# Predicting the Stability of a Compressible Periodic Parallel Jet Flow 

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#### Abstract

It is known that mixing enhancement in compressible free shear layer flows with high convective Mach numbers is difficult. One design strategy to get around this is to use multiple nozzles. Extrapolating this design concept in a one dimenional manner, one arrives at an array of parallel rectangular nozzles where the smaller dimension is $w$ and the longer dimension $b$ is taken to be infinite. In this paper, the feasibility of predicting the stability of this type of compressible periodic parallel jet flow is discussed. The problem is treated using Floquet-Bloch theory. Numerical solutions to this eigenvalue problem are presented. For the case presented, the interjet spacing, $s$, was selected so that $s / w=2.23$. Typical plots of the eigenvalue and stability curves are presented. Results obtained for a range of convective Mach numbers from 3 to 5 show growth rates $\omega_{i}=\frac{k c_{i}}{2}$ range from 0.25 to 0.29 . These results indicate that coherent two-dimensional structures can occur without difficulty in multiple parallel periodic jet nozzles and that shear layer mixing should occur with this type of nozzle design.


## 1 Introduction

Interest in proving the economic and environmental feasibility of a high-speed civil transport has stimulated again studies in supersonic nozzle design. The primary process which must be understood and controlled to meet design objectives involves supersonic mixing. Improving the efficiency of supersonic combustors is another area where mixing enhancement at supersonic speeds must be understood and controlled to meet design objectives.

A reduction in mixing and a growth rates of supersonic shear layers with increasing Mach number has been demonstrated experimentally by many investiga-
tors (Refs. [1], [2], [3] ). In addition, linear stability analysis shows results that are similar to the experimental studies (Refs. [4], [5], [6] ). However, these linear stability investigations have been of unbounded flows that are uniform at infinity. No attention has been given to supersonic flows periodic in the direction normal to the stream. In this paper, the feasibility of predicting the stability of a compressible periodic parallel jet flow is discussed. While this problem has not been dealt with analytically, one design strategy to get around the decrease in mixing with increasing convective Mach number is to use multiple nozzles. Recent flow studies using multiple rectangular nozzles are presented in Refs. [7] and [8]. Extrapolating this design concept in a one dimensional manner, one arrives at an array of parallel rectangular nozzles where the smaller dimension is $w$ and the longer dimension $b$ is taken to be infinite. In addition, one starting point for a computational fluid dynamic study of multiple nozzles might be a study of a compressible periodic parallel jet flow. Consequently, the work discussed might be of interest to both the experimental and computational nozzle design community. Furthermore, this type of study might also be useful in understanding screech from multiple nozzles since it provides information on the flow disturbances leaving the nozzle.

In this paper the linear stability of a spatially periodic supersonic jet flow having velocity profile characterized by an abrupt rise and a sharp fall is discussed. In contrast to the uniform velocity distribution of free flows such as jets, wakes, and free shear layers in the far field, the periodic flow repeats exactly its velocity profile endlessly. The periodic flow is, therefore, expected to have different stability characteristics than those of free flows. In particular, the variation growth rates with Mach number needs to be examined. In this paper, the
inter jet spacing, $s$, was selected so that $s / w=2.23$ and the ratio of the periodic nozzle wavelength, $\lambda=s+w$, to $w$ was 3.23. Note that $w$ is the smaller dimension of the rectangular nozzle. Results obtained for a range of convective Mach numbers from 3 to 5 will be presented.

## 2 Formulation of the problem

Let $(U(y), 0,0)$ be the velocity of a steady plane-parallel flow, where the $x$-axis is in the direction of the flow and

$$
U(y)=\bar{U}+\frac{\Delta U}{2} h(y)
$$

where $U_{1}$ is the velocity outside the jet, $U_{2}$ is the mean centerline jet velocity, $\bar{U}=\frac{U_{1}+U_{2}}{2}, \Delta U=U_{2}-U_{1}$, and $h(y)$ is the velocity profile function which varies from -1 to 1 .

The flow field is perturbed by introducing wave disturbances in the velocity and pressure with amplitudes that are a function of $\hat{y}$. Thus,

$$
\begin{aligned}
& (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) \\
= & (\breve{u}(\hat{y}), \breve{v}(\hat{y}), \breve{w}(\hat{y}), \breve{p}(\hat{y})) \exp [i(\hat{k} \hat{x}+\hat{l} \hat{z}-\hat{\omega} \tau)] .
\end{aligned}
$$

Where

$$
\begin{aligned}
\hat{k} & =k L^{*} \\
\hat{l} & =\ell L^{*} \\
\hat{\omega} & =\frac{\omega L^{*}}{\Delta U} \\
\frac{\hat{\omega}}{\hat{k}} & =\frac{\omega}{k \Delta U}=\frac{c}{\Delta U}=\check{c}
\end{aligned}
$$

and we define $\hat{c}$ as follows

$$
\check{c}=\frac{c}{\Delta U}=\frac{\bar{U}}{\Delta U}+\frac{\hat{c}}{2}
$$

By definition $\hat{k}$ is real positive number that represents the wavenumber in the $x$-direction, $\hat{\ell}$ is the wavenumber in the $z$-direction, $\hat{c}_{r}$ is the relative phase velocity, and $\omega_{i}=\frac{k \varepsilon_{i}}{2}$ is the amplification rate of the disturbance.

