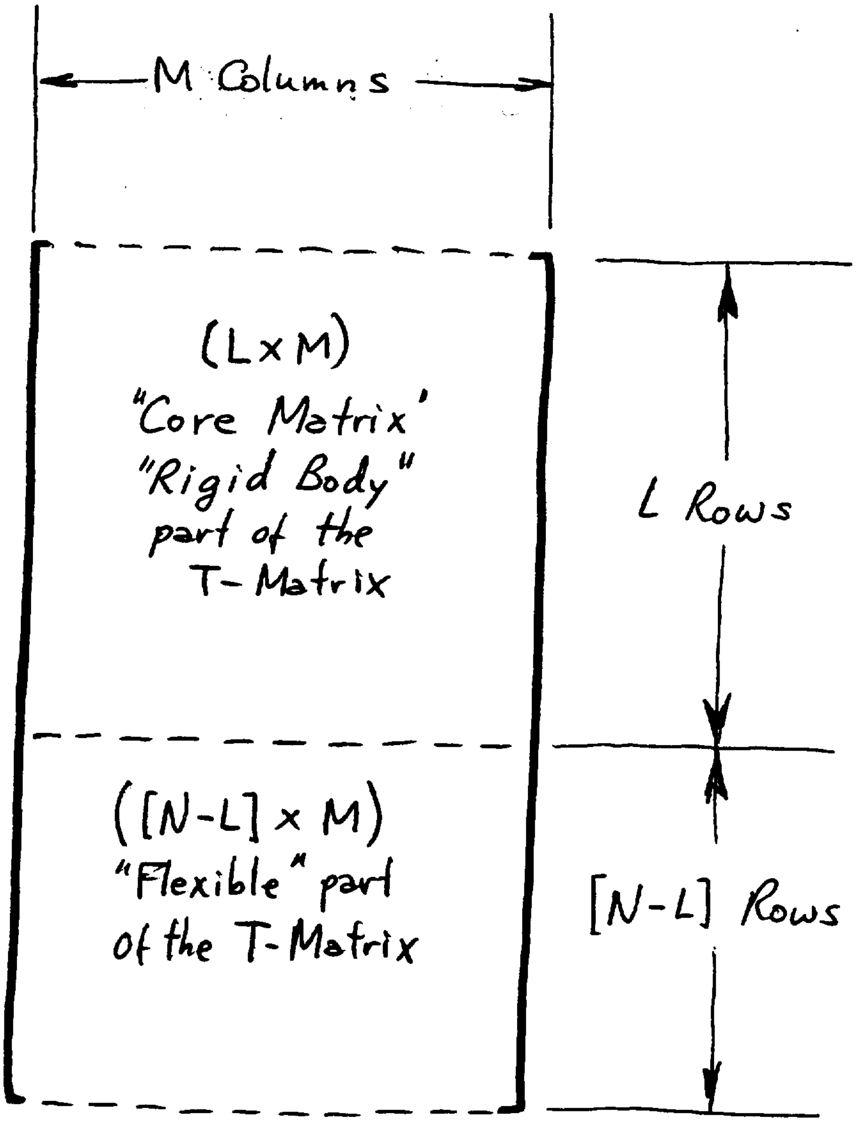


Figure 2. General scheme for the closed-loop vibration control system.



where

L	dimension of the "reference base set of sensors", the "base measurement dimension", the number rows in the "Core Matrix"
N	total number of sensors, number of rows in the T-Matrix, dimension of the Z- vector
M	the number of control variables to be optimised, dimension of the θ-vector (Control Vector), number of columns in the T- Matrix

