# SOME CONSIDERATIONS IN THE DETERMINATION OF THE ACCURACY OF A MEASUREMENT IN SPACE OF THE NEWTONIAN GRAVITATIONAL CONSTANT (G)


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## ABSTRACT

A commonly suggested method for determining the Newtonian constant of universal gravitation (G) is to observe the motion of two bodies of known mass moving about each other in an orbiting laboratory. In low Earth orbit (LEO), bodies constructed of even the densest material available experience a gravitational attraction that is several times smaller than the "tidal" forces (due to their proximity to the Earth), which tend to pull them apart. While the tidal forces do not preclude stable orbits of the two objects about each other, they and the Coriolis force (in the rotating laboratory) dominate the motion, and the gravitational attraction of the two bodies may be considered a weak (but significant) contribution to the motion. As a result, compared to an experiment that would be performed in a laboratory far from the Earth, greater accuracy of measuring the motion of the two bodies may be required for a given accuracy in the determination of G. We find that the accuracy with which positions must be determined is not much different in an experiment in LEO than in one performed far from the Earth, but that rotational periods must be determined more accurately. Using a curvature matrix analysis, we also find that a value of G may be extracted (with some loss in accuracy, but probably some practical gain) from an analysis of the time dependence of the distance between the bodies rather than of a full specification (distance and direction) of their relative positions. A measurement of the gravitational constant to one part in $10^4$ continues to be thinkable, but one part in $10^5$ will be very difficult.

## INTRODUCTION

A commonly suggested method for determining the Newtonian constant of universal gravitation (G) is to observe the motion of two bodies of known mass moving about each other in an orbiting laboratory. In low Earth orbit (LEO), bodies constructed of even the densest material available experience a gravitational attraction that is several times smaller than the "tidal" forces (due to their proximity to the Earth), which tend to pull them apart. While the tidal forces do not preclude stable orbits of the two objects about each other, they and the Coriolis force (in the rotating laboratory) dominate the motion, and the gravitational attraction of the two bodies may be considered a weak (but significant) contribution to the motion. As a result, compared to an experiment that would be performed in a laboratory far from the Earth, greater accuracy of measuring the motion of the two bodies may be required for a given accuracy in the determination of G. In this report, we show that the accuracy with which positions must be determined is not much different in an experiment in LEO than in one performed far from the Earth, but that rotational periods must be determined more accurately. In the final section of this report, we briefly describe a curvature matrix analysis which shows that a value of G may be extracted (with some loss in accuracy, but probably some practical gain) from an analysis of just the time dependence of the distance between the bodies rather than of a full specification (distance and direction) of their relative positions.

If a system of two balls is far from any other objects, the balls can move around each other in elliptical Keplerian orbits, where the relationship between the gravitational constant G, the period T of the motion, the average (half the sum of the minimum and the maximum) separation a of the balls, and the total mass m of the system is given by

$$G = 4\pi^2 a^3/mT^2 \quad . \tag{1}$$

This leads us to the following result. If a, T, and m are measured with accuracies $\Delta a$, $\Delta T$, and $\Delta m$, respectively, we may determine the fractional error in G from the fractional errors in a, T, and m as follows:

$$\left(\frac{\Delta G}{G}\right) = 2\left(\frac{\Delta T}{T}\right) + 3\left(\frac{\Delta a}{a}\right) + \left(\frac{\Delta m}{m}\right) \quad . \tag{2}$$

Thus, to achieve a given fractional accuracy in the determination of G, the fractional accuracy of the period must be at least two times better, the fractional accuracy of the orbit size must be at least three times better, and the fractional accuracy of the mass must be at least as good.

## SIMPLE ORBITS IN LOW EARTH ORBIT

For the motion of two balls in an orbiting laboratory in a low Earth orbit (LEO), we must take into account the tidal forces and the Coriolis forces and integrate numerically the equations of motion (assuming the laboratory is in a circular orbit about a spherically symmetric Earth). It turns out that two balls may still orbit about one another in a stable fashion, but the motion is complicated by the presence of the Earth. I will lay out the equations of motion here, and I will then comment on them in the following paragraph.

The equations of motion are

$$\ddot{x} = 2\dot{z} - \rho\, x\, [x^2 + y^2 + z^2]^{-3/2}$$

$$\ddot{y} = -y - \rho\, y\, [x^2 + y^2 + z^2]^{-3/2}$$

$$\ddot{z} = 3z - 2\dot{x} - \rho\, z\, [x^2 + y^2 + z^2]^{-3/2} \qquad (3)$$

where x is along an axis in the direction of motion of the orbiting laboratory, z is along an axis directed toward the Earth from the laboratory (NASA convention), and y is the third Cartesian coordinate. The distances x, y, and z are measured in dimensionless units relative to a characteristic length d in the laboratory, time is measured in dimensionless units relative to the reciprocal of the angular velocity of the laboratory about the Earth, and the quantity $\rho$ is defined:

$$\rho \equiv (m/M)(A/d)^3 \qquad (4)$$

where m is the mass of the two-ball system, M is the mass of the Earth, and A is the radius of the laboratory's orbit about the Earth.

