NASA-TM-112176

THE EFFECT OF NONLINEAR CRITICAL LAYERS ON BOUNDARY LAYER TRANSITION

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Abstract

Asymptotic methods are used to describe the nonlinear self-interaction between pairs of oblique instability modes that eventually develops when initially linear and spatially growing instability waves evolve downstream in nominally two-dimensional and spanwise periodic laminar boundary layers. The first nonlinear reaction takes place locally within a so-called "critical layer" with the flow outside this layer consisting of a locally parallel mean flow plus an appropriate superposition of linear instability waves. The amplitudes of these waves are determined by either a single integro-differential equation or by a pair of integro-differential equations with quadratic to quartic-type nonlinearities.

1. Introduction

Transition to turbulence in boundary layers frequently begins with initially linear and noninteracting instability waves that grow to nonlinear amplitudes as they propagate downstream. This phenomenon is usually studied experimentally by artificially exciting the flow with small-amplitude, nearly two-dimensional and single-frequency excitation devices. Lack of two-dimensionality in the excitation device can produce streamwise vortices, which can easily be accounted for if the cross-flow velocities are sufficiently small by calculating the mean flow from the three-dimensional boundary region equations of Davis and Rubin (1980), rather than from the usual two-dimensional boundary layer equations (Wundrow and Goldstein, 1994).

The initial unsteady motion should then have harmonic time dependence and be well described by linear instability theory. As the instability waves propagate downstream, they continue to grow, and if they get large enough, nonlinear effects will come into play. This discussion is concerned with this first nonlinear stage of evolution, which is usually characterized by the rapid growth of three-dimensional disturbances due to resonant interactions between instability waves and between instability waves and streamwise vortices.

The instability-wave growth rates should be small compared to the inverse boundary layer thickness Δ^{-1} in subsonic two-dimensional, flat-plate boundary layers, but can be of the same order as Δ^{-1} in two-dimensional supersonic boundary layers as well as in boundary layers with sufficiently strong streamwise vortices at any Mach number. However, the growth rates will frequently be small (relative to Δ^{-1}) by the time nonlinear effects set in, even in these more unstable flows. This can result from flow divergence effects in two-dimensional mean flows because the local growth rate will usually increase, reach a maximum, and then go to zero as the local Strouhall number increases (as shown in figure 1), while the excitation is usually located in the vicinity of the peak local growth rate in most experiments. The growth rate should therefore decrease as the (constant frequency) instability waves propagate downstream into a region where (in most cases) the boundary-layer thickness Δ will have increased. The ultimate viscous decay of the streamwise vortex system will, of course, cause the instability wave growth to approach zero in three-dimensional flows where the wave growth is produced by a streamwise vortex field (Goldstein and Wundrow, 1995).

figure 1

This suggests that the method of matched asymptotic expansions can be used to describe these flows: with an "inner" nonlinear region, in which the instability-wave growth rate is small, and a much larger "outer" region in which the unsteady flow is governed by linear dynamics, but in which mean-flow divergence effects are important. (Goldstein and Leib, 1988; Hultgren, 1992) Once the solutions in these two regions have been found, a uniformly valid composite solution that applies everywhere in the linear and nonlinear regions can be obtained in one of the usual ways--say, by multiplying the linear and nonlinear solutions together and then dividing through by their common part in the overlap domain (that always exists between the inner and outer regions).

2. The Outer Linear Flow

We first consider the initial linear region just downstream of the excitation device where the instability waves are still small enough so that no significant modal interactions take place. At supersonic Mach numbers--below about 6 or so--where the so-called, first-mode instability is dominant (Mack, 1984 and 1987), the most rapidly growing modes on a two-dimensional, flat-plate boundary layer are the oblique instability waves, and the first modal interaction to take place is likely to be the self-interaction between symmetric pairs of oblique instability waves (Leib and Lee, 1995). In which case, it is appropriate to suppose that the unsteady motion is initiated from a pair of oblique (equi-amplitude) instability wave modes with the same streamwise wave number α_r and scaled angular frequency $\omega \dagger \Delta / U_\infty = \alpha_r c_r$ and equal and opposite spanwise wave numbers ($\pm\beta$). (U_∞ is the characteristic velocity of the flow, and the subscript r is used to denote the real part of the wave number α and the phase speed c, as well as all other quantities to which it is appended.) These two waves combine to form a standing wave in the spanwise direction that propagates only in the direction of flow--which is the situation that most frequently occurs in wave excitation experiments that typically involve longish excitation devices placed perpendicular to the flow.