Figure 3, General (N x M) T-MATRIX

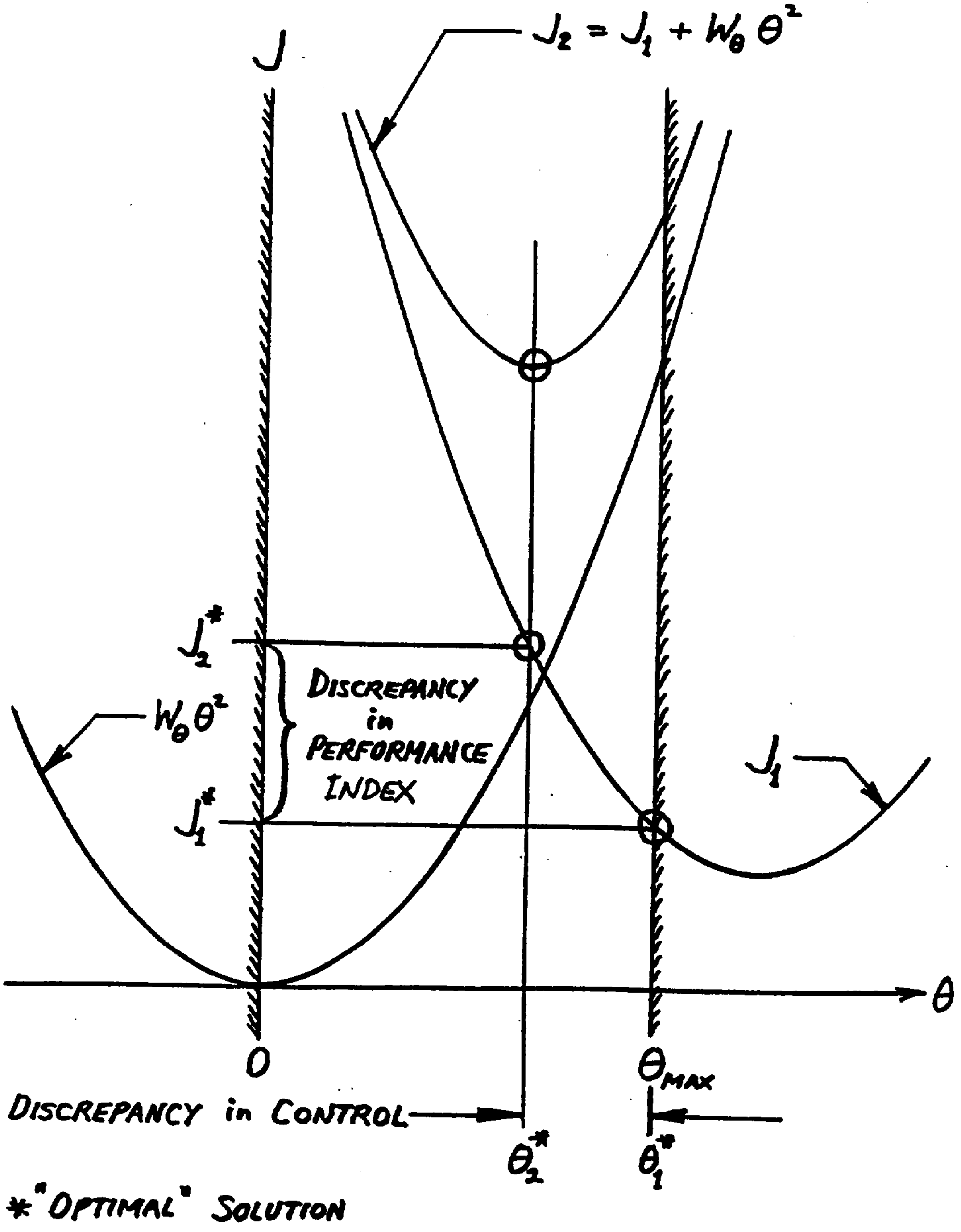


Figure 4 SCALAR EXAMPLE

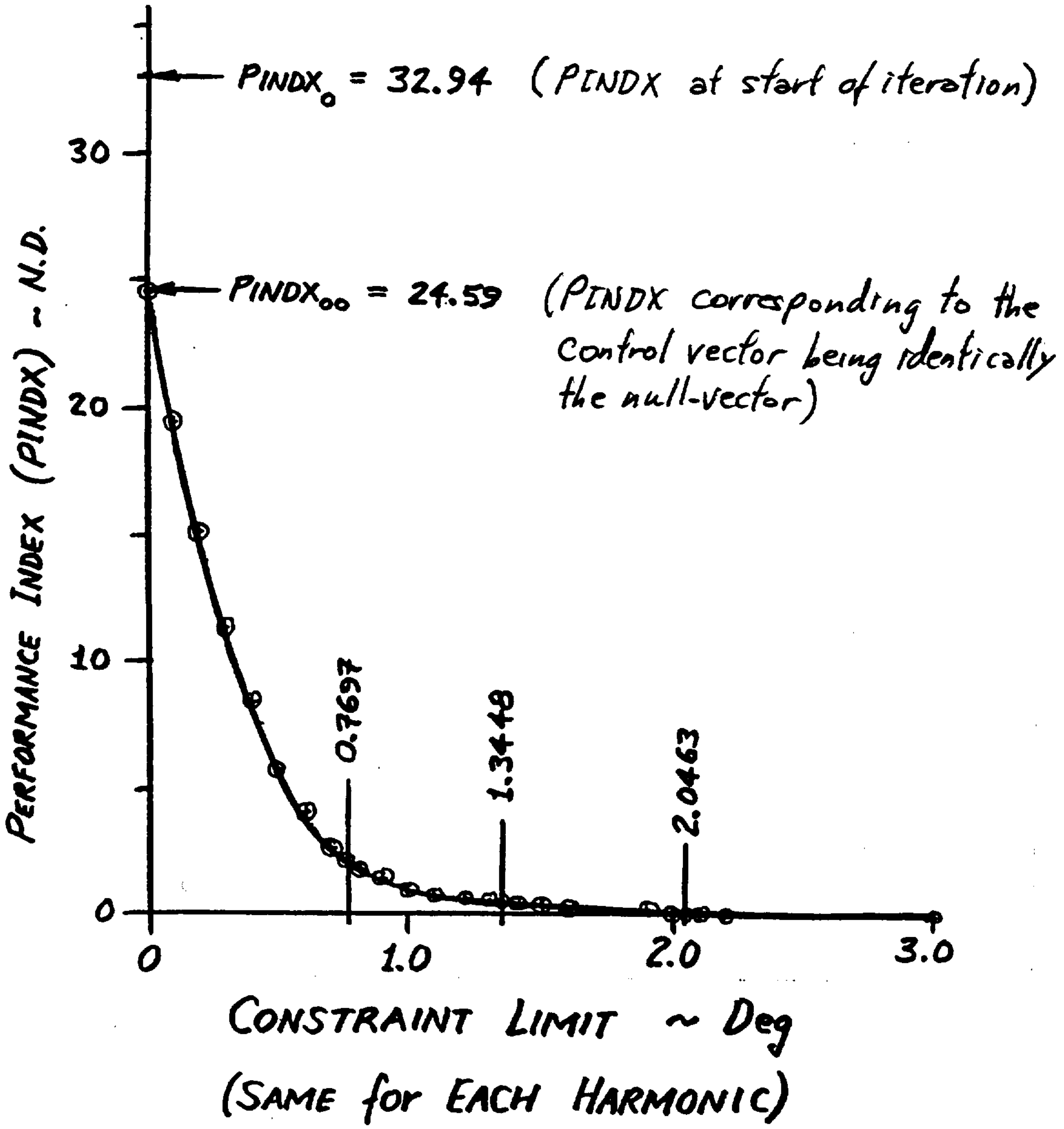


Figure 5 Performance Index Reduction

