HOMOTOPY SOLUTIONS OF KEPLER'S EQUATIONS

/ 3 / 553

Norman Fitz-Coy† and Jiann-Woei Jang★
AeMES Department
University of Florida
Gainesville, FL

Abstract Kepler's Equation is solved using an integrative algorithm developed using homotopy theory. The solution approach is applicable to both elliptic and hyperbolic forms of Kepler's Equation. The results from the proposed algorithm compare quite favorable with those from existing iterative schemes.

I Introduction

The problem of relating orbital position and time in the two-body problem requires the solution of Kepler's Equation, i.e.,

$$M = E - e\sin E \tag{1}$$

where E is the eccentric anomaly, and M is the mean anomaly for the elliptic orbits. The applications of Kepler's Equation can be classified into two categories [1],

(1) given a probe's position in the orbit, determine the time when the probe is at that position

or

(2) given the time since last periapsis passage, determine a probe's position in the orbit at that time.

The first case is rather trivial. By substituting the current eccentric anomaly E into Eq. (1), one can immediately determine the mean anomaly M, and hence t. The second case, however, is more complex than the first one, since Kepler's Equation has no closed-form solution for the eccentric anomaly E. In the literature, most solution approaches for E use iterative methods [1-5]. Recently, a non-iterative method was proposed in Ref. [6]

In this paper, an alternate solution approach based on homotopy theory is presented. In section II, a preliminary introduction to homotopy theory is given. The homotopy algorithm for the elliptic form of Kepler's Equation is developed in sections III and the algorithm for the hyperbolic form is discussed in section IV. A summary and conclusion is presented in section V. It is not our intent to claim that our approach is superior (or inferior) to the existing iterative approaches, but rather to demonstrate that alternative approaches do exist. However, in order to demonstrate that our approach is indeed viable, we do make comparisons with some existing iterative methods [1-5]. These comparison results are included in sections III and IV.

II Homotopy Theory

Mathematical Model

The homotopy theory introduced in this paper is based primarily on the research work of Zangwill and Garcia [7]. Given a system of nonlinear equations to be solved, the homotopy algorithm starts with a simple solution (a solution to a similar but easily solved system of linear/non-nonlinear equations) and through a series of integrations, reaches the exact solution of the original system of nonlinear equations.

Let $\Re^n$ be an Euclidean n space. Given $f(x) : \Re^n \rightarrow \Re^n$; i.e., $x \in \Re^n$, we want to find the solution, $x^* = [x_1^*, \ ..., \ x_n^*]^T$, of

† Assistant Professor, Member: AAS, AIAA
★ Post Doc., Student Member: AIAA

$$f(x) = 0 \tag{2}$$

The homotopy procedure for solving Eq. (2) can be stated as follows:

(1) Identify an easily solved system of equations, say

$$g(x) = 0 \tag{3}$$

where $g(x)$: $\Re^n \to \Re^n$. Determine the solution x^0 of Eq. (3).

(2) Define a homotopy function $H(x, \lambda)$, such that

$$H(x, 0) = g(x) \tag{4}$$

$$H(x, 1) = f(x) \tag{5}$$

(3) Generate a path (from $\lambda = 0$ to $\lambda = 1$) which leads the solution from x^0 to x^*.

Homotopy Functions

Although there exist numerous forms of the homotopy function, the following three are the most commonly used forms:

(1) Newton homotopy

$$H(x, t) = f(x) - (1 - \lambda)f(x^0) \tag{6}$$

(2) Fix-point homotopy

$$H(x, t) = (1 - \lambda)(x - x^0) + tf(x) \tag{7}$$

(3) Linear homotopy

$$\begin{aligned} H(x, t) &= \lambda f(x) + (1 - \lambda)g(x) \\ &= g(x) + \lambda(f(x) - g(x)) = 0 \end{aligned} \tag{8}$$

The homotopy paths start from an arbitrary point, x^0, for either the Newton homotopy function or the fix-point homotopy function; hence, they are the most popular forms of the homotopy function currently in use. However, if an easily solved system of equations can be identified, and it is very "close" to the original system of equations, the linear homotopy function might be a better mathematical model to use. However, the above homotopy forms can be converted to each other; for example, $f(x) - f(x^0)$ in Eq. (6) and $x - x^0$ in Eq. (7) are equivalent to $g(x)$ in Eq. (8).