The two-dimensional mode usually exhibits the most rapid growth at subsonic speeds provided, of course, that the mean flow is sufficiently two-dimensional. However, even very weak spanwise periodic mean-flow distortions (i.e., streamwise vortices) can cause the oblique modes to grow faster than the plane wave at the high Reynolds numbers being considered herein. In fact, all instability waves will behave like oblique modes when the streamwise vortices are sufficiently strong (Wundrow and Goldstein, 1994). In which case, the first nonlinear interaction will again be a self-interaction between oblique modes, which is of the same type as in the previous case (Wundrow and Goldstein, 1994).

But, even when no streamwise vortices are present (or when they are very weak) and the mean flow is effectively two-dimensional, the oblique modes can eventually exhibit the most rapid growth upon entering some intermediate (or parametric resonance) stage. The oblique modes can then become large enough to interact with themselves nonlinearly upon passing through this stage, and the resulting nonlinear interaction will be the same as in the previous two cases.

The resonant interaction stage can be treated simultaneously with the self-interaction stage if the unsteady motion is initiated from a resonant triad of instability waves in the initial linear region--a plane fundamental frequency wave, with scaled angular frequency $2\omega \dagger \Delta / U_\infty$ and a pair of oblique equi-amplitude subharmonic waves, (again) with the same streamwise wave number and angular frequency, α_r and $\alpha_r c_r$, respectively, but equal and opposite spanwise wave numbers $\pm\beta$.

In the present context, the importance of the "resonance" condition is that it implies, among other things, that the three waves all have the same phase speed c_r. This occurs (for the small growth rates and large Reynolds numbers that are of interest here) when

$$\beta = \sqrt{3}\,\alpha_r \,, \tag{1}$$

which means that the oblique instability waves make a 60° angle with the direction of flow. We can, of course, allow this angle to be arbitrary in flows where an oblique mode can grow more rapidly than the plane wave and resonant interaction with the latter is not required to enhance the growth rate of the former.

It is only possible to develop a systematic asymptotic theory of these phenomena when the Reynolds number R is assumed to be large. Then, since we also require that the instability-wave growth rates be small in the nonlinear region of the flow, the initial modal and nonlinear interactions will be confined to a localized region centered around the "critical level" (Lin, 1957) where the mean-flow velocity, say U_c, is equal to the common phase velocity c_r of the two or three modes that interact there. (See Figure 2.) This may explain why energy exchange between resonant modes (which share a common critical layer) is much more efficient than between nonresonant modes.

figure 2

The flow outside the "critical layer" is still governed by linear dynamics, which means that it is given by a locally parallel mean flow plus an appropriate superposition of linear instability waves. In the most general case, the mean flow will be a unidirectional transversely sheared flow, say U(y,z), which, as indicated in figure 3, means that its velocity is in a single direction but can vary in magnitude in both transverse directions. It is also appropriate to require that the mean flow be periodic in the spanwise direction z in order to represent the streamwise vortices. Thus

$$U(y, z + 2\pi n/\beta_0) = U(y,z) \quad for \quad n = 1, 2, \ldots \,, \tag{2}$$

where $2\pi/\beta_0$ is the spanwise period of the flow. The unsteady pressure fluctuation, say p, will then be determined by the generalized Rayleigh equation (Goldstein, 1976, pp. 6-10) figure) 3

$$\frac{D}{Dt}\left(\nabla^2 p - \frac{1}{C_0^2}\,\frac{D^2}{Dt^2}p\right) - 2\nabla U \cdot \nabla \frac{\partial p}{\partial x} = 0\ , \tag{3}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\ , \tag{4}$$

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial Z} \tag{5}$$

and C_0 is the (assumed constant) sound speed for the flow. The streamwise, transverse, and spanwise coordinates, normalized by the boundary layer thickness Δ, are x, y, and z, respectively, and t denotes the normalized time.

The relevant solutions to Equation (3) are the so-called, normal-mode solutions which are of the form (Henningson, 1987)

$$p = Re\, e^{i(\alpha x - ct)}\,\Phi(y,z) \tag{6}$$

where Re denotes the real part and the normal shape Φ is determined by the reduced generalized Rayleigh equation

$$\nabla_T \frac{1}{(U-c)^2} . \nabla_T \Phi + \left[\frac{1}{C_0^2} - \frac{\alpha^2}{(U-c)^2}\right]\Phi = 0\ , \tag{7}$$

where

$$\nabla_T \equiv \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{8}$$

and the solution Φ is not necessarily periodic in the z-direction with the period $2\pi/\beta_0$ of the mean flow.