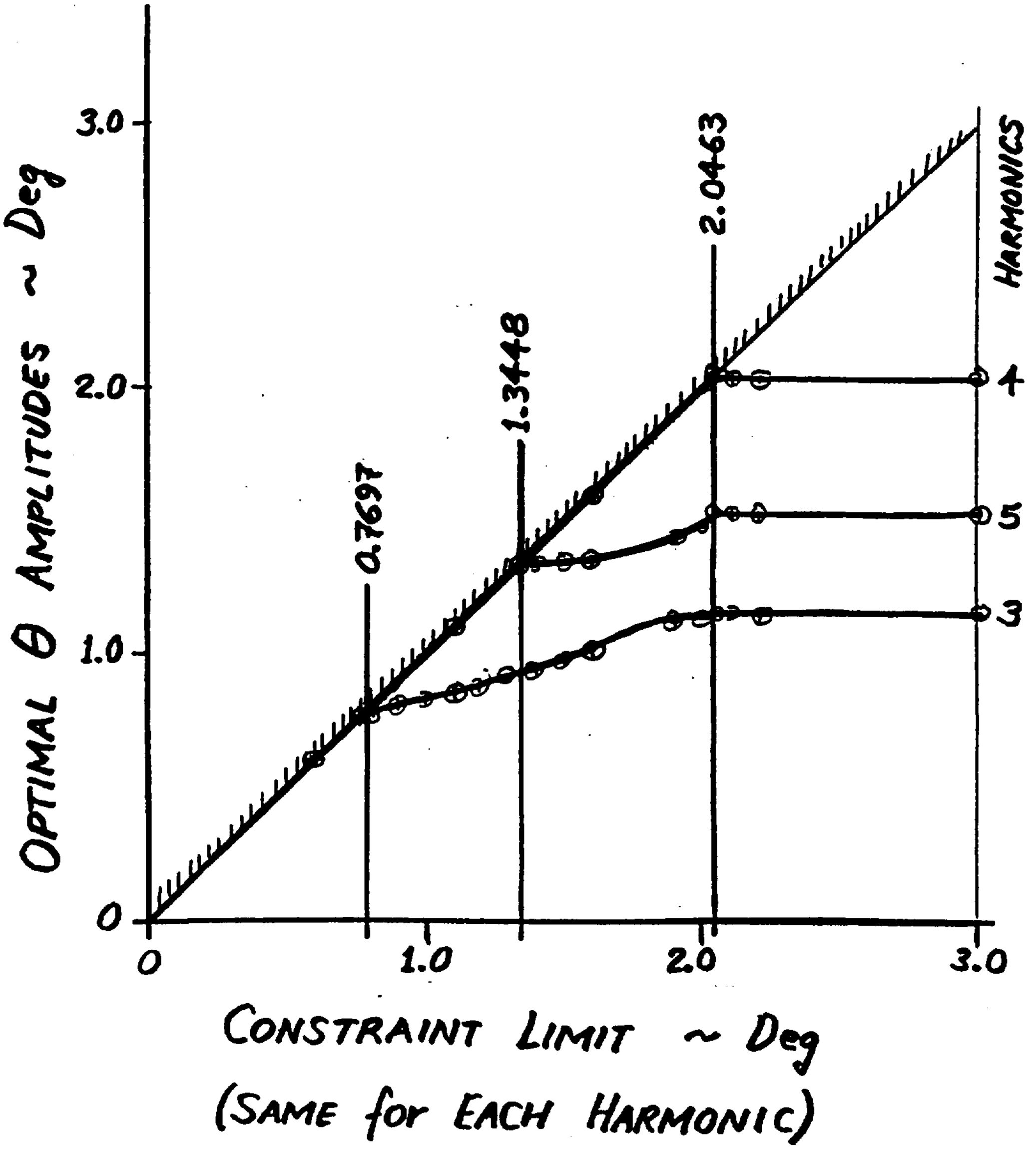


Figure 6 Optimum Θ Magnitude

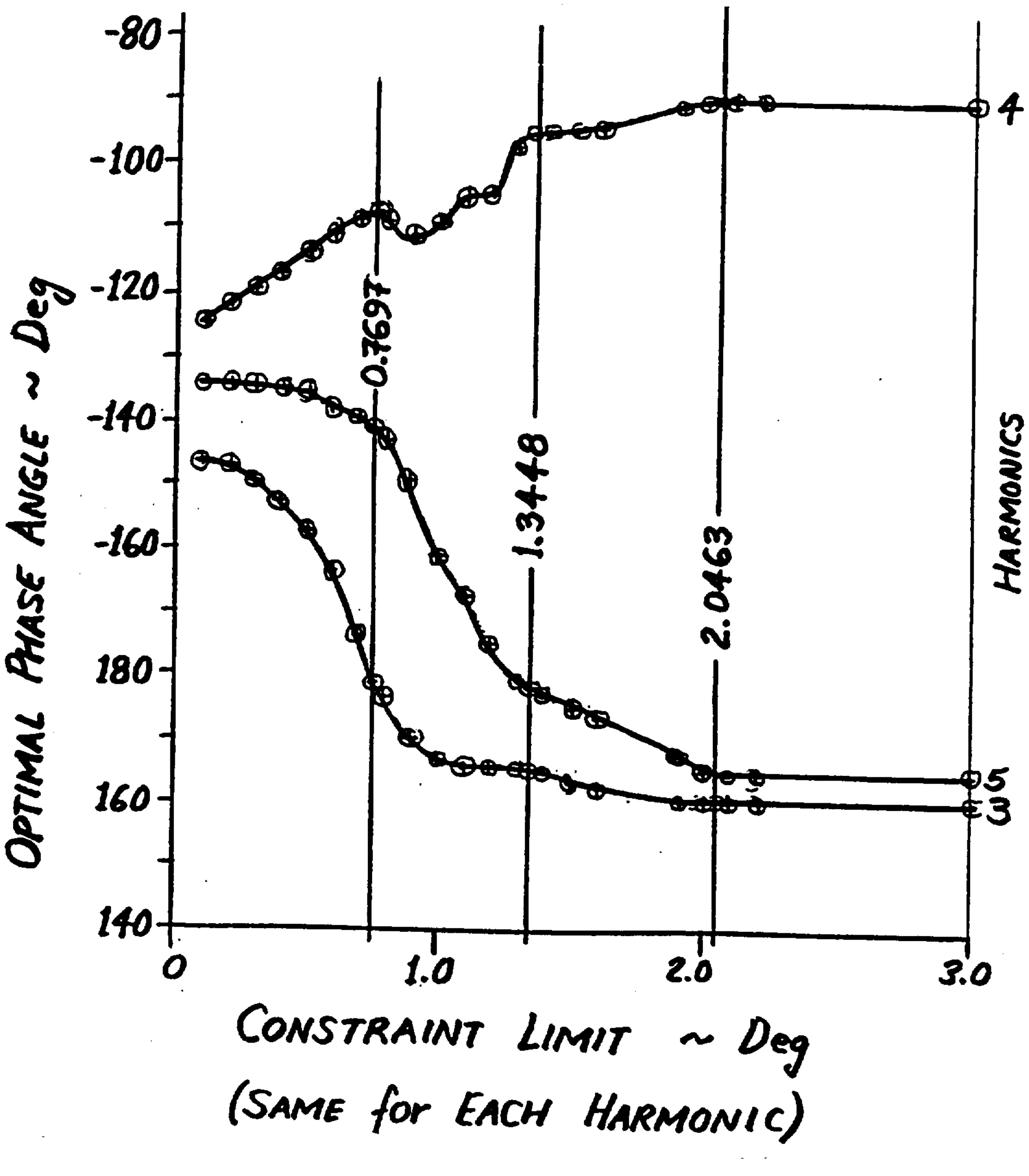


Figure 7 Optimum Θ Phase Angle

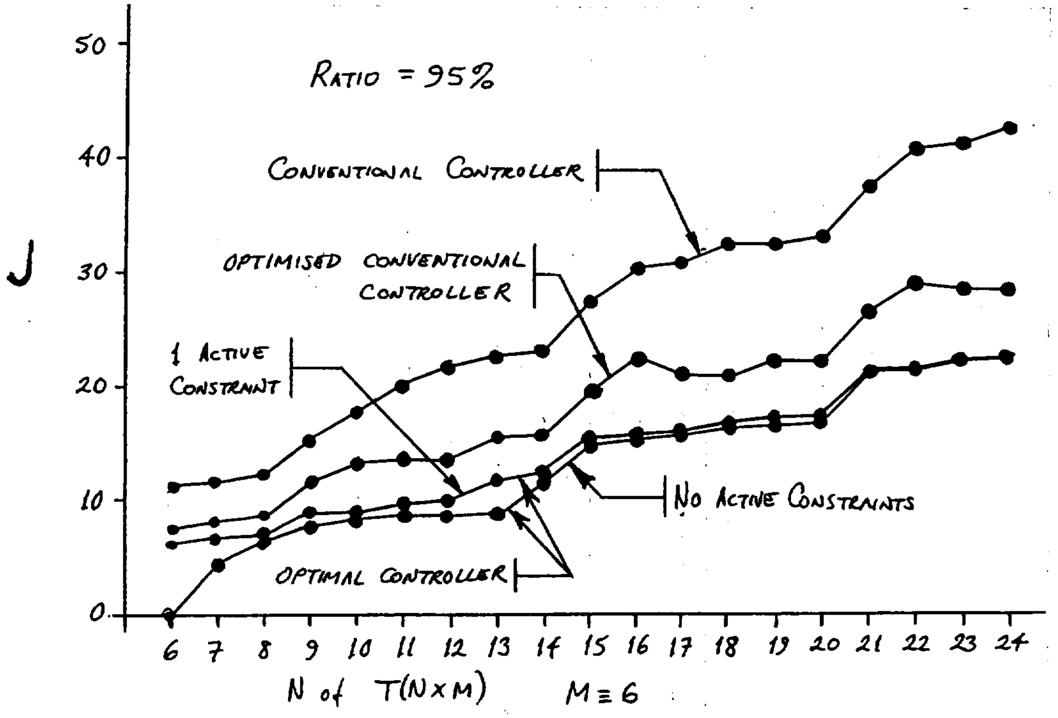