Path Following Algorithm and the Homotopy Differential Equation

The differentiation of any homotopy function with respect to the homotopy variable, λ, is 0, since a homotopy function is defined to be zero in (x, λ); i.e.,

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial x}\frac{\partial x}{\partial \lambda} = 0 \tag{9}$$

Equation (9) is referred to as the Davidenko differential equation in the literature [8]. Manipulation of Eq. (9) yields

$$\frac{dx}{d\lambda} = -\left(\frac{\partial H}{\partial x}\right)^{-1}\frac{\partial H}{\partial \lambda} \tag{10}$$

which, along with the condition $x(0) = x^0$, defines an initial value problem for the path from x^0 to x^*. Thus, integrating Eq. (10) over λ from 0 to 1 yields the desired solution of Eq. (2). Let $x \in D \subset \Re^n$, and $T = \{\lambda \mid 0 \le \lambda \le 1\}$. In order to guarantee the path existence, the regularity of H or the inversion of the Jacobian matrix, $\frac{\partial H}{\partial x}$, in Eq. (10) should always hold for all (x, λ) in H^{-1}, where H^{-1} is defined as the set of all the solutions of $H(x, \lambda)$; i.e.,

$$H^{-1} = \{(x, \lambda) \in D \times \Lambda \mid H(x, \lambda) = 0\} \tag{11}$$

Path existence for homotopy methods are guaranteed by Sard's Theorem for almost all cases. The path existence fails only for points which are in a set of Lebesgue measure zero.

Theorem 1 Sard's Theorem (Zangwill [7], Ch. 22)

Let $H : \mathcal{H} \subset \Re^q \to \Re^n$, where $\mathcal{H}$ is the closure of an open set and H be C^k. If $K \ge 1 + \max\{0, q - n\}$, then H is regular for almost all ε, except for ε on a set of Lebesgue measure zero, where

$$H(\cdot) = \varepsilon \qquad \square$$

Corollary 1 Extended Sard's Theorem (Zangwill [7], Ch. 22)

Let $D \subset \Re^n$ be the closure of an open set, $H : D \times \Lambda \rightarrow \Re^n$ be $\mathcal{C}^2$, and $F : D \rightarrow \Re^n$ be $\mathcal{C}^1$, then the following three statements are equivalent,

(1) H will be regular for almost all ε.

(2) H will be regular at $\bar{\lambda}$ for almost all ε.

(3) F will be regular for almost all ε. $\square$

For a homotopy method, $q = n + 1$, hence if k is greater than or equal to 2, then $H(x, t)$ is regular for almost all ε. In other words, as long as $H(x, t) \subset \mathcal{C}^2$, Sard's theorem states that for an arbitrary ε, $H(x, t)$ is almost assuredly regular. It can be shown that the second derivative of Kepler's Equation is continuous. This implies that even if the integration path bifurcates during the solution of Kepler's Equation via the homotopy method, a slight perturbation of the stating point of the homotopy equation guarantees that a new integration path exists.