The (external) pressure fluctuation p is then given (in the general case) by

$$p = \epsilon Re\left[A(x_0)\Phi(y,Z)e^{iX} + \left(\frac{\epsilon}{\sigma}\right)^{1/3} A_0(x_0)\Phi_0(y,Z)e^{i2X}\right], \tag{9}$$

where

$$X = \sigma\bar{\alpha}(x - \sigma\bar{c}t), \quad Z = \sigma\bar{\beta}z, \tag{10}$$

$$\alpha = \sigma\left[\bar{\alpha} + 0\left(\frac{\epsilon}{(1+\lambda)\sigma}\right)^{\frac{1}{3}}\right], \quad c = \sigma\left[\bar{c} + 0\left(\frac{\epsilon}{(1+\lambda)\sigma}\right)^{\frac{1}{3}}\right], \quad c_0 = \sigma\left[\bar{c} + 0\left(\frac{\epsilon}{(1+\lambda)\sigma}\right)^{\frac{1}{3}}\right], \tag{11}$$

$$x_0 = \sigma\left(\frac{\epsilon}{\sigma}\right)^{1/3} x, \tag{12}$$

and the scale factor $\sigma \leq 1$ has to be inserted in order to simultaneously cover the 0(1) and long wavelength cases. The scaled spanwise wave number, streamwise wave number and phase speed $\bar{\beta}$, $\bar{\alpha}$, and $\bar{c}$, respectively, are purely real.

The first term on the right-hand side of Equation (9) represents the subharmonic mode, while the second represents the fundamental mode. Φ and Φ_0, are the linear normal mode shapes which can be determined from Equation (7) in the general case and for a two-dimensional mean flow are given by

$$\Phi = \phi(y)\cos Z \quad \textit{and} \quad \Phi_0 = \phi_0(y) \tag{13}$$

which shows that the fundamental mode is independent of Z, i.e., that it is a plane wave and that the subharmonic mode is a supposition of oblique modes that combine to form a standing wave in the spanwise direction.

A and A_0, which depend only on the streamwise coordinate (and then only through the scaled streamwise variable x_0, which varies on the length scale of the nonlinear region which, not very surprisingly, turns out to be the reciprocal instability-wave growth rate) determine the overall growth of the instability waves. They are completely determined by the nonlinear dynamics within the critical layer and are, in practice, found by equating the velocity jump across the critical layer, as calculated from the external linear solution (i.e., the solution to Rayleigh's equation), to the velocity jump calculated from the internal nonlinear solution within the critical layer. ϵ and $\epsilon(\epsilon/\sigma)^{1/3}$ are the amplitude scale factors for the oblique and plane waves, respectively, where ϵ is always much less than σ.

Notice that the growth-rate and subharmonic amplitude scalings $\sigma(\epsilon/\sigma)^{1/3}$ and ϵ, respectively, are related. This relation ensures that linear growth and nonlinear (or modal interaction) effects will both impact the external linear solution at the same asymptotic order. It is dictated by the requirement that the nonlinear stage correspond to the first stage of evolution beyond the initial linear region, i.e., that the nonlinear solutions match onto the upstream linear solutions in the matched asymptotic sense. The Benney-Bergeron (1969) parameter $\lambda \equiv 1/\epsilon\sigma^3 R$, where R is the Reynolds number based on the boundary layer thickness Δ, is (in the present context) a measure of the relative importance of viscous to growth-rate effects within the critical layer. The wavelength scale factor σ can be set to unity when the initial linear instability wave has order-one wavelength.

3. Critical Layer Dynamics and the Amplitude Equations

The lowest order critical-layer equations turn out to be linear and (in the most general case) correspond to a balance between growth (i.e., nonequilibrium), mean-flow convection, and viscous-diffusion effects. The nonlinear and modal interaction effects are weak in the present description. Which means that they do not affect the lowest order equations, but enter only through inhomogeneous terms in a higher order problem. This ultimately implies that the scaled subharmonic amplitude function A can be determined from a single amplitude equation or that the amplitude functions A and A_0 can be determined from a pair of amplitude equations--depending on whether or not the parametric resonance interaction plays a role in the development. In the former case, the relevant equation corresponding to the generalized scaling (10) through (12) is given by Goldstein and Choi (1989), Wu, Lee, and Cowley (1993), Goldstein and Wundrow (1995), Leib and Lee (1995), and Wundrow and Goldstein (1994) as

$$\frac{d\tilde{A}}{d\tilde{x}} = \tilde{\kappa}\tilde{A} + i\tilde{\gamma}\int_{-\infty}^{\tilde{x}}\int_{-\infty}^{x_1} K\tilde{A}(x_1)\tilde{A}(x_2)\tilde{A}^*(x_1 + x_2 - \tilde{x})dx_2dx_1 \tag{14}$$

and, in the latter case, are given by Goldstein and Lee (1992, 1993), Wu (1992), Goldstein (1994), and Mallier and Maslowe (1994) as