Figure 8 Performance Index (J) vs. Number of Measurements (N) for 95% Row Duplication

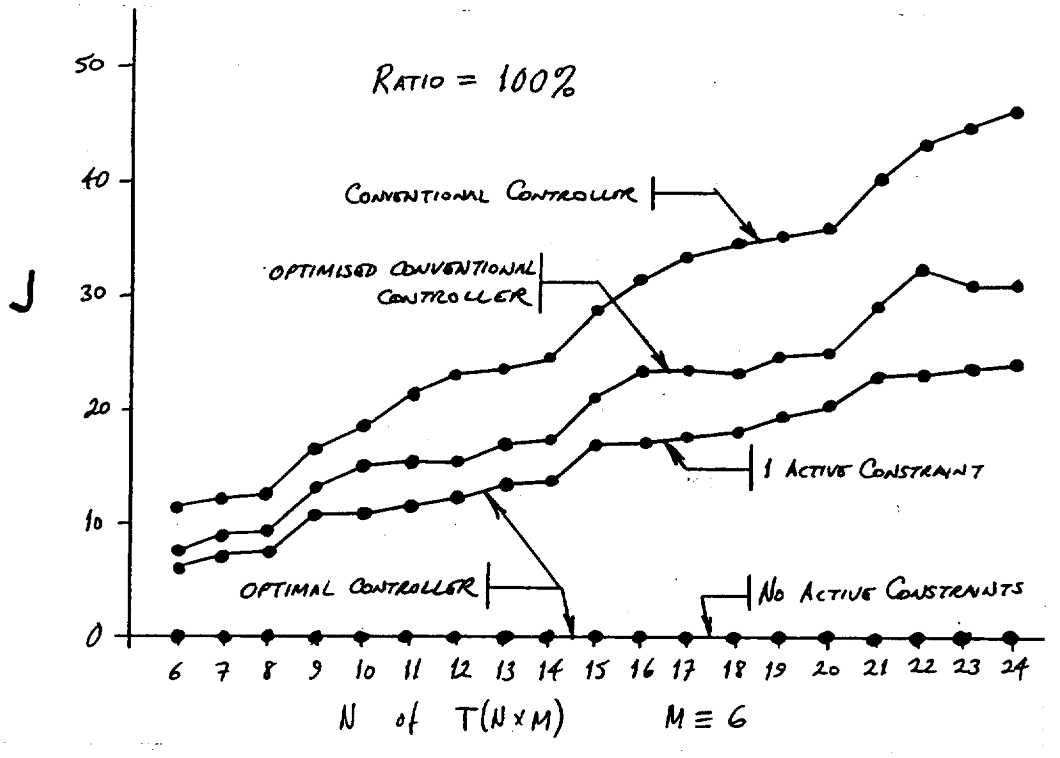


Figure 9 Performance Index (J) vs. Number of Measurements (N) for 100% Row Duplication