III Kepler's Equation in Elliptic Orbits

Homotopy Function for Kepler's Equation: Elliptic Orbits

Homotopy methods has been used to solve a system of nonlinear equations [7][8]. To apply the homotopy method, first rewrite Kepler's Equation (Eq. (1)) as

$$f(E) = E - e \sin E - M = 0 \tag{12}$$

An easily solved equation is chosen as

$$g(E) = E - M_0 = 0 \tag{13}$$

where

$$M_0 = M + \frac{e \sin M}{1 - sin(M + e) + \sin M}$$

The linear homotopy function is

$$\begin{aligned} H(E,\lambda) &= g(E) + \lambda(f(E) - g(E)) \\ &= E - M_0 + \lambda(M_0 - M - e\ sin(E)) = 0 \end{aligned} \tag{14}$$

Differentiating the above equation with respect to the homotopy variable, λ yields

$$(1 - \lambda e \cos E)\frac{dE}{d\lambda} = e \sin E \tag{15}$$

or

$$\frac{dE}{d\lambda} = \frac{e \sin E}{1 - \lambda e \cos E} \tag{16}$$

Notice that the denominator in Eq. (16) is nonzero since $\lambda e \cos E < 1$ for all E. Hence the the homotopy integration path can never bifurcate.

An algorithm for the solution of Kepler's Equation using homotopy methods is as follows:

Step 1. Identify an appropriate function $g(E)$ (e.g., Eq. (13)) and develop the homotopy function $H(E, \lambda)$ (e.g., Eq. (14)).

Step 2. Form the initial value problem; i.e., differentiate Eq. (14) to obtain Eq. (16) and solve Eq. (13) for the initial condition.

Step 3. Numerically integrate the initial value problem from $\lambda = 0$ to $\lambda = 1$.

Step 4. Stop.

Numerical Results

In order to demonstrate the viability of the homotopy approach, Kepler's Equation was solved over a range of M and e values using the proposed approach and also using the Newton-Raphson method presented in Ref. [1]. The mean anomaly, M, was varied from 0 to 2π in increments of $\pi/250$ and the eccentricity, e, was varied from 0 to 1 in increments of 0.02. The numerical integration was accomplished using MATLAB's ode45 function with $tol = 5 \times 10^{-7}$, $h = hmax = 1$ and $hmin = 1/20000$. The terminal condition for the Newton-Raphson scheme was $|f| < 10^{-12}$, where $f(E) = E - e\sin E - M$.

The contour plot in Fig. 1 shows the number of integration steps required to solve Kepler's Equation as a function of mean anomaly M and eccentricity e using the homotopy approach. There are two "4+" regions in Fig. 1; the maximum number of integration steps for the left "4+" region is 11 and the maximum number of integration steps for the right "4+" region is 29. The maximum number of integration steps in the right "4+" occurs when $e = 1$; excluding the case of $e = 1$, the maximum number is 27. The contour plot in Fig 2 shows the number of iterative loops required to solve Kepler's Equation as a function of mean anomaly M and eccentricity e using the Newton-Raphson algorithm. In Fig. 2, there also exist two "4+" regions; the maximum number of iterations in the left "4+" region is 10, and the maximum number of iterations in the right "4+" region is 48,818. In the right "4+" region, the maximum occurs when $e = 1$ (note: if $e = 1$ is excluded, then the maximum number of iterations is 957).

It is important to observe that, for the Newton-Raphson approach, each iteration involves two function evaluations, whereas, for the homotopy approach, each integration involves six function evaluations (using ode45.m). Improvements in the performance of the homotopy approach can be achieved by an appropriate selection of the function $g(E)$. Our preliminary results indicate that the number of integration steps can be reduced significantly (i.e., the contour in the Me-plane is predominantly a 1 integration step region). Our experiments here were rather ad hoc and we suspect that a more systematic approach may indeed yield an "optimal" choice for $g(E)$.

