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Conversion of Osculating Orbital Elements to Mean Orbital Elements

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Abstract

Orbit determination and ephemeris generation or prediction over relatively long elapsed times can be accomplished with mean elements. The most simple and efficient method for orbit determination, which is also known as epoch point conversion, performs the conversion of osculating elements to mean elements by iterative procedures. Previous epoch point conversion methods are restricted to shorter elapsed times with linear convergence. The new method presented in this paper calculates an analytic initial guess of the unknown mean elements from a first order theory of secular perturbations and computes a transition matrix with accurate numerical partials. It thereby eliminates the problem of an inaccurate initial guess and an identity transition matrix employed by previous methods. With a good initial guess of the unknown mean elements and an accurate transition matrix, converting osculating elements to mean elements can be accomplished over long elapsed times with quadratic convergence.

Basic Concepts

This paper presents new methods to solve the following problems:

- A user's propagator requires a mean orbital element set (e.g., NORAD--North American Aerospace Defense Command two-card element set or any other mean element set) as input but the element set is not available.
- For ground based radar acquisition and space sensor surveillance, a single osculating state vector of an object at the current time is available, but the mean elements corresponding to an epoch a few days, weeks or months earlier are not. If the set of mean elements at an epoch can be computed, then the object can be identified with respect to a known catalog (e.g. NORAD element set for Resident Space Objects). The mean elements at an epoch are needed to efficiently provide radar or sensor pointing commands.

The osculating orbital elements represent, in a general sense, the true position and velocity vectors of a satellite, but are poorly behaved over time as a basis for prediction. The mean orbital elements do not represent the true position and velocity vectors of a satellite, but are well behaved over time. An orbit described by a set of mean orbital elements is said to be perturbed or non-Keplerian.

By way of notation all vectors are in **bold** unless specified. For the sake of simplicity, the osculating elements and the mean elements are respectively denoted as:

$$\mathbf{y}(t) = \begin{bmatrix} a & e & i & \Omega & \omega & M \end{bmatrix}^T \qquad \text{and} \qquad \bar{\mathbf{y}}(t) = \begin{bmatrix} \bar{a} & \bar{e} & \bar{i} & \bar{\Omega} & \bar{\omega} & \bar{M} \end{bmatrix}^T$$

where a is the semimajor axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of perigee and M is the mean anomaly. The six elements of $\mathbf{y}(t)$ or $\bar{\mathbf{y}}(t)$ can be chosen in a variety of ways and the classical orbital elements are chosen to enhance theoretical understanding. The singularities of small eccentricity, small inclination or critical inclination do not exist in the conversion of osculating elements to mean elements, but the reverse is not true. The transformations between classical orbital elements, $\mathbf{y}(t)$, and predicted or true position and velocity vectors, $\mathbf{r}(t)$ and $\mathbf{v}(t)$, are simple. The transformation of the mean elements, $\bar{\mathbf{y}}(t)$, to the mean position and velocity vectors, $\bar{\mathbf{r}}(t)$ and $\bar{\mathbf{v}}(t)$, is straightforward, but the reverse is difficult. By way of definition, a *transformation* between vectors is an instantaneous conversion. A *conversion* between vectors involves an elapsed time greater than or equal to zero.

The ballistic coefficient of a satellite is defined as $B = \frac{C_{d0}A}{2m}$, where C_{d0} is the zero drag coefficient, A is the reference area of the satellite and m is the mass of the satellite. The problem of converting osculating orbital elements at a final time, t, to mean orbital elements at an initial time, t_0, can be stated as

Given: B, t_0, t and y(t)

Find: $\bar{y}(t_0)$

Depending on the perigee altitude of the satellite, previous methods presented by References 1 and 2 that use iterative procedures, are restricted to a short elapsed time, $(t - t_0)$. Reference 3, which uses a combination of Kozai's and Izsak's theories, calculates the difference between the osculating elements and mean elements with zero elapsed time. The difference, $y(t_0) - \bar{y}(t_0)$, is the very small variation of orbital elements due to short-periodic perturbations. The method presented in this paper allows the elapsed time to be extended over much longer intervals. Only the case of non-negative elapsed time, $(t - t_0) \geq 0$, is formulated, however, the method also holds for negative elapsed time as well.

To understand our approach to solve the above problem, a background on some basic concepts is required. The state prediction problem for a satellite orbiting about a central body such as the Earth is to find the position and velocity vectors, r(t) and v(t), at time t that satisfy the vector equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_d \qquad (1)$$

subject to the given initial conditions $\mathbf{r} = \mathbf{r}_0$ and $\mathbf{v} = \mathbf{v}_0$ at time $t = t_0$. In Equation (1), μ is the gravitational constant and $\mathbf{a}_d$ is the total disturbed acceleration vector due to disturbed gravity, atmospheric drag, lunisolar gravitational attractions, solar radiation pressure, tidal friction, n-body gravitational attractions and thrust. The first term on the right-hand-side of Equation (1) is the acceleration vector due to central gravity.

If $\mathbf{a}_d$ is zero, then Equation (1) can be solved analytically by one of Kepler's methods and the osculating elements (a, e, i, Ω, ω) describing the size, shape and orientation of the satellite orbit remain constant in time. The osculating mean anomaly, M, defines the angular position of the satellite in its orbit with respect to time. For a typical non-thrusting, near-Earth satellite between the theoretical atmospheric altitude of 91 km (300,000 ft) and a 12-hour orbit altitude of approximately 20,000 km, the disturbed acceleration vector, $\mathbf{a}_d^*$, is due mainly to the Earth zonal gravitational harmonics, J_2, J_3 and J_4, and atmospheric drag. Lunisolar gravitational attractions and solar radiation pressure, whose effects may be formulated similar to atmospheric drag, are neglected in this study. The orbital elements affected by the disturbed acceleration vector, $\mathbf{a}_d^*$, are defined as the *osculating elements*, y(t). [Strictly speaking the *osculating elements* are affected by $\mathbf{a}_d$.] The disturbed accelerations of $\mathbf{a}_d^*$ cause secular, short-periodic and long-periodic variations in the classical orbital elements (a, e, i, Ω, ω, M). Short periods are on the order of time of one satellite passage around the Earth. Long periods are on the order of time of one complete perigee passage around the Earth. If the periodic effects are removed, then the new orbital elements are defined as *mean elements*, $\bar{y}(t)$. That is, the mean elements are affected only by secular perturbations.

Periodic variations occur in all osculating elements and are induced by all zonal gravitational harmonics. However, the variations in osculating elements induced by periodic perturbations are much smaller than those induced by secular perturbations as the elapsed time increases. The variations in osculating elements induced by secular perturbations are constant or non-periodic, and only even zonal gravitational harmonics and atmospheric drag give rise to secular effects. Atmospheric drag can be a significant part of the secular perturbations if the perigee altitude of a typical satellite orbit is less than 500 km. In summary:

- Secular perturbations >> Periodic perturbations (for long elapsed time)
- Secular perturbations = Disturbed acceleration due to gravity (J_2 and J_4) and Drag (perigee altitude < 500 km)

Figures 1 to 6 show the variations of the osculating elements due to the disturbed acceleration vector, $\mathbf{a}_d^*$, for a non-thrusting Low Earth Orbit satellite with a perigee altitude of approximately 200 km. Osculating elements are computed by four methods: numerical integration, two NORAD propagators (SGP and SGP4) and a first order theory of secular perturbations. The osculating elements predicted by the first order theory of secular perturbations are depicted by the thick solid line. As shown in Figures 1 to 3, the first order secular effects on the osculating elements a, e and i are almost negligible and are induced only by drag, and therefore the averaged time derivatives $\left.\frac{da}{dt}\right|_{av}$, $\left.\frac{de}{dt}\right|_{av}$ and $\left.\frac{di}{dt}\right|_{av}$ are almost zero (the thick solid lines are almost horizontal). The mean elements, $\bar{y}(t_0)$, are the initial values of osculating elements at time $t_0 = 0$ on the thick solid line. The solutions computed by numerical integration, SGP and SGP4, are depicted respectively by the thin solid line, the triangle and the square.