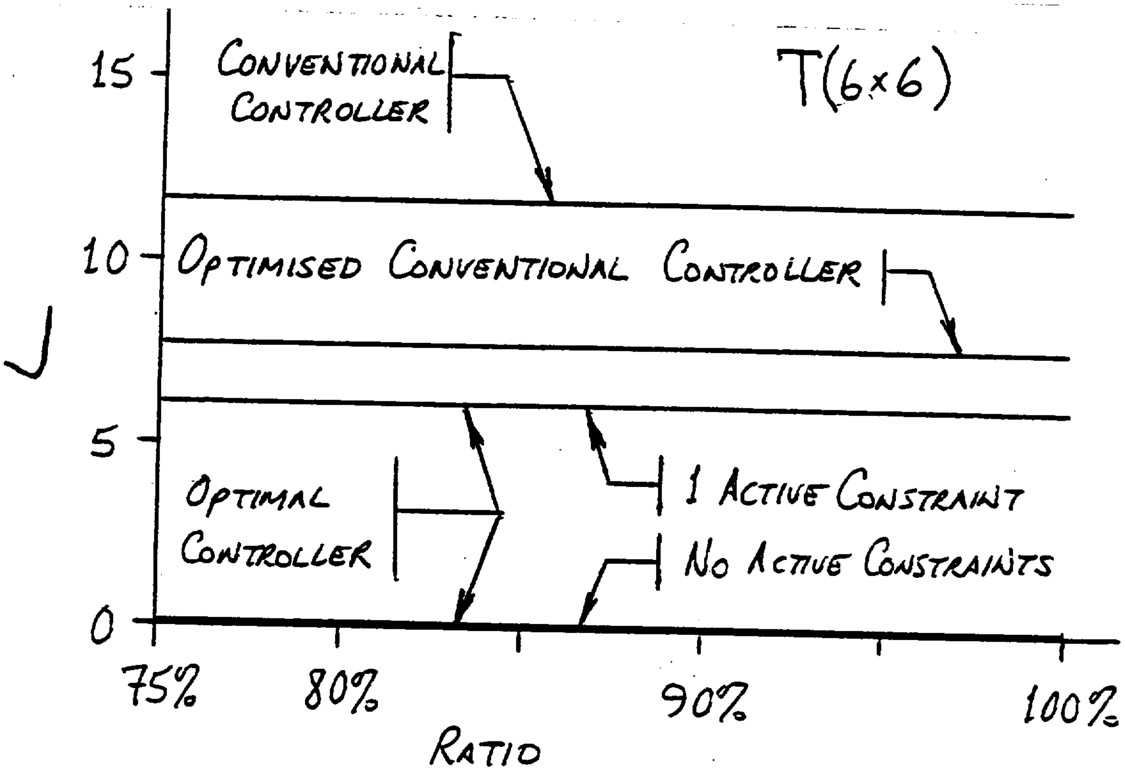


Figure 10 Controller Performance for T(6×6)

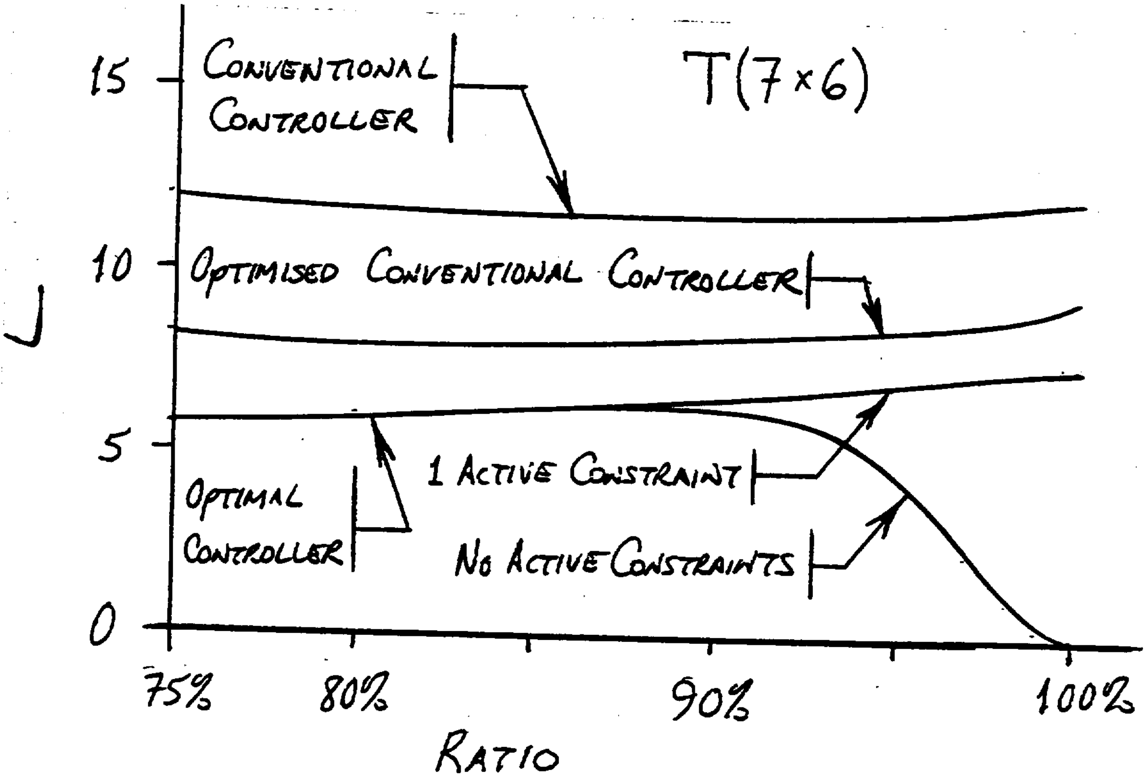


Figure 11 Controller Performance for T(7×6)