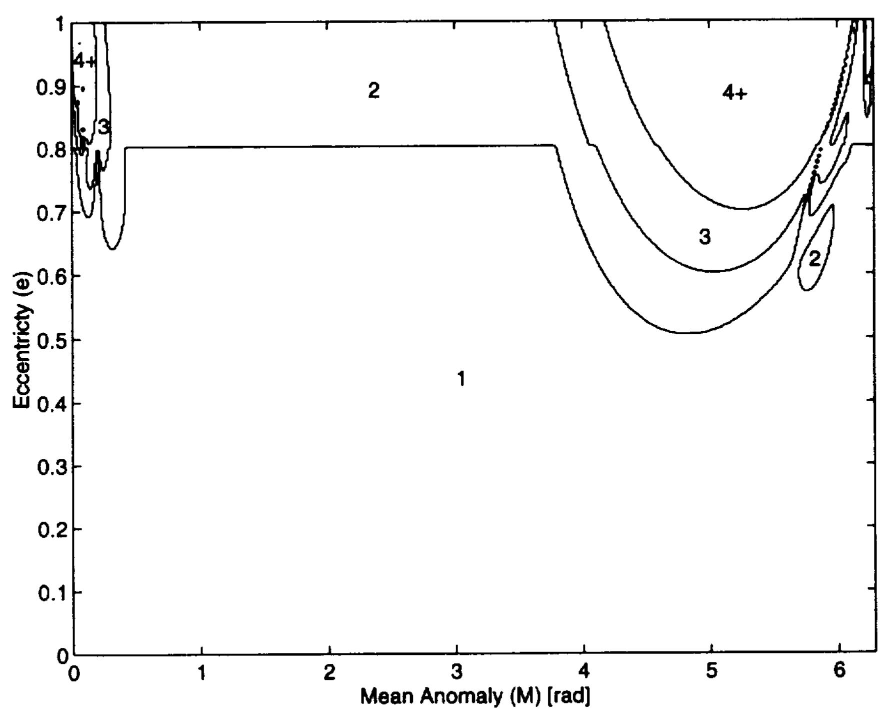

Figure 1 Elliptic Case using a Homotopy Algorithm

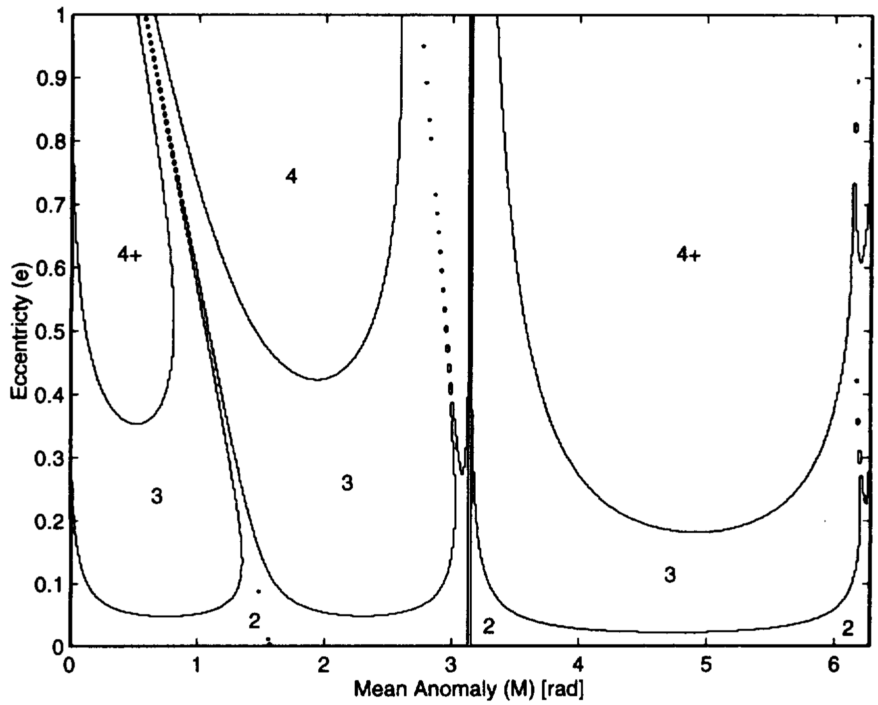

Figure 2 Elliptic Case using a Newton-Raphson Algorithm

IV Kepler's Equation in Hyperbolic Orbits

Homotopy Function for Kepler's Equation in Hyperbolic Orbits

For hyperbolic orbits, Kepler's Equation is

$$M = e \sinh F - F \tag{17}$$

where F is the hyperbolic anomaly, and as before, M is the mean anomaly. To apply the homotopy method, Eq. (17) is rewritten as