As shown in Figures 4 to 6, the secular effects on the osculating elements Ω, ω and M are significant and are induced by both gravity and drag. The averaged time derivatives $\left.\frac{d\Omega}{dt}\right|_{av}$, $\left.\frac{d\omega}{dt}\right|_{av}$ and $\left.\frac{dM}{dt}\right|_{av}$ are almost constant to first order even if the time dependent contributions due to drag are included. At 200 km altitude, the disturbed acceleration due to J_2 is at least one or two orders of magnitude greater than that due to J_4. It is well known in general perturbations theory that the osculating elements (a, e, i, Ω, ω) are "slow" variables and that M is a "fast" variable. This implies that $\left.\frac{dM}{dt}\right|_{av}$ is much greater than $\left.\frac{d\Omega}{dt}\right|_{av}$ and $\left.\frac{d\omega}{dt}\right|_{av}$. If the elapsed time, $(t - t_0)$, is long, then the averaged time derivative due to drag, $\left.\frac{dM}{dt}\right|_{drag}$, must be included for a typical Low Earth Orbit satellite with a perigee altitude of 200 km even though References 4 to 6 and many excellent textbooks have recommended otherwise.

Since the satellite orbit of this example is almost circular, the first order estimates of the osculating eccentricity and argument of perigee are not equal to their averaged values during the two periods of the satellite orbit as shown in Figures 2 and 5. However, the first order estimates of the other osculating elements are very close to the averaged values of the osculating elements as predicted by the NORAD propagator SGP.

If a NORAD propagator is used, then the mean mean motion, $\bar{n}$, which replaces $\bar{a}$ of $\bar{y}(t)$, must be carefully computed. The osculating mean motion, n, is determined from the equation: $\mu = n^2 a^3$. If the elapsed time is short (less than a day), the errors arising from interchanging the osculating mean motion and mean mean motion are negligible. If the elapsed time is long (on the order of days), the mean mean motion, $\bar{n}$, must be used in evaluating the averaged time derivatives (except for initialization). It should be clear that $\mu \neq \bar{n}^2 \bar{a}^3$.

From general perturbations theory, the osculating elements and the mean elements have a first order secular relationship given by

$$y(t) = \bar{y}(t_0) + \left.\frac{dy}{dt}\right|_{av} (t - t_0) \tag{2}$$

where the averaged time derivatives valid between t_0 and t, are defined as

$$\left.\frac{dy}{dt}\right|_{av} = \frac{1}{2\pi}\int_0^{2\pi} \frac{dy}{dt}\, dM = \frac{1}{2\pi}\int_0^{2\pi} (1 - e\cos E)\frac{dy}{dt}\, dE \tag{3}$$

and E is the eccentric anomaly. The instantaneous time derivative, $\frac{dy}{dt}$, can be obtained from Lagrange's planetary equations for secular perturbations. Analytic solutions of $\left.\frac{dy}{dt}\right|_{av}$, which are not difficult to obtain, are relatively accurate except for Low Earth Orbit satellites. If the elapsed time is long and the disturbed acceleration due to drag is significant (e.g. for a Low Earth Orbit satellite), then the analytic averaged time derivatives of the semimajor axis and eccentricity can be an order of magnitude in error with respect to their numerically integrated values. This can happen even if the elapsed time is less than one period for some satellite orbits.

From Reference 7, the averaged time derivatives due to the disturbed gravity (J_2 and J_4) for secular perturbations are given by

$$\left.\frac{da}{dt}\right|_{gravity} = 0 \tag{4}$$

$$\left.\frac{de}{dt}\right|_{gravity} = 0 \tag{5}$$

$$\left.\frac{di}{dt}\right|_{gravity} = 0 \tag{6}$$

$$\left.\frac{d\Omega}{dt}\right|_{gravity} = -\frac{3\bar{n}\cos\bar{i}\, r_e^2}{2\bar{p}^2}\left\{\begin{array}{l} J_2 - \dfrac{J_2^2 r_e^2}{16\bar{p}^2}\left[\left(12-80\sin^2\bar{i}\right)-\bar{e}^2\left(4+15\sin^2\bar{i}\right)\right] \\ -\dfrac{5J_4 r_e^2}{16\bar{p}^2}\left[\left(4-7\sin^2\bar{i}\right)\left(2+3\bar{e}^2\right)\right] \end{array}\right\} \tag{7}$$

$$\begin{aligned}\left.\frac{d\omega}{dt}\right|_{gravity} = &\ \frac{3\bar{n}J_2 r_e^2}{4\bar{p}^2}\left[4-5\sin^2\bar{i}\right] \\ &+\frac{9\bar{n}J_2^2 r_e^4}{384\bar{p}^4}\left[10\sin^2\bar{i}\left(76-89\sin^2\bar{i}\right)+\bar{e}^2\left(56-36\sin^2\bar{i}-45\sin^4\bar{i}\right)\right] \\ &-\frac{15\bar{n}J_4 r_e^4}{32\bar{p}^4}\left[\left(16-62\sin^2\bar{i}+49\sin^4\bar{i}\right)+\bar{e}^2\left(18-63\sin^2\bar{i}+\frac{189}{4}\sin^4\bar{i}\right)\right]\end{aligned} \tag{8}$$

$$\begin{aligned}\left.\frac{dM}{dt}\right|_{gravity} = &\ \bar{n} + \frac{3n_0 J_2 r_e^2\left(1-\bar{e}^2\right)^{1/2}}{4\bar{p}^2}\left[2-3\sin^2\bar{i}\right] \\ &+\frac{9\bar{n}J_2^2 r_e^4\left(1-\bar{e}^2\right)^{1/2}}{96\bar{p}^4}\left[\sin^2 i_o\left(100-131\sin^2\bar{i}\right)+\bar{e}^2\left(20-98\sin^2\bar{i}+67\sin^4\bar{i}\right)\right] \\ &-\frac{45\bar{n}J_4 r_e^4\left(1-\bar{e}^2\right)^{1/2}}{128\bar{p}^4}\left[\bar{e}^2\left(8-40\sin^2\bar{i}+35\sin^4\bar{i}\right)\right]\end{aligned} \tag{9}$$

where $\bar{p} = \bar{a}(1-\bar{e}^2)$. In satellite state prediction, an accurate solution must always be computed numerically. Therefore, the averaged time derivatives due to drag should be integrated numerically for the semimajor axis and eccentricity. From References 8, the averaged time derivatives due to drag for secular perturbations are given by