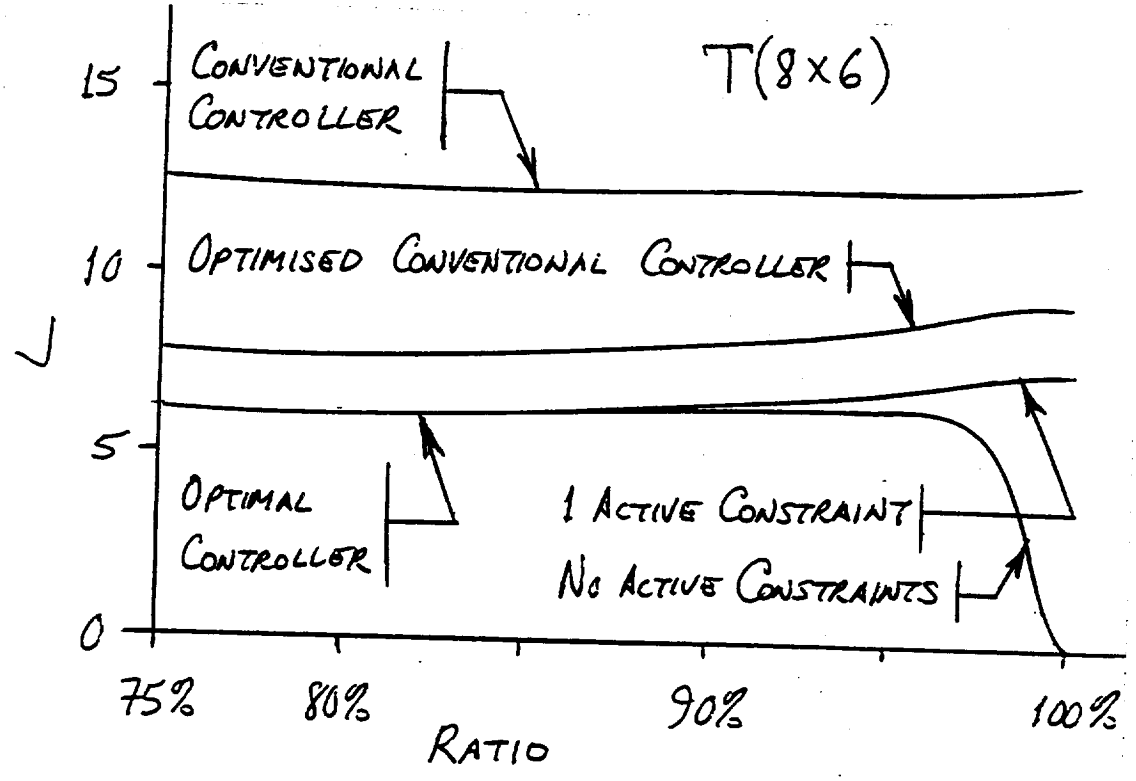


Figure 12 Controller Performance for T(8x6)

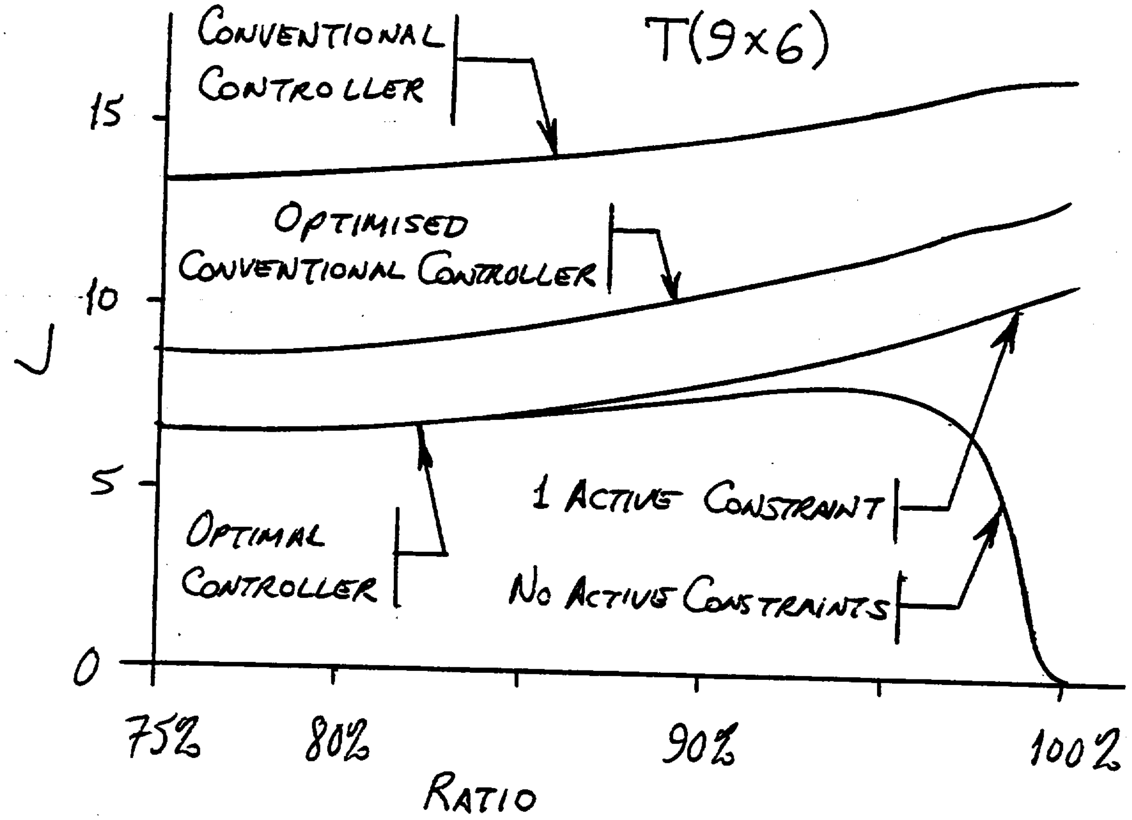


Figure 13 Controller Performance for T(9x6)