$$f(F) = e \sinh F - F - M = 0 \tag{18}$$

and an easily solved equation is chosen as

$$g(F) = e \sinh F - M = 0 \tag{19}$$

Following the algorithm developed in section III, the linear homotopy function is

$$\begin{aligned} H(F,\lambda) &= g(F) + \lambda(f(F) - g(F)) \\ &= e \sinh F - M + \lambda(-F) = 0 \end{aligned} \tag{20}$$

which yields the following Davidenko's differential equation.

$$(e \cosh F - \lambda)\frac{dF}{d\lambda} = F \tag{21}$$

Thus, the homotopy differential equation for the hyperbolic case is

$$\frac{dF}{d\lambda} = \frac{F}{e \cosh F - \lambda} \tag{22}$$

Once again, observe that the denominator of Eq. (22) is nonzero since $e \cosh F > \lambda$ for all F. Hence the the homotopy integration path can never bifurcate.

Numerical Results

For this case, the homotopy algorithm was evaluated against two Newton-Raphson algorithms; the first algorithm was the "conventional" quadratic convergence algorithm and the second was a quartic convergence algorithm [3]. The mean anomaly M was varied from 0 to 3π in increments of $\pi/150$ and the eccentricity e was varied from 1 to 6 in increments of 0.01. The numerical integrations associated with the homotopy algorithm were again accomplished using MATLAB's ode45 function with $tol = 5 \times 10^{-7}$, $h = hmax = 1$ and $hmin = 1/20000$ and the termination condition for both Newton-Raphson algorithms was $|f| < 10^{-12}$, where $f(F) = e \sinh F - F - M$. The results are shown in Figs. 3-5.

The contour plot in Fig. 3 shows the number of integration steps required to solve Kepler's Equation as a function of mean anomaly M and eccentricity e using the homotopy algorithm. The maximum number of integration steps for the "5+" region is 9. If instead of Eq. (19), the "known-solution" function was chosen as

$$g(F) = e \sinh F - (M + 0.01) = 0, \tag{23}$$

the maximum number of integration steps for the "5+" region is reduced to 7.

The results shown in Figs. 4 and 5 are based on an initial guess determined from

$$F_0 = \ln(\frac{2M}{e} + 1.8)$$

Figure 4 shows the results for the Newton-Raphson algorithm with quadratic convergence. The maximum number of iterations in the "6+" region is 21. Figure 5 shows the results for the quartic convergence Newton-Raphson algorithm. In Fig. 5, there are two "4+" regions; the maximum number of iterations in the left "4+" region is 11 and the maximum iteration number in the right "4+" region is 5.

Again, it is important to observe that the quartic convergence Newton-Raphson algorithm requires two function evaluations for each iteration, whereas the homotopy algorithm requires six function evaluations for each integration. Also, the Newton-Raphson algorithm with local quartic convergence [3] requires five function evaluations for each