$$\left.\frac{da}{dt}\right|_{drag} = -2B\bar{n}\bar{a}^2\left[\frac{1}{\pi}\int_0^{\pi}\rho Q_1^2(1+\bar{e}\cos E)\sqrt{\frac{(1+\bar{e}\cos E)}{(1-\bar{e}\cos E)}}\,dE\right] \tag{10}$$

$$\left.\frac{de}{dt}\right|_{drag} = -2B\bar{n}\bar{p}\left[\frac{1}{\pi}\int_0^{\pi}\rho Q_1 Q_2\sqrt{\frac{(1+\bar{e}\cos E)}{(1-\bar{e}\cos E)}}\,dE\right] \tag{11}$$

$$\left.\frac{di}{dt}\right|_{drag} = 0 \tag{12}$$

$$\left.\frac{d\Omega}{dt}\right|_{drag} = 0 \tag{13}$$

$$\left.\frac{d\omega}{dt}\right|_{drag} = 0 \tag{14}$$

$$\left.\frac{dM}{dt}\right|_{drag} = \frac{\dot{n}}{2}(t - t_0) = -\frac{3\bar{n}}{4\bar{a}}\left.\frac{da}{dt}\right|_{drag}(t - t_0) \tag{15}$$

where the rotating Earth factors are:

$$Q_0 = \frac{\omega_e \cos\bar{i}\sqrt{1-\bar{e}^2}}{\bar{n}}$$

$$Q_1 = 1 - Q_0\frac{(1-\bar{e}\cos E)}{(1+\bar{e}\cos E)}$$

$$Q_2 = \cos E - \frac{Q_0}{2(1-\bar{e}^2)}(1-\bar{e}\cos E)(2\cos E - \bar{e} - \bar{e}\cos^2 E)$$

and ω_e is the constant scalar Earth rotational rate and ρ is the intantaneous density at a reference altitude with respect to the mean anomaly E.

The drag-induced averaged time derivatives of inclination, longitude of the ascending node and argument of perigee are normally much smaller than those of their gravity counterparts, and therefore are neglected. The averaged second time derivatives of the semimajor axis and eccentricity are at least three orders of magnitudes less than the averaged time derivatives and therefore are also neglected.

Since the disturbed accelerations are additive, the averaged time derivatives of the osculating elements y(t) due to secular perturbations are the sum of the gravity and drag components. Substituting the averaged time derivatives of Equations (4) through (15) into Equation (2) and rearranging, a good initial guess of the unknown mean elements $\bar{y}(t_0)$ at time t_0 is given by

$$\begin{bmatrix} \bar{a}(t_0) \\ \bar{e}(t_0) \\ \bar{i}(t_0) \\ \bar{\Omega}(t_0) \\ \bar{\omega}(t_0) \\ \bar{M}(t_0) \end{bmatrix} = \begin{bmatrix} a(t) \\ e(t) \\ i(t) \\ \Omega(t) \\ \omega(t) \\ M(t) \end{bmatrix} - \begin{bmatrix} \left.\frac{da}{dt}\right|_{drag}(t - t_0) \\ \left.\frac{de}{dt}\right|_{drag}(t - t_0) \\ 0 \\ \left(\left.\frac{d\Omega}{dt}\right|_{gravity}\right)(t - t_0) \\ \left(\left.\frac{d\omega}{dt}\right|_{gravity}\right)(t - t_0) \\ \left(\left.\frac{dM}{dt}\right|_{gravity} + \left.\frac{dM}{dt}\right|_{drag}\right)(t - t_0) \end{bmatrix} \tag{16}$$

The mean elements $\bar{a}$, $\bar{e}$, $\bar{i}$ and $\bar{n}$ in the averaged time derivatives of Equations (4) through (15) are initially replaced by the respective osculating elements to start the initial guess process of Equation (16). Fortunately, the variations of these mean elements are normally slow. Now, given B, t_0, t and y(t), the right-hand-side of Equation (16) can be computed to give a good initial guess of the unknown mean elements $\bar{y}(t_0)$ at time t_0.

Methods for Converting Mean Elements to Osculating Elements

A method of general perturbations for satellite theory seeks the solution of Equation (1) by series expansion and term-by-term analytic integration of the disturbed acceleration. General perturbations methods circumvent numerical integration but must initiate with mean elements. The NORAD element set propagators, which are general perturbations methods, start with a given mean vector, $\bar{y}(t_0) = (n_o, e_o, i_o, \Omega_o, \omega_o, M_o)$ at a given epoch time, t_0. The given mean mean motion, n_o, can be converted to the mean semimajor axis, a_o, as described in Reference 9. Reference 10 documented five NORAD models for propagating Resident Space Objects (RSO) and satellites around the Earth. The Simplified General Perturbations (SGP) model, which contains most of the first order gravitational terms as described in References 11 and 12, computes the drag terms as linear functions of time. The SGP4 model, which uses the gravitational model of References 13 and 14, calculates the drag terms by a power density function of the atmosphere. The SGP4 model is customarily used for near-Earth satellites. A space object is classified by NORAD as near-Earth if its period is less than 225 minutes; it is classified as deep-space otherwise. The SDP4 model, which includes the gravitational terms due to third-body effects of the Sun and Moon and the Earth sectorial and tesseral harmonics, is an extension of SGP4 for deep-space objects. The SGP8 model is an extension of SGP4 with the same gravitational and drag models, but predicts state vectors more accurately especially when the satellite altitude is under 200 km. The SDP8 model is an extension of SGP8 and SDP4 for deep-space objects. The two higher order propagators, SGP8 and SDP8 were briefly considered as replacements for the SGP4 and SDP4, but the increased computational time and only slight improvement in state prediction accuracy have discouraged the changeover until the present time.

New and improved versions of SGP4 and SDP4 may be obtained directly from NORAD. However, using a UNIX workstation which is linked to Internet, a version of the five NORAD propagators (SGP, SGP4, SDP4, SGP8 and SDP8) can be downloaded from a computer at the Air Force Institute of Technology. The comprehensive instructions of Reference 10 and all the necessary algorithms downloaded from Internet do not guarantee that the reader can use the NORAD propagators immediately. Reference 9 discusses the problems and solutions.

The NORAD propagators were developed from the satellite theories of Kozai and Brouwer to propagate mean elements to osculating elements. Other mean element to osculating element propagators such as those described in References 16 and 17 will not be considered since their improvements in computational speed, memory storage and state vector accuracy are insignificant for our purpose. A method of converting osculating elements to mean elements requires a "forward" propagator which converts mean elements to osculating elements. Figures 1 to 6 show that the predicted osculating elements propagated by SGP are closer to those predicted by the first order theory of secular perturbations (the thick solid line) for most satellite orbits. If the forward propagator is SGP, then $\frac{\dot{n}}{2} = -\frac{3\bar{n}}{4\bar{a}}\left.\frac{da}{dt}\right|_{drag}$ and $\frac{\ddot{n}}{6} = -\frac{5\dot{n}}{12\bar{a}}\left.\frac{da}{dt}\right|_{drag}$ are required and can be computed from Equation (10). If an accurate $\frac{\ddot{n}}{6}$ is required, then $\left.\frac{d^2a}{dt^2}\right|_{drag}$ needs to be computed. If the forward propagator is SGP4 or SDP4, then $B^*_{sgp4} = 6378137.0\, K\, \rho_o\, B$, where $2 \geq K \geq 1$ is a constant related to the density model used to calculate ρ_o at the altitude of 120 km. The units of ρ_o and B are respectively kg/m^3 and m^2/kg. As a concrete example, a SGP or NORAD propagator has been chosen as the forward propagator in this paper even though the choice is arbitrary.