The tidal forces (the first terms on the right side of the expressions for $\ddot{y}$ and $\ddot{z}$ ) and the Coriolis forces (the velocity dependent terms) can be large compared to the gravitational attraction (the last term in all three expressions). It turns out that if the two balls are of equal mass and are made of material the density of tungsten, and if d (the length scale) is chosen to be the distance between their centers when they are in contact, the quantity $\rho$ is very nearly equal to unity in LEO (for sintered tungsten balls of 10 kg each, density about 19.1 $g/cm^3$, the distance between their centers when they are touching is about 10 cm.) Thus, in LEO, for real balls moving about one another, the gravitational attraction term will be small compared to the average values of the other terms in Eq. (3), and the motion will be only gently (but for our purposes still importantly) affected by the gravitational attraction between the two balls.

An example of a possible stable orbit of two balls about each other is shown in Figure 1, which shows the motion of one ball relative to the other. Parameters of the motion are described in the caption. Generally, the orbits (unlike the Keplerian orbits) are not closed. Moreover, the initial conditions must be chosen carefully to give motion in which the particles remain close for a long time. Fig 2 shows an example of the relative motion of the balls in which they do not stay close together for long. For the purposes of the next section, it was necessary to determine the parameters for the special case of closed orbits, several of which are shown in Fig 3. Here we see that large orbits have periods approaching $2\pi$ and a motion with is simple harmonic in x and in z, with a ratio of amplitudes of 2. Smaller orbits are more nearly circular and have shorter periods. They are ovals but they are not ellipses.

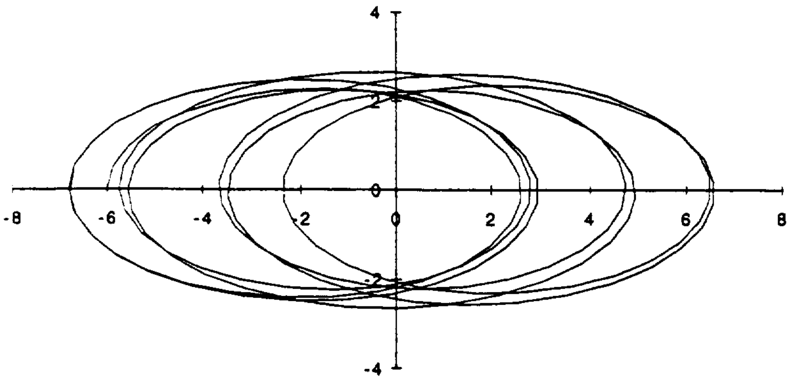


Figure 1.— Relative position in xz plane of two balls moving about each other in LEO. The x direction (horizontal axis) is in the direction of motion of the laboratory, and the z direction (vertical axis) is toward the Earth from the laboratory. $\rho = 1$, $x_o$=-6.0, $v_{ox} = 0.1$, $v_{oz} = 2.5$, all other initial values of coordinates and velocity components are zero. The motion is stable in the sense that the balls remain within about 7 length units indefinitely. The trajectory shown is for the time interval between $t = 0$ and $t = 40\pi/3$ (6.67 orbits of the laboratory about the Earth).

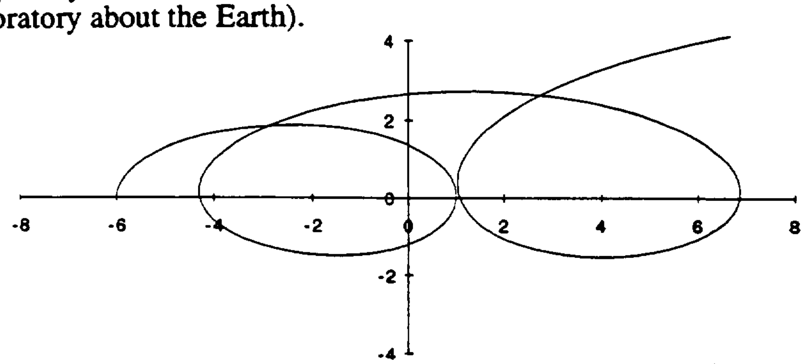


Figure 2.— Relative position in xz plane of two balls moving about each other in LEO. $\rho = 1$, $x_o$=-6.0, $v_{ox} = 0.1$, $v_{oz} = 2.1$, all other initial values of coordinates and velocity components are zero. the motion is unstable, and after just one loop around the origin, the trajectory continues to the right in a cycloidal fashion. The trajectory shown is for the time interval between $t = 0$ and $t \cong 4\pi$ (about two orbits of the laboratory about the Earth).