From the equations of motion if nonlinear and viscous terms are neglected one can obtain an equation for the $y$-component of the perturbation velocity as follows:

$$
\begin{aligned}
& \breve{v}^{\prime \prime}-\breve{v}^{\prime}\left(\frac{\hat{T}^{\prime}}{\hat{T}}+\frac{A^{\prime}}{A}\right) \\
& (2.1)-\breve{v}\left[\frac{h^{\prime \prime}}{(h-\hat{c})}+A i \hat{k}-\left(\frac{\hat{T}^{\prime}}{\hat{T}}+\frac{A^{\prime}}{A}\right) \frac{h^{\prime}}{(h-\hat{c})}\right]=0
\end{aligned}
$$

where the primes denote differentiation with respect to $\hat{y}$

$$
\begin{gathered}
A=-i \hat{k}-i \frac{\hat{l}^{2}}{\hat{k}}+m^{2} i \hat{k} \frac{(h-\hat{c})^{2}}{4} \\
A^{\prime}=2 m^{2} i \hat{k} \frac{(h-\hat{c}) h^{\prime}}{4} \\
m^{2}=\frac{m_{1}^{2}}{\hat{T}}
\end{gathered}
$$

and from Crocco's Equation [9]

$$
\begin{aligned}
\hat{T}(y)= & \frac{T(y)}{T_{1}} \\
= & \frac{T_{2}}{T_{1}}+\frac{(1+h(y))}{2}\left(1-\frac{T_{2}}{T_{1}}\right) \\
& -\frac{1}{2}\left(m_{1}\right)^{2}(\gamma-1) \frac{(h(y)+1)(h(y)-1)}{4}
\end{aligned}
$$

where

$$
m_{1}=\frac{\Delta U}{a_{1}}=\frac{\Delta U}{a_{2}} \frac{\sqrt{T_{2}}}{\sqrt{T_{1}}}=m_{2} \frac{\sqrt{T_{2}}}{\sqrt{T_{1}}}
$$

In this paper, the velocity profile function, $h(y)$, is periodic such that

$$
h(y+2 \pi)=h(y)
$$

The velocity profile $h(y)$ is not any exact solution of the Navier-Stokes equation, but it can be considered as a simple model of some real periodic flow.

The velocity profile $h(y)$ discussed herein is given by

$$
h(y)=1-2 f(y)
$$

where the function $f(y)$ is given by

$$
\begin{gathered}
f(y)=\frac{1}{\left[1+\left(\sinh \left(\frac{\eta}{\sinh (1)}\right)\right)^{18}\right]} \\
\eta=\Lambda\left(-1+\frac{y}{\pi}\right)
\end{gathered}
$$

$\Lambda=1.5$ and $y$ goes from 0.0 to $2 \pi$. The profile function $f(y)$ is adapted from an equation used by Monkewitz in Ref. ([10]) in a study of the absolute and convective instability of two-dimensional wakes. Only two-dimensional disturbances will be considered. A schematic of the nozzle geometry is shown in Figure 1. The velocity profile is shown in Figure 2.

## 3 Floquet-Bloch theory

Since the basic flow velocity profile, $f(y)$, is periodic, equation (2.1) is an example of a Floquet-Bloch problem. The mathematics of solving Floquet-Bloch type problems is discussed in Refs. [11], [12], and [13]. Applications to solid state physics are discussed in Refs. [14], [15], and [16]. Applications to spatially periodic flow is discussed in Refs. [17], [18], [19], [20], and [21].

The paper by Beaumont (1981) Ref. [19] and the description of the Floquet-Bloch theorem by Hochstadt (1963) in Ref.[12] were particularly useful in guiding this research.

A survey of the spatially periodic flow literature is presented by K. Gotoh and M.Y. Yamada in Ref. [22] .