The NORAD element set propagators all start with a given mean vector, $\bar{y}(t_0)$, and their output is the predicted position and velocity vectors, **r**(t) and **v**(t), which can then be transformed to the osculating elements, **y**(t). Osculating elements are the ones that are usually available and the reconstruction of mean elements must begin with osculating elements.

Previous Methods for Converting Osculating Elements to Mean Elements (Ref. 1 to 3)

A method for converting osculating elements to mean elements which uses a transition matrix

$$T = \left[\frac{\partial\ y(t)}{\partial\ \bar{y}(t_0)} \right] \tag{17}$$

for the equation

$$\delta y(t) = T\ \delta\bar{y}(t_0) \tag{18}$$

is known as a method of differential corrections or a transition matrix algorithm in applied optimal control theory. A transition matrix algorithm usually begins with a guessed nominal $\bar{y}(t_0)$ at time t_0, and then propagates forward to a nominal y(t) at time t, which in turn gives $\delta y(t)$. Therefore the differential corrections at time t_0 is

$$\delta\bar{y}(t_0) = T^{-1}\ \delta y(t) \tag{19}$$

using Equation (18). In practice, accurate transition matrices are computed numerically. Chapter 7 of Reference 18 provides the algorithm to numerically compute a transition matrix such as that of Equation (17). Reference 19 describes the technique of accurate numerical partials.

Reference 1 suggests an iterative procedure by using the given osculating elements y(t) as the initial guess of the mean elements $\bar{y}(t_0)$ and assuming an identity transition matrix. That is, the averaged time derivatives of Equation (16) are zero for any elapsed time and Equation (19) is reduced to

$$\delta\bar{y}(t_0) = \delta y(t) \tag{20}$$

since T = I. If the elapsed time is less than one day, this iterative procedure may convert y(t) to $\bar{y}(t_0)$ for some Low Earth Orbits. This method converges linearly at best. If the elapsed time is greater than a day, this method fails for most Low Earth Orbits. The cause of failure is a combination of:

- The neglected drag terms for semimajor axis, eccentricity and mean anomaly are not small.
- The identity matrix is a poor approximation to the transition matrix and Equation (20) is not valid.

Recalling that the problem is to find $\bar{y}(t_0)$ given B, t_0, t and y(t). The iterative procedures of Reference 1 may be summarized as follows:

1. Let the initial guess of the mean elements at the given time t_0 be the same as the given osculating elements, y(t). That is

$$\bar{y}_k(t_0) = y(t)$$

where k is the iteration number (k = 0 at this point).

2. Propagate forward (using a SGP propagator) from $\bar{y}_k(t_0)$ to $\mathbf{x}_k(t)$ and then transform $\mathbf{x}_k(t)$ to $\mathbf{y}_k(t)$. The difference in osculating elements at time t is

$$\delta y(t) = y(t) - y_k(t) = \delta\bar{y}(t_0)$$

using Equation (20).

3. Compute the new guess of the mean elements at time t_0 as

$$\bar{y}_{k+1}(t_0) = \bar{y}_k(t_0) + \delta\bar{y}(t_0)$$

For $\delta y = \left| y(t) - y_k(t) \right| > 10^{-10}$, then procedure 2 is repeated with the new guess $\bar{y}_{k+1}(t_0)$; otherwise the desired mean elements at time t_0 is

$$\bar{y}(t_0) = \bar{y}_{k+1}(t_0)$$

References 2 and 3 are transformations between osculating and mean elements with $t = t_0$. Reference 2 is an iterative method that uses the Frazer elements (position and velocity vectors and their variations), includes the short- and long-periodic perturbations. Reference 3, which uses a combination of Kozai's and Izsak's theories, calculates the difference between the osculating elements and mean elements only due to short-periodic perturbations. The methods of these two references are only instantaneous conversions or transformations, and therefore are not described in this paper.

A New Method for Converting Osculating Elements to Mean Elements

This method uses a good initial guess derived from a first order theory of secular perturbations and a transition matrix computed by accurate numerical partials. As shown in Figures 1 to 6, the osculating elements computed from a first order theory of secular perturbations (the thick solid straight line) behave linearly with respect to time and are close to the average values of the SGP solutions. This implies that the initial guess of the mean elements computed from Equation (16) will be close to the desired mean elements, $\bar{y}(t_0)$, at time t_0. A traditional transition matrix algorithm requires 7 forward propagations (1 nominal and 6 neighboring trajectories). The stepsize h_i is usually set to 10^{-6} of the i^{th} mean element; this may be too small for eccentricity but too large for the semimajor axis. The transition matrix algorithm based on accurate numerical partials requires 25 forward propagations (1 nominal and 24 neighboring trajectories). The iterative procedures of this method may be summarized as follows:

1. Compute the averaged time derivatives of Equation (4) to (15). Let the given osculating elements, y(t), be **Y**, then the initial estimate of the mean semimajor axis, mean eccentricity, mean inclination and mean mean motion are given by:

$$\bar{a} = a(t) - \lambda \left.\frac{da}{dt}\right|_{drag} (t - t_0)$$

$$\bar{e} = e(t) - \lambda \left.\frac{de}{dt}\right|_{drag} (t - t_0)$$

$$\bar{i} = i(t)$$

$$\bar{n} = n(t)$$

where the empirical constant λ is 0.5 for Low Earth Orbits and zero otherwise.

2. Initialize $\frac{\dot{n}}{2}$ and $\frac{\ddot{n}}{6}$ for the forward propagator SGP and the initial guess of the mean mean anomaly, $\overline{M}$, by numerically integrating $\left.\frac{da}{dt}\right|_{drag}$ of Equation (10), then $\frac{\dot{n}}{2} = -\frac{3\bar{n}}{4\bar{a}}\left.\frac{da}{dt}\right|_{drag}$ and $\frac{\ddot{n}}{6} = -\frac{5\dot{n}}{12\bar{a}}\left.\frac{da}{dt}\right|_{drag}$. If the forward propagator is SGP4 or SDP4, then B^*_{sgp4} is required.

3. Compute the nominal mean elements $\bar{y}^*(t_0)$ guess from Equation (16) and then propagate forward to t giving the nominal osculating elements $y^*(t)$

4. Compute the nominal differential correction of osculating elements at t as

$$\delta y(t) = \mathbf{Y} - y^*(t)$$

5. Compute the transition matrix, $T = \left[\frac{\partial y(t)}{\partial \bar{y}(t_0)}\right]$ by accurate numerical partials.