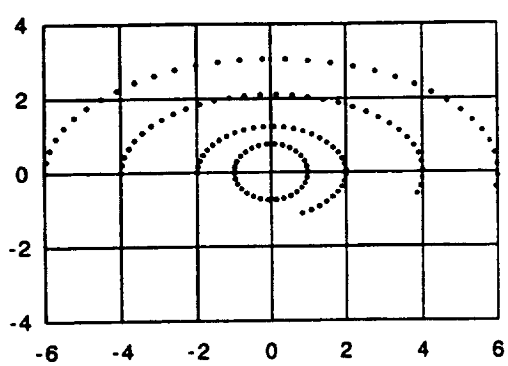


Figure 3.— Relative position in xz plane of two balls moving about each other in LEO. $\rho = 1$. The initial values of coordinates and velocity components have been chosen to give closed orbits. The time interval between dots is 0.10; for the largest orbits, the period is close to $2\pi$, for smaller orbits the period is smaller.

## ERRORS ASSOCIATED WITH SIMPLEST ORBITS

Equations (3) tell us that when the particles are close together, the gravitational attractions between them is more important than the tidal and Coriolis forces. (While this is not possible in LEO, it would be possible if the laboratory were farther from the Earth.) The motion of the two balls is in Keplerian orbits, and the relationship between period (dimensionless units), maximum separation a (dimensionless units), and $\rho$ is

$$a^3/T^2 = \rho/4\pi^2 \tag{5}$$

where $\rho = (m/M)(A/d)^3$. Note the similarity to the expression for the Kepler motion of an isolated binary; Gm is simply replaced by $\rho$.

The task of measuring G becomes the task of measuring $\rho$. From the above relation, Eq.(5), we see that, far from the Earth,

$$\left(\frac{\Delta\rho}{\rho}\right) = 2\left(\frac{\Delta T}{T}\right) + 3\left(\frac{\Delta a}{a}\right) \tag{6}$$

The issue is: How does $\left(\frac{\Delta\rho}{\rho}\right)$ depend on $\left(\frac{\Delta T}{T}\right)$ and $\left(\frac{\Delta a}{a}\right)$ when the experiment is carried out in LEO? That is, what happens to the factors 2 and 3 in the equation above? (Of course, getting from $\rho$ to G also involves knowing the mass of the binaries, the length scale, and the radius of the orbit of the laboratory about the Earth. Since we should be able to know these to the $1:10^6$ level, we will ignore them now, so the accuracy with which $\rho$ is determined is practically the accuracy with which G is determined. )

Here is a description of the calculation. With $\rho$ set equal to 1, for each of several values of a (maximum separation) we found the orbit which was closed and determined the period of the orbit. We also determined numerically the partial derivative of T with respect to a, $(\partial T/\partial a)$, and the partial derivative of T with respect to $\rho$, $(\partial T/\partial \rho)$. We can then determine the fractional change in $\rho$ which results from given fractional changes in a and T, as follows.

We may write

$$dT = \frac{\partial T}{\partial a}\, da + \frac{\partial T}{\partial \rho}\, d\rho\ , \tag{8}$$

which may be rearranged to give

$$d\rho = \frac{dT}{\left(\frac{\partial T}{\partial \rho}\right)} - \frac{\left(\frac{\partial T}{\partial a}\right)}{\left(\frac{\partial T}{\partial \rho}\right)} da \ . \qquad (9)$$

Dividing all terms by ρ and multiplying numerator and denominator of the two terms on the right side of the equal sign by T or a, respectively, we obtain

$$\left(\frac{\Delta\rho}{\rho}\right) = \left(\frac{dT}{T}\right)\left(\frac{T}{\rho\left(\frac{\partial T}{\partial \rho}\right)}\right) - \left(\frac{da}{a}\right)\left(\frac{a\left(\frac{\partial T}{\partial a}\right)}{\rho\left(\frac{\partial T}{\partial \rho}\right)}\right) = \mathcal{A}\left(\frac{dT}{T}\right) + \mathcal{B}\left(\frac{da}{a}\right) \qquad (10)$$

Eq. (10) indicates how the coefficients $\mathcal{A}$ and $\mathcal{B}$ are calculated. If the orbits were Keplerian (small a, assuming ρ is unity), the values of $\mathcal{A}$ and $\mathcal{B}$ would be 2 and 3, respectively, as we see in Eq. (6). The actual values for larger, and potentially usable values of a, are shown in Figure 4. What we find is that the ρ becomes much more sensitive to the time measurement. This is reasonable because as the orbits become larger, the effect of the gravitational attraction of the masses becomes smaller, and the period becomes practically independent of the size of the orbit. What may be surprising (it is surprising to me) is that the fractional precision with which the size of the orbit must be determined varies very little from the value of 3 for the values of a in our region of interest.