The second order differential equation can be described by a system of first order differential equations. Let

$$
\begin{aligned}
\breve{v} & =x_{1} \\
\breve{v}^{\prime} & =x_{2}
\end{aligned}
$$

so that Eq. 2.1 can be rewritten as the system

$$
X^{\prime}=\left(\begin{array}{ll}
0 & 1  \tag{3.2}\\
D & C
\end{array}\right) X
$$

where

$$
C=\left(\frac{\hat{T}^{\prime}}{\hat{T}}+\frac{A^{\prime}}{A}\right)
$$

and

$$
D=\left[\frac{h^{\prime \prime}}{(h-\hat{c})}+A i \hat{k}-\left(\frac{\hat{T}^{\prime}}{\hat{T}}+\frac{A^{\prime}}{A}\right) \frac{h^{\prime}}{(h-\hat{c})}\right]
$$

If $\boldsymbol{\Phi}(y)$ is a fundamental matrix solution of equation ( (3.2)) such that

$$
\Phi(0)=1
$$

where $I$ is the identity matrix, then from the Floquet-Bloch theorem

$$
\Phi(y+2 \pi)=\Phi(y) \Phi(2 \pi)
$$

We now introduce two solutions of equation (3.2) with initial values at $y=0.0$

$$
\Phi(0)=\left[\begin{array}{ll}
\phi_{1}(0) & \phi_{2}(0) \\
\phi_{1}^{\prime}(0) & \phi_{2}^{\prime}(0)
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

Next we seek the eigenvalues of $\boldsymbol{\Phi}(2 \pi)$

$$
\begin{aligned}
& |\Phi(2 \pi)-\mu \mathbf{I}| \\
= & \left|\begin{array}{cc}
\phi_{1}(2 \pi)-\mu & \phi_{2}(2 \pi) \\
\phi_{1}^{\prime}(2 \pi) & \phi_{2}^{\prime}(2 \pi)-\mu
\end{array}\right| \\
= & \mu^{2}-\left(\phi_{1}(2 \pi)+\phi_{2}^{\prime}(2 \pi)\right) \mu \\
& +\left(\phi_{1}(2 \pi) \phi_{2}^{\prime}(2 \pi)-\right. \\
& \left.\phi_{2}(2 \pi) \phi_{1}^{\prime}(2 \pi)\right) \\
= & \mu^{2}-\left(\phi_{1}(2 \pi)+\phi_{2}^{\prime}(2 \pi)\right) \mu+1
\end{aligned}
$$

Since

$$
\phi_{1}(2 \pi) \phi_{2}^{\prime}(2 \pi)-\phi_{2}(2 \pi) \phi_{1}^{\prime}(2 \pi)=|\Phi(2 \pi)|=|\Phi(0)|=1
$$

The independent solutions of equation (3.2) have the form

$$
\phi=X(y) \exp \left(\frac{\log (\mu)}{2 \pi} y\right)=X(y) \exp (\Gamma y)
$$

The parameter $\Gamma$ specifies the period of the eigenfunction $\phi$. If $\Gamma$ is real the eigenfunction grows or decays at infinity. Consequently, only imaginary values of $\Gamma$ are acceptable. Thus the eigenfunction oscillates in space and is called a continuous mode. The disturbance with $\Gamma_{i}=1 / n$, where $n$ is a nonzero integer, has a period $2 n \pi$. One with $\Gamma_{i}=0$ has the same period $2 \pi$ as the main flow, while an irrational value of $\Gamma_{i}$ means the disturbance is aperiodic. Note that the parameter $\Gamma$ does not appear in the flow equation, but is due to the Floquet-Bloch theorem.

Solutions of 3.2 are thus of the form

$$
\begin{aligned}
X_{1}(y+2 \pi) & =\mu_{1} X_{1}(y) \\
X_{2}(y+2 \pi) & =\mu_{2} X_{2}(y)
\end{aligned}
$$

where $\mu_{1}$ and $\mu_{2}$ represent the zeros of (3.3), provided they are distinct.

In general, these solutions will not be periodic.
Conditions for periodic solutions can be found as follows

Let $\mu_{1}=e^{i \theta_{i}}$ and $\mu_{2}=e^{-i \theta_{i}}$.
Then from equation (3.3)

$$
\cos \left(\theta_{i}\right)=\phi_{1}(2 \pi)+\phi_{2}^{\prime}(2 \pi)=\delta / 2
$$

Consequently, for a solution to be periodic $\delta$ must be real and $|\delta|$ smaller than 2.

The constants $\mu$ are termed the characteristic multipliers of the Floquet-Bloch system (3.2) and the corresponding characteristic exponents are determined by the relation $\Gamma=\Gamma_{r}+i \Gamma_{i}=\frac{\log (\mu)}{2 \pi}=\frac{\theta_{r}}{2 \pi}+i \frac{\theta_{i}}{2 \pi}$.