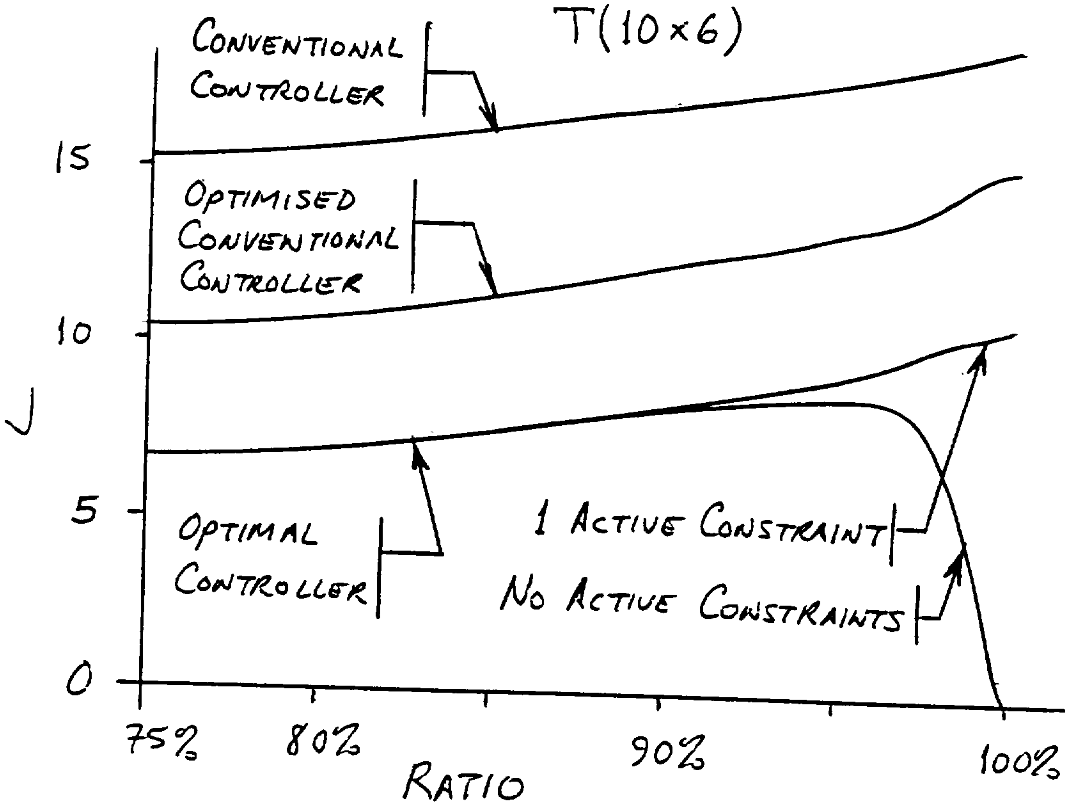


Figure 14 Controller Performance for T(10×6)