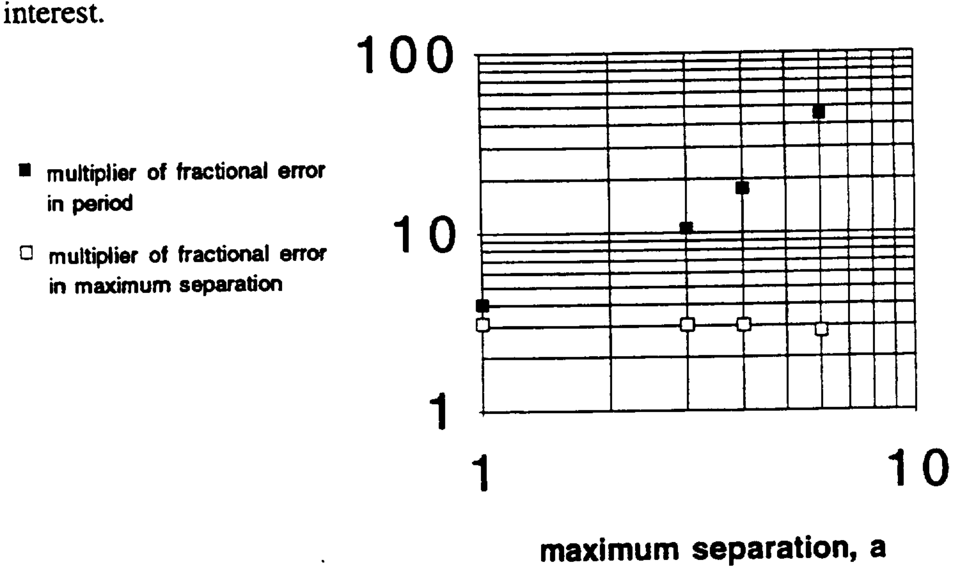


Figure 4.— Values of the coefficients $\mathcal{A}$ (solid squares) and $\mathcal{B}$ (open squares) versus the size of the orbit of the relative motion of the two balls. See text for explanation.

We can now understand the effect of LEO on the accuracy with which we need to measure the orbital motion to extract G. What we find is that the period must be measured with much greater fractional accuracy, whereas the size of the orbit still needs to be measured with about the same fractional accuracy as before.

## CURVATURE MATRIX ERROR ANALYSIS

In the analysis of an actual experiment, data points, consisting of position and time measurements, will be fit to a numerical model of the motion. The parameters of the model (initial position and velocity, strength of the gravitational interaction, effect of other masses) are determined by a least squares search process and the precision of the parameters is deduced form the curvature of the chi-square hypersurface in parameter space. We may first consider a slightly simpler problem. We calculate the orbits with parameters and produce positions at regular intervals of time. By changing each of the parameters and each pair of parameters by small amounts, we can calculate a chi-square which characterizes the deviation of the modified set of positions from the original set. From these chi-squares, we determine the curvature matrix which describes the dependence of chi-square on the parameters. Inverting this curvature matrix then gives an indication of the sensitivity of the motion to each of the parameters, in particular to the gravitational attraction between the members of the orbiting binary. One would expect this analysis, which is more sophisticated than the analysis of the simplest, closed orbits, described in the previous section, will agree with the error analysis of the simple closed orbits but can be extended to orbits which are not closed and which reflect more complex environments.

We have analyzed an orbit of two 10 kg tungsten balls with parameters d = 10 cm, $\rho = 1$, $x_0=-6.0$, $v_{OX} = 0.1$, $v_{OZ} = 2.1$, all other initial values of coordinates and velocity components being zero. Supposing the position is determined at 100 points, each with a precision of about 10 µm, at time intervals of 0.25 (for a total time of about 6 orbits of the laboratory about the Earth), we find that the precision with which $\rho$ (for these assumptions) is determined is about $2.7 \times 10^{-5}$, consistent with the simpler analysis described above. This indicates that a determination of G to one part in $10^4$ is thinkable (but one part in $10^5$ will be difficult).

It is interesting to note that one may also perform such an analysis with the chi-square calculated not from the relative positions of the particles (separation distance and direction) but from the deviation of just the separation of the particles (in analogy with the analysis of lunar laser ranging data). Some information is lost in doing this, but for the same conditions as in the previous paragraph, the precision with which $\rho$ is determined is about $4.4 \times 10^{-5}$, not much worse than before.

In conclusion, a measurement (of the kind discussed above) in LEO of the gravitational constant to one part in $10^4$ continues to be thinkable, but one part in $10^5$ will be very difficult.