Table I Eigenvalues at maximum growth rate

| $\mathrm{m}_{2}$ | $k$ | $\omega_{i}=\frac{k c_{i}}{2}$ | $\Gamma_{i}$ | $c_{r}$ |
| :---: | :---: | :---: | :---: | :---: |
| 3 | 2.855 | 0.2855 | $2.158 \mathrm{e}-01$ | -0.2195 |
| 3.2 | 2.735 | 0.2735 | $1.359 \mathrm{e}-01$ | -0.2005 |
| 3.4 | 2.495 | 0.2495 | $9.450 \mathrm{e}-02$ | 0.4963 |
| 4. | 2.470 | 0.2470 | 7.632e-02 | 0.5262 |
| 5 | 2.675 | 0.2675 | $4.010 \mathrm{e}-03$ | 0.6367 |

Table II Cray YMP CPU
computation time, hrs. ( $\Lambda=1.5 s / h=2.23$ )

| $m_{2}$ | $c_{i}=0.1$ | $c_{i}=0.2$ | $c_{i}=0.3$ | Total |
| :---: | :--- | :--- | :--- | :--- |
| 3 | 20.863 | 12.768 | 3.049 | 36.68 |
| 3.2 | 20.654 | 11.697 | 6.913 | 39.265 |
| 3.4 | 19.236 | 12.724 | 5.571 | 37.531 |
| 4 | 26.147 | 11.847 | 6.525 | 44.519 |
| 5 | 25.555 | 13.17 | 5.212 | 43.937 |
| Total |  |  |  | 201.932 |

## 4 Numerical calculations

In obtaining the results presented here equation 3.2 was integrated from $y=0$ to $y=2 \pi$ using a standard Runge-Kutta procedure (IMSL math library routine IVPRK) for fixed values of $\hat{c}_{i}=0.1,0.2$, and 0.3 for $m_{2}$ values of $3,3.2,3.4,4$ and 5.

To investigate stability, iteration was used to vary the real value of the phase velocity, $\hat{c}_{r}$. At each iteration, the eigenvalues, $\mu$, of the matrix $\Phi(2 \pi)$ were then found and the corresponding characteristic multipliers $\Gamma$ calculated. The iteration was successful if $\delta_{i}=0$ so that $\Gamma_{r}=0$. Also at each iteration a check was made that $\operatorname{det} \Phi(2 \pi)=1$.

Typical eigenvalue plots of the phase velocity, $\hat{c}_{r}$ verses growth rate $\omega_{i}$ and stability plots of the characteristic multipliers $\Gamma_{i}$ verses $\omega_{i}$ are presented in Figures 3-8.

## 5 Results

Results for $m_{2}=3$ are given for $\hat{c}_{i}=0.1,0.2$, and 0.3 in Figures 3-5. The (a) part of the figure shows eigenvalue plots of the phase velocity, $\hat{c}_{r}$ verses growth rate $\omega_{i}$ while the (b) part of the figure shows stability plots of the characteristic multipliers $\Gamma_{i}$ verses $\omega_{i}$. Results using the same format are presented for $m_{2}=4.0$ in Figures 6-8.

Table I shows the values at the growth rate maximurns for the cases investigated. Results obtained for a range of convective Mach numbers from 3 to 5 show growth rates $\omega_{i}=\frac{k c_{i}}{2}$ range from 0.2 to 0.3 .

Table II. shows the CRAY YMP CPU computational time needed to search the phase speed, $c$,
wavenumber $k$ space in order to see if solutions exist. The wave number spacing used was 0.005 . The total computation time was slightly more than 200 hours. The results were slowly accumulated over a period of seven months.

## 6 Conclusions

A numerical method for performing the stability analysis of a compressible periodic parallel jet flow has been developed and tested. Obtaining a set of solutions is a tedious expensive process. However, the resulting solutions provide detailed information on the growth, phase speed and periodicity of small disturbances propagating with the flow near the nozzle exit.

These results indicate that coherent structures can occur without difficulty in multiple parallel periodic jet nozzles and that shear layer mixing should occur with this type of nozzle design. Using these results, it should be possible to determine which nozzle design yields solutions with high growth rates and to pick out likely screech frequencies due to feedback from shock cells to the nozzle.

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Figure 1.—Nozzle configuration.


Figure 2.-Velocity profile ( $\Lambda=1.5$ ).


Figure 3.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{j} / 2\left(c_{i}=0.1, m_{2}=3, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.1, m_{2}=3, s / w=2.23\right)$.


Figure 4.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.2, m_{2}=3, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.2, m_{2}=3, s / w=2.23\right)$.


Figure 5.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.3, m_{2}=3, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.3, m_{2}=3, s / w=2.23\right)$.


Figure 6.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.1, m_{2}=4, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.1, m_{2}=4, s / w=2.23\right)$.



Figure 7.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.2, m_{2}=4, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.2, m_{2}=4, s / w=2.23\right)$.


Figure 8.-(a) Eigenvalue $c_{r}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.3, m_{2}=4, s / w=2.23\right)$. (b) $\Gamma_{i}$ verses growth rate $\omega_{i}=k c_{i} / 2\left(c_{i}=0.3, m_{2}=4, s / w=2.23\right)$.