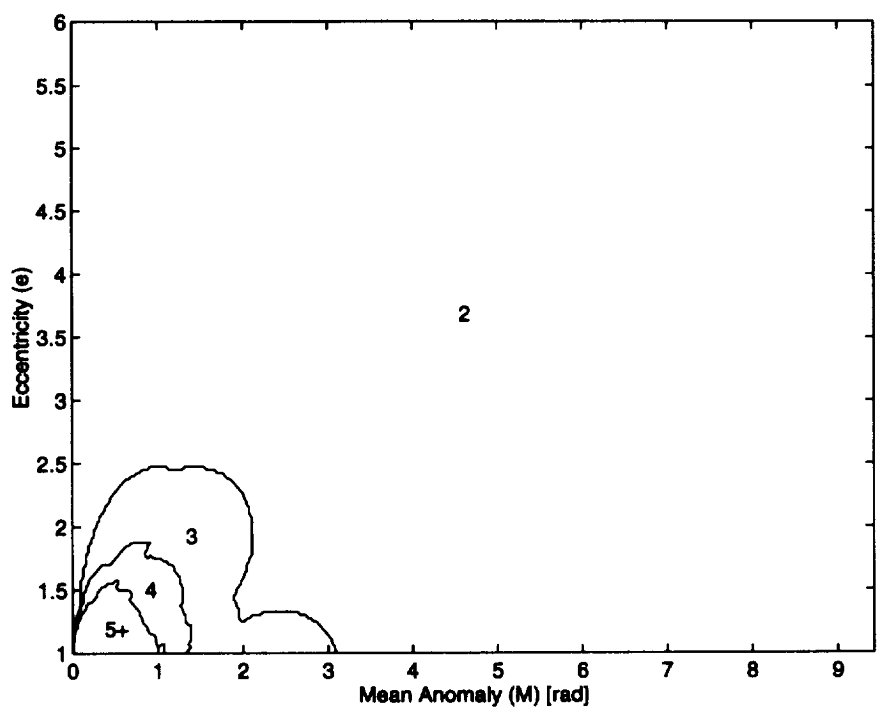

Figure 3 Hyperbolic Case using a Homotopy Algorithm

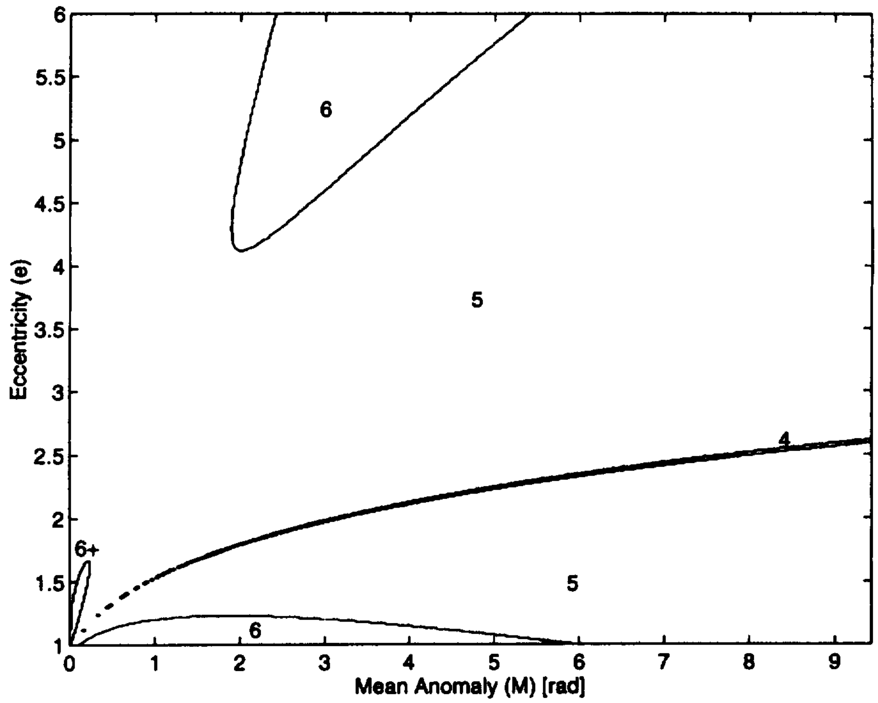

Figure 4 Hyperbolic Case using a Newton-Raphson Algorithm

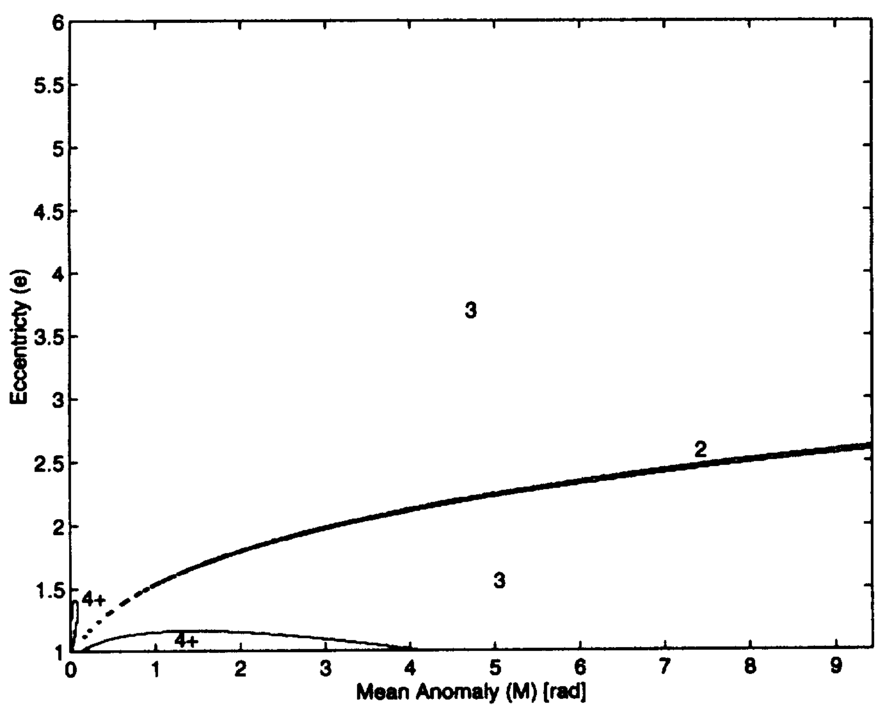

Figure 5 Hyperbolic Case using a Newton-Raphson Algorithm with Quartic Convergence

iteration step. Thus, for the *Me*-plane analyzed in this paper, the homotopy approach appears to have a computational advantage over the quartic convergence iterative approach. Again, additional improvements in the performance of the homotopy approach can be achieved by proper selection of $g(F)$.

VI Summary

In this paper, we present a new algorithm to solve Kepler's Equation based on homotopy theory. The procedure transforms the root finding problem into an initial value problem which can be integrated to yield the desired solution. Although no attempts were made to optimize the developed algorithm, it compared quite favorably with some existing iterative algorithms. Improvements in the algorithm can be obtained by investigating appropriate "known functions" ($g(E)$ or $g(F)$) and by investigation other numerical integration schemes.

References