Propagate forward by a SGP propagator 6 times in the neighborhood of the nominal trajectory (step 3) using

$$\bar{y}(t_0) = \bar{y}^*(t_0) + \delta\bar{y}(t_0)$$

For the i^{th} neighboring trajectory (i = 1, 2, .. , 6), four "neighboring" neighboring trajectories are computed from 4 stepsizes of $\frac{h_i}{2}$, $-\frac{h_i}{2}$, $\frac{\rho h_i}{2}$ and $-\frac{\rho h_i}{2}$. The corresponding 4 osculating elements computed by a SGP propagator at time t are y_1, y_2, y_3 and y_4. The partial derivatives of the i^{th} column of T is given by

$$\frac{\partial\, y(t)}{\partial\, \bar{y}_i(t_0)} = \frac{(y_3 - y_4) - \rho^3(y_1 - y_2)}{\rho h_i(1-\rho^2)}$$

The 6 x 6 transition matrix can be approximated as

$$T = \left[\frac{\partial\, y(t)}{\partial\, \bar{y}(t_0)}\right] = \left[\frac{\partial\, y(t)}{\partial\, \bar{y}_1(t_0)} \; \frac{\partial\, y(t)}{\partial\, \bar{y}_2(t_0)} \; \cdot \; \cdot \; \frac{\partial\, y(t)}{\partial\, \bar{y}_n(t_0)}\right]$$

In computing T, 24 propagation by the forward propagator SGP are required. A good choice of ρ is 1/2 and that of the stepsize h_i is 10^{-4} of each mean element.

6. Update the nominal $\bar{y}^*(t_0)$ at t_0 as

$$\bar{y}^*(t_0)\Big|_{new} = \bar{y}^*(t_0)\Big|_{old} + T^{-1}\,\delta y(t)$$

Step 3 to 6 are repeated until the magnitude of $\delta y(t)$ is reduced to an acceptably small value (10^{-12}).

Examples

Two examples are given to illustrate the performance of the new method for satellites at a Low Earth Orbit and a High Earth Orbit. Using a first order theory of secular perturbations, the initial guess of the slow variables, $(\bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega})$, can be estimated in the vicinity of their unknown values at time t_0. If the initial guess of the fast variable, $\bar{M}$, can be predicted to within approximately 30 degrees of its unknown value, then convergence is fast. This is not a problem for almost any satellite orbit if the elapsed time is less than one day.

The variations of the osculating mean anomaly, M, at time t for the example Low Earth Orbit satellite are shown in Figures 6. The initial guess of $\bar{M}$ at time t_0 is related to M at time t by Equation (16). For short elapsed times, the 30 degree requirement can be satisfied easily. The primary reason is that the unknown mean semimajor axis, $\bar{a}$, has changed only slightly in a short elapsed time, and as a consequence the effects on the terms due to $\left.\frac{dM}{dt}\right|_{gravity}$ and $\left.\frac{dM}{dt}\right|_{drag}$ are small. The right-hand-side of Equation (16) can then be computed quite accurately giving a good initial guess of $\bar{M}$.

In the following examples, first we assumed to know the mean elements, $\bar{y}(t_0)$, at time t_0, and then we used a forward propagator (SGP4 for *Example 1* and SDP4 for *Example 2*) to get the osculating elements, y(t), at time t. In what follows, we discard the mean elements, $\bar{y}(t_0)$, and no knowledge of the mean elements at time t_0 will be used. The problem is:

Given B, t_0, t and y(t); retrieve $\bar{y}(t_0)$.

Example 1

The mean elements, $\bar{y}(t_0)$, at time t_0 are taken out of the SGP4 example of Reference 10 with minor adjustments for double precision computation. Also the ballistic coefficient, B, is replaced by that of the LANDSAT-D. The osculating elements, y(t), at time t are computed by using SGP4 with B^*_{sgp4} computed from Equation (18).

Given: $C_{d0} = 2.0$, $A = 12.2778$ meter^2, $m = 1710.0$ kg, $B = \frac{C_{d0}A}{2m} = 0.00718 \; \frac{\text{meter}^2}{\text{kg}}$

$t_0 = 0.0$, $t = 0, 1, 5$ days

$$y(0\ \text{day}) = \begin{bmatrix} 6641.774062 \\ .0096661858 \\ 72.85385095 \\ 115.9622955 \\ 59.40458042 \\ 103.8371428 \end{bmatrix}, \quad y(1\ \text{day}) = \begin{bmatrix} 6633.640850 \\ .0083375605 \\ 72.85513567 \\ 113.4116703 \\ 56.77943987 \\ 130.9131346 \end{bmatrix}, \quad y(5\ \text{days}) = \begin{bmatrix} 6589.666059 \\ .0053578232 \\ 72.84433117 \\ 103.0554340 \\ 54.07490762 \\ 349.1054119 \end{bmatrix}$$

Following the procedures 1 to 3 of the new method, the initial guesses of the nominal mean elements, $\bar{y}^*(t_0)$, at time t_0 for the cases of $t = 0, 1, 5$ days are computed as:

$$\bar{y}^*(t_0)\Big|_{0\ \text{day}} = \begin{bmatrix} 6641.7740 \\ .00966618 \\ 72.853850 \\ 115.96229 \\ 59.404580 \\ 103.83714 \end{bmatrix}, \quad \bar{y}^*(t_0)\Big|_{1\ \text{day}} = \begin{bmatrix} 6638.4631 \\ .00869400 \\ 72.855135 \\ 115.97053 \\ 59.232901 \\ 105.09619 \end{bmatrix}, \quad \bar{y}^*(t_0)\Big|_{5\ \text{days}} = \begin{bmatrix} 6641.9257 \\ .00834486 \\ 72.844331 \\ 115.93401 \\ 66.308201 \\ 138.72967 \end{bmatrix}$$

Comparison of results:

	Method of Reference 1 (Walter)		New Method (Der&Danchick)	
Days from t_0	# of iterations required: N_1	# of SGP4 calls 7 N_1	# of iterations required: N_2	# of SGP4 calls 25 N_2
0	12	84	3	75
1	40	280	3	75
5	not converged	---	4	100

Found: The mean elements, $\bar{y}(t_0)$, at time t_0 for the cases of $t = 0, 1, 5$ days:

$$\bar{y}(t_0) = \begin{bmatrix} 6637.68397 \\ .0086731 \\ 72.84350 \\ 115.9689 \\ 52.69880 \\ 110.5714 \end{bmatrix}$$

One reason to change the ballistic coefficient from that of Reference 10 is to investigate how close the new method works during the satellite orbital decay. Using the lifetime equation of Reference 4, this satellite has a lifetime of approximately 9000 minutes or a little over six days from time t_0. Numerical integration shows that the lifetime is approximately 10 days from time t_0. After 5 days, the perigee altitude of the satellite is close to 150 km and the state vectors predicted by the SGP4 propagator become inaccurate. Nevertheless, the new method converges very close to $\bar{y}(t_0)$ in the last few days before satellite re-entry.

Example 2

The mean elements, $\bar{y}(t_0)$, at time t_0 are constructed from a Molniya orbit with perigee altitude of 500 km and apogee altitude of 40000 km. Critical inclination (63.4 degrees) is chosen for this HEO to demostrate that there is no singularity at any inclination for the conversion from osculating elements to mean elements. The ballistic coefficient, B, which is not important for this HEO, is chosen to be the same as that of example 1. The osculating elements, y(t), at time t are computed by using the SDP4 propagator. The effect of drag is negligible on this HEO satellite.