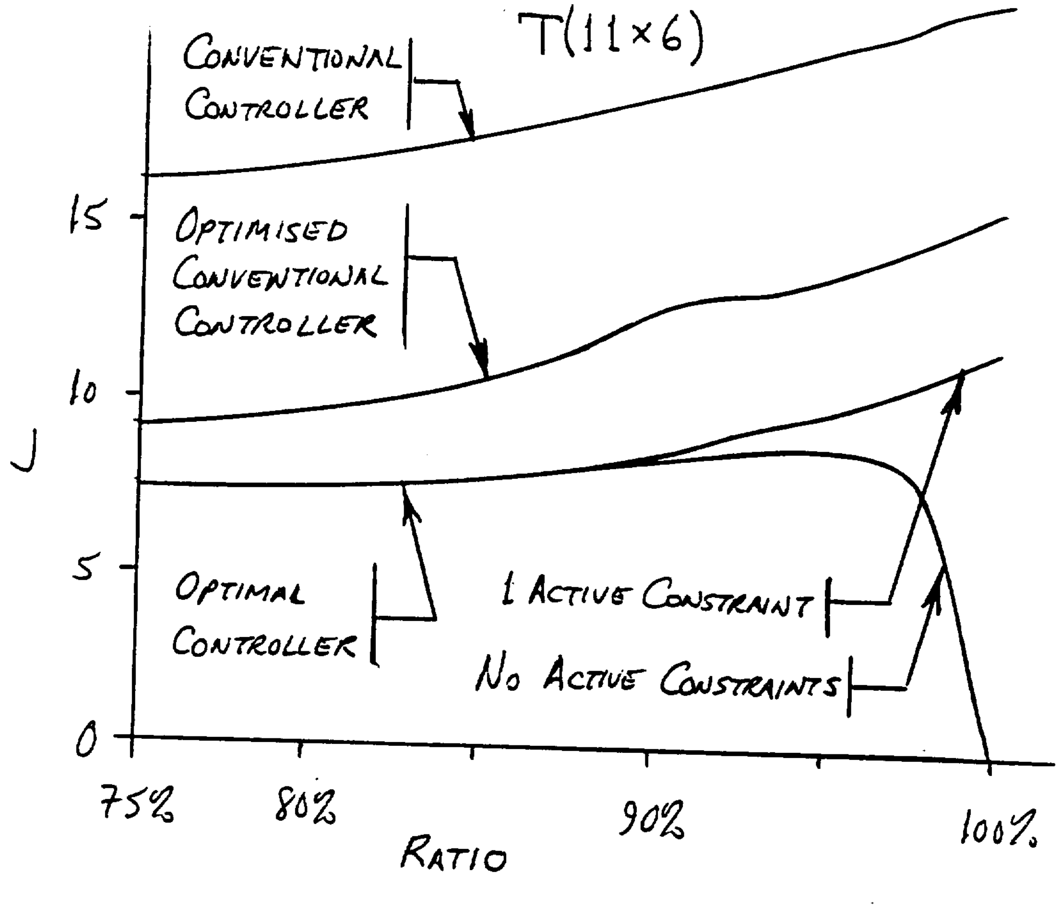


Figure 15 Controller Performance for T(11×6)

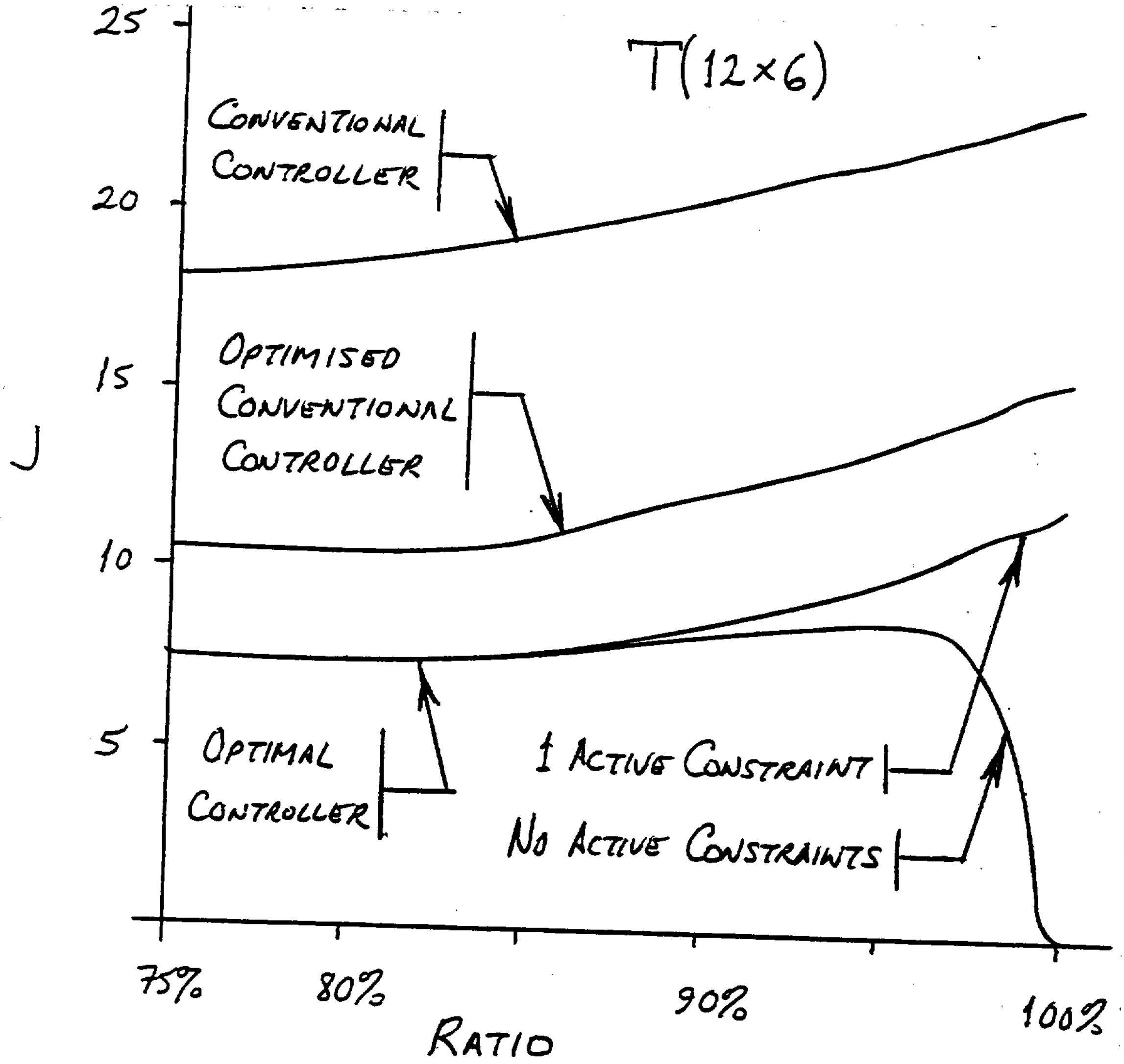


Figure 16 Controller Performance for T(12×6)

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13. ABSTRACT ***(Maximum 200 words)***

Three methods to optimise rotorcraft aeromechanical behaviour for those cases where the rotorcraft plant can be adequately represented by a linear model system matrix were identified and implemented in a stand-alone code. These methods determine the optimal control vector which minimises the vibration metric subject to constraints at discrete time points, and differ from the commonly used non-optimal constraint penalty methods such as those employed by conventional controllers in that the constraints are handled as actual constraints to an optimisation problem rather than as just additional terms in the performance index. The first method is to use a Non-linear Programming algorithm to solve the problem directly. The second method is to solve the full set of non-linear equations which define the necessary conditions for optimality. The third method is to solve each of the possible reduced sets of equations defining the necessary conditions for optimality when the constraints are pre-selected to be either active or inactive, and then to simply select the best solution. The effects of manoeuvres and aeroelasticity on the systems matrix are modelled by using a pseudo-random pseudo-row-dependency scheme to define the systems matrix. Cases run to date indicate that the first method of solution is reliable, robust, and easiest to use, and that it was superior to the conventional controllers which were considered.

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