[1]. Prussing, J. E., and Conway, B. A., Orbital Mechanics, Oxford University Press, New York, 1993.

[2]. Danby, J. M. A., and Burkardt, T. M, "The Solution of Kepler's Equation I," Celestial Mechanics, Vol. 31, 1983, pp. 95-107.

[3]. Burkardt, T. M, and Danby, J. M. A., "The Solution of Kepler's Equation II," Celestial Mechanics, Vol. 31, 1983, pp. 317-328.

[4]. Conway, B. A., "An Improved Algorithm Due to Laguerre for the Solution of Kepler's Equation," Celestial Mechanics, Vol. 39, 1986, pp. 199-211.

[5]. Odell, A., and Gooding, R. H., "Procedures for Solving Kepler's Equation," Celestial Mechanics, Vol. 38, 1986, pp. 307-334.

[6]. Markley, F. L., "Kepler's Equation Solver," Proceedings of the 1995 Flight Mechanics/Estimation Theory Symposium, NASA CP-3299, pp. 47-57.

[7]. Zangwill, W. I., and Garcia. C. B., Pathways to Solutions, Fixed Points, and Equilibria, Prentice-Hall, Englewood Cliffs, N.J., 1981.

[8]. Collins, Jr., E., Davis, L. and Richter, S., "Homotopy Algorithm for Maximum Entropy Design," Journal of Guidance, Control and Dynamics, Vol. 17, No. 2, March-April, 1994, pp. 311-321.

FLIGHT MECHANICS/ESTIMATION THEORY SYMPOSIUM

MAY 14-16, 1996

SESSION 5