Given: $C_{d0} = 2.0$, $A = 12.2778 \;\; \text{meter}^2$, $m = 1710.0 \;\; \text{kg}$, $B = \dfrac{C_{d0}A}{2m} = 0.00718 \;\; \dfrac{\text{meter}^2}{\text{kg}}$,

$t_0 = 0.0$, $t = 10, \;\; 100, \;\; 200 \;\; \text{days}$

$$y(10\text{ days}) = \begin{bmatrix} 26626.70165 \\ .7415398328 \\ 63.45549845 \\ 118.5145116 \\ .0695130482 \\ 137.6237041 \end{bmatrix}, \quad y(100\text{ days}) = \begin{bmatrix} 26630.49652 \\ .7364330943 \\ 63.36886054 \\ 105.0926980 \\ .4435084366 \\ 77.71324416 \end{bmatrix}, \quad y(200\text{ days}) = \begin{bmatrix} 26626.65222 \\ .7323070455 \\ 63.72013696 \\ 90.19640476 \\ .7794822322 \\ 26.78924405 \end{bmatrix}$$

Following the procedures 1 to 3 of the new method, the initial guesses of the nominal mean elements, $\bar{y}^*(t_0)$, at time t_0 for the cases of $t = 10, \;\; 100, \;\; 200 \;\; \text{days}$ are computed as:

$$\bar{y}^*(t_0)\Big|_{10\text{ days}} = \begin{bmatrix} 26626.798 \\ .74154077 \\ 63.455498 \\ 119.93263 \\ .07822485 \\ 144.12848 \end{bmatrix}, \quad \bar{y}^*(t_0)\Big|_{100\text{ days}} = \begin{bmatrix} 26630.570 \\ .73643493 \\ 63.368860 \\ 119.43259 \\ .43299544 \\ 158.12109 \end{bmatrix}, \quad \bar{y}^*(t_0)\Big|_{200\text{ days}} = \begin{bmatrix} 26626.686 \\ .73230790 \\ 63.720136 \\ 117.80345 \\ 1.5247843 \\ 156.05599 \end{bmatrix}$$

Comparison of results:

	Method of Reference 1 (Walter)		New Method (Der&Danchick)	
Days from t_0 (days)	# of iterations required: N_1	# of SDP4 calls 7 N_1	# of iterations required: N_2	# of SDP4 calls 25 N_2
10	13	91	3	75
100	20	140	4	100
200	not converged	---	7	175

Found: The mean elements, $\bar{y}(t_0)$, at time t_0 for the cases of $t = 10, \;\; 100, \;\; 200 \;\; \text{days}$:

$$\bar{y}(t_0) = \begin{bmatrix} 26626.96632 \\ .7416966 \\ 63.33610 \\ 120.0032 \\ .0077000 \\ 143.8417 \end{bmatrix}$$

Conclusions

- If the constants given in the data block of the five NORAD propagators are defined in double precision, then the osculating state vectors can be predicted much more accurately especially for satellite orbits without the influence of atmospheric drag. With this simple modification, the five NORAD propagators can be used as forward propagators for the conversion of osculating elements to mean elements.
- The method presented by this paper achieves quadratic convergence due to accurate numerical partials. It has uniformly good performance over the three test cases, succeeds where the Reference 1 method failed, and is generally more computationally efficient.

- For Low Earth Orbits (the example described by Figures 1 to 6), the initial guess of the fast changing mean mean anomaly, $\overline{M}$, at time t_0 must include the term computed from the averaged time derivative due to drag. In this case, the conversion of osculating elements to mean elements can be extended from an elapsed time of one day to the last few days before satellite re-entry
- For High Earth Orbits, the initial guess of the fast changing mean mean anomaly, $\overline{M}$, at time t_0 can be predicted accurately even for long elapsed times. In this case, the elapsed time for the conversion of osculating elements to mean elements can be extended to months.
- For Geosynchronous Earth Orbits (results not included in this paper), the initial guess of the fast changing mean mean anomaly, $\overline{M}$, at time t_0 can also be predicted accurately even for long elapsed times. In this case, the elapsed time for the conversion of osculating elements to mean elements can be extended to years.
- The determination of mean orbital elements can be radically streamlined with the new method and made applicable for mean elements orbit determination with observations made by heterogeneous sensor types over long spans of elapsed time.

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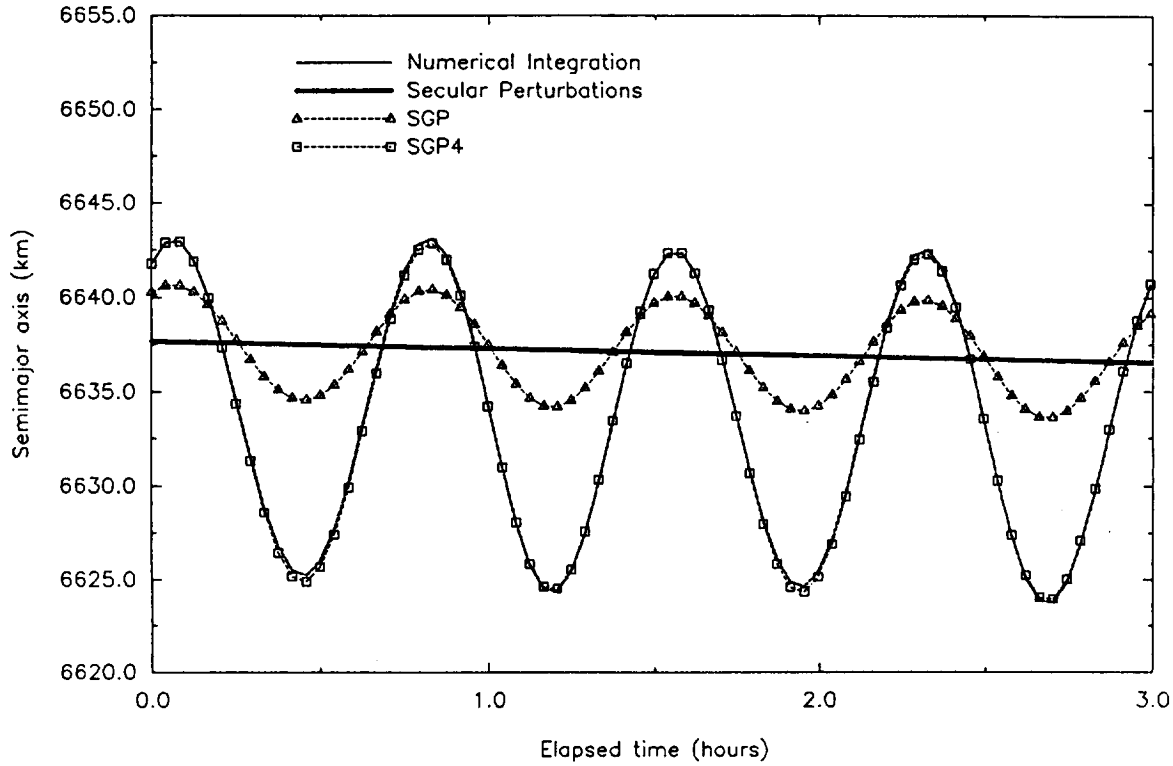

Figure 1. Effects of gravity and drag on semimajor axis for 2 Low Earth orbits

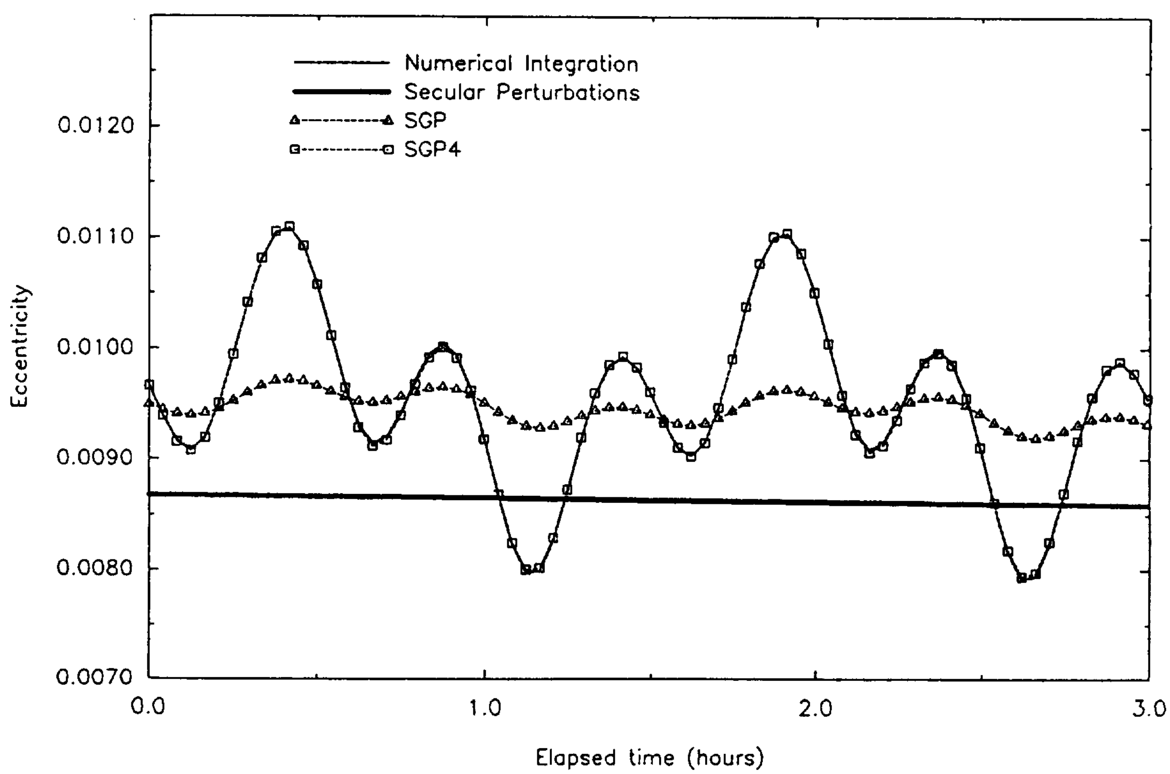

Figure 2. Effects of gravity and drag on eccentricity for 2 Low Earth Orbits

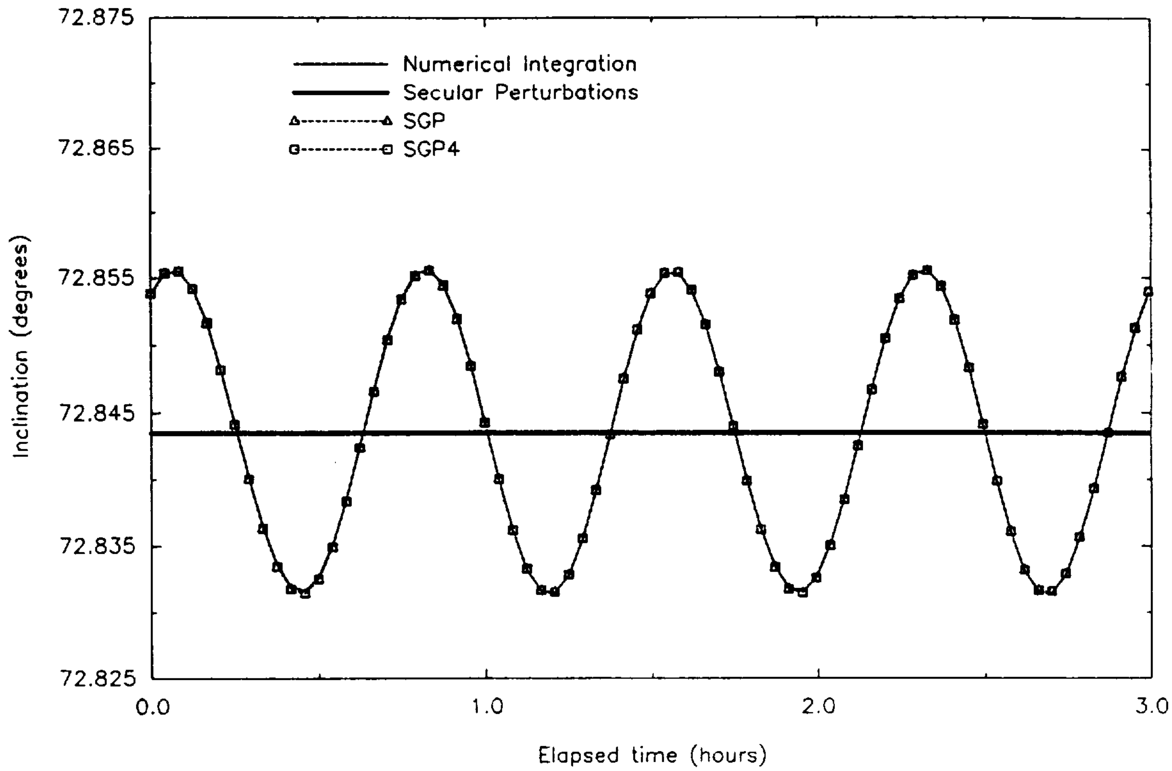

Figure 3. Effects of gravity and drag on inclination for 2 Low Earth Orbits

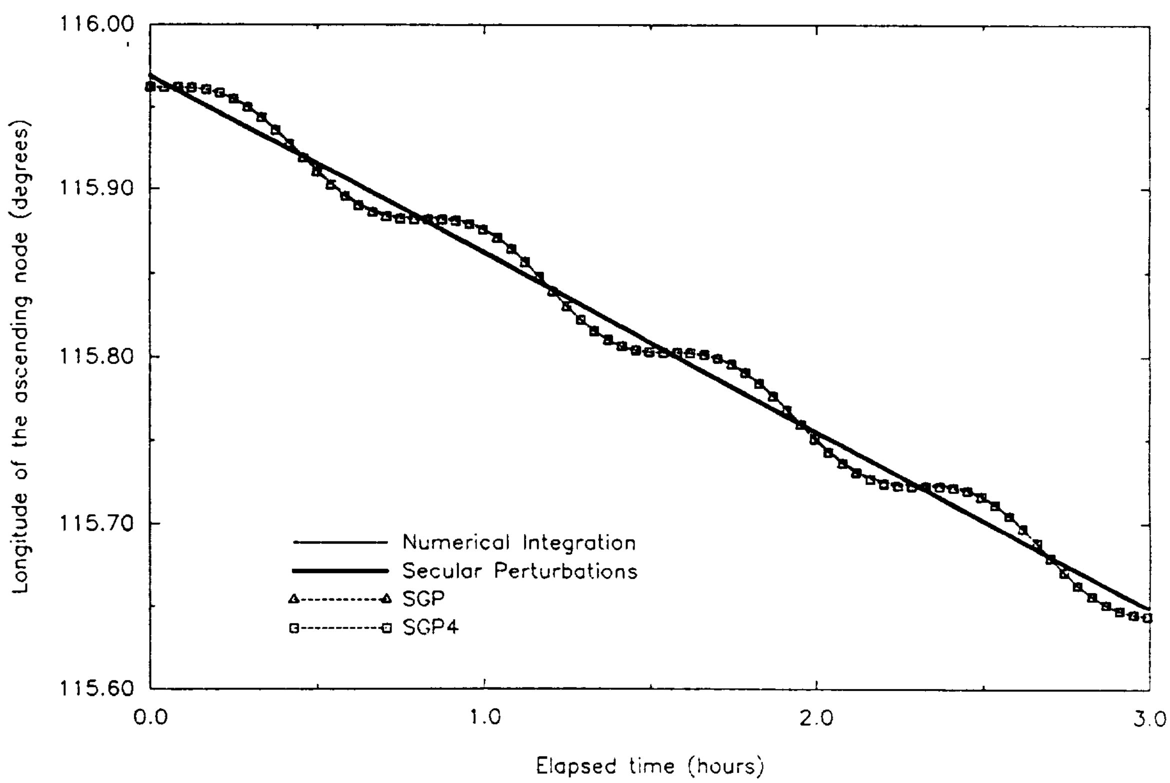

Figure 4. Effects of gravity and drag on longitude of the ascending node for 2 Low Earth Orbits

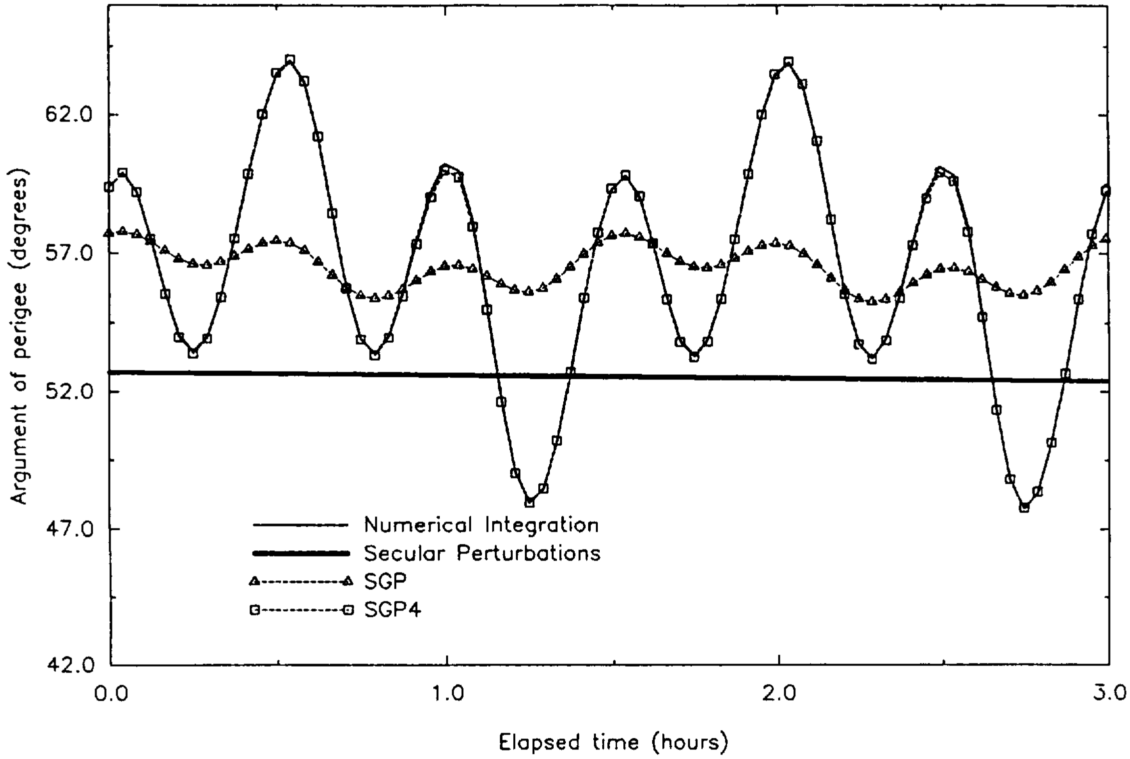

Figure 5. Effects of gravity and drag on argument of perigee for 2 Low Earth Orbits

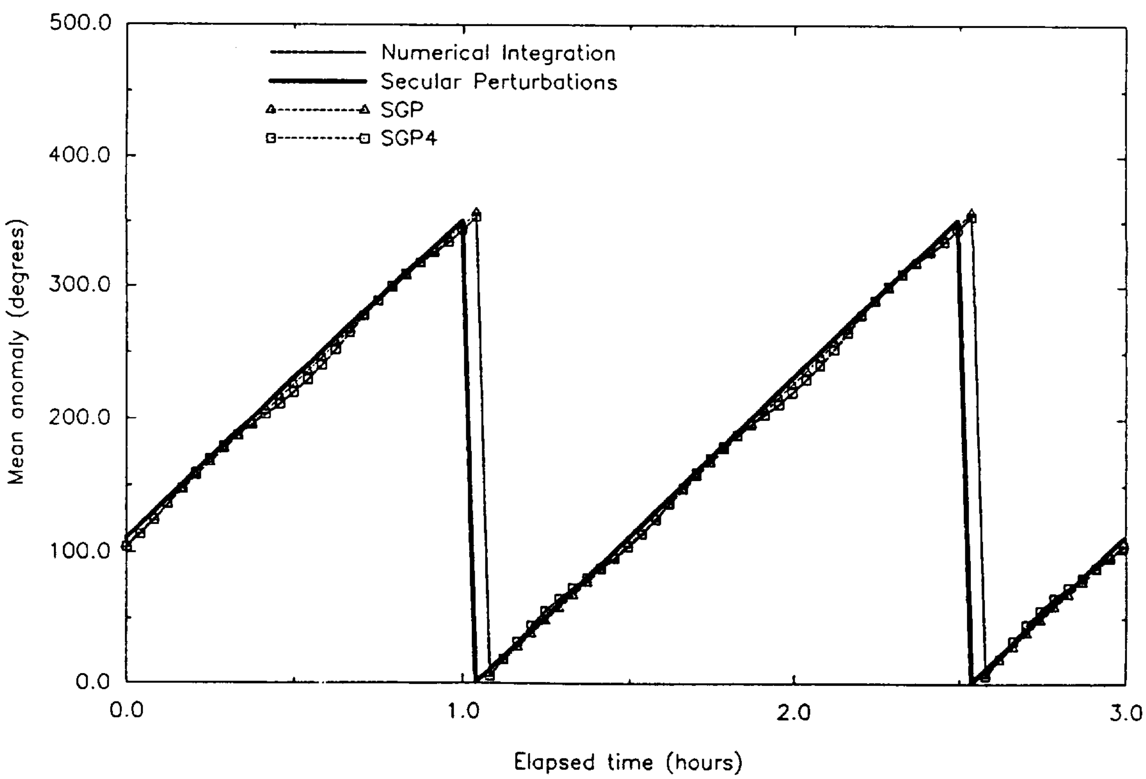

Figure 6. Effects of gravity and drag on mean anomaly for 2 Low Earth Orbits