$$\frac{d\tilde{A}(\tilde{x})}{d\tilde{x}} = \tilde{\kappa}\tilde{A}(\tilde{x}) + i\int_{-\infty}^{\tilde{x}} K_1\tilde{A}_0(x_1)\tilde{A}^*(2x_1 - \tilde{x})dx_1$$

$$+ i\tilde{\gamma}\int_{-\infty}^{\tilde{x}}\int_{-\infty}^{x_1} K_0\tilde{A}(x_1)\tilde{A}(x_2)\tilde{A}^*(x_1 + x_2 - \tilde{x})dx_2dx_1 , \tag{15}$$

$$\frac{d\tilde{A}_0(\tilde{x})}{d\tilde{x}} = \tilde{\kappa}_0 \tilde{A}_0(\tilde{x}) +$$

$$i\tilde{\rho}\tilde{\gamma}\int_{-\infty}^{\tilde{x}}\int_{-\infty}^{x_1}\left[K_2\tilde{A}_0(x_1)\tilde{A}(x_2)\tilde{A}^*(2x_1+x_2-2\tilde{x})+K_3\tilde{A}(x_1)\tilde{A}_0(x_2)\tilde{A}^*(x_1+2x_2-2\tilde{x})\right]dx_2dx_1$$

$$+\, i\tilde{\rho}\tilde{\gamma}^2\int_{-\infty}^{\tilde{x}}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}K_4\tilde{A}(x_1)\tilde{A}(x_2)\tilde{A}(x_3)\tilde{A}^*(x_1+x_2+x_3-2\tilde{x})dx_3dx_2dx_1\,,$$

(16)

where the asterisks denote complex conjugates; $\tilde{x}$, $\tilde{A}$ and $\tilde{A}_0$ are suitably renormalized, and shifted variables corresponding to x_0, A, and A_0, respectively; and $\tilde{\rho}$ and $\tilde{\gamma}$ are complex parameters which are dependent on the basic mean flow. The real part of $\tilde{\kappa}$ is the linear growth rate of the subharmonic mode. The real part of $\tilde{\kappa}_0$ is the scaled linear growth rate of the fundamental mode. The imaginary part of κ_0, $\tilde{\kappa}_{0i}$, represents the initial wave-number shift between the subharmonic and fundamental modes.

Notice that these are integro-differential equations of the type first proposed for Rossby waves on two-dimensional flows by Hickernell (1984), rather than the usual ordinary differential equations of the classical Stuart (1960)-Watson (1960)-Landau (Landau and Lifshitz, 1987) theory. The integrals arise because the evolution or growth effects have a dominant (i.e., first order) effect on the flow within the critical layer, but only weakly affect the flow outside the critical layer. Therefore, the nonlinear (or wave interaction) terms are influenced by the growth effects (which produce the integrals) when these terms are generated within the critical layer (as they are in the present approach). The integrals will not, of course, appear when the nonlinear effects are generated outside the critical layer, as in the classical Stuart-Watson-Landau theory.

The nonlinear kernel functions K, K_0, and K_1 through K_4 turn out to be simple polynomial functions of the streamwise (and corresponding integration) variables in the inviscid limit $\lambda \rightarrow 0$ and in the general case involve integrals of exponentials and polynomials of the streamwise coordinates. K_0 is a special case of K corresponding to the resonant condition (1) or its generalization for a spanwise variable mean flow.

To be consistent with our requirement that the solutions evolve from an initially linear stage, the amplitude Equations (14) or (15) and (16) usually have to be solved subject to the upstream boundary conditions

$$\tilde{A} \rightarrow a^{(0)}e^{\tilde{\kappa}\tilde{x}},\ \tilde{A}_0 \rightarrow e^{\tilde{\kappa}_0\tilde{x}} \text{ as } \tilde{x} \rightarrow -\infty, \tag{17}$$

that they match onto the linear, small growth-rate solution far upstream--but see Goldstein (1995) and Wundrow, Hultgren, and Goldstein (1994) for an important exception to this. Notice that only the first term on each of the right-hand sides of Equations (15) and (16) contributes to these equations when A and A_0 are sufficiently small--as they are initially--and that (17) is then an exact solution to the resulting equations. We include the linear wave-number shift $\tilde{\kappa}_{0i}$ to allow for an appropriate amount of wave-number detuning in the analyses, which means that the resonance (1) does not necessarily have to be exact and that the analysis actually applies to a relatively broad wave-number range about this resonance condition.

4. The Mean Flow Change

An important feature of the present high-Reynolds-number approach is that the nonlinear critical-layer interaction produces a spanwise-variable, mean-flow change

$$u = \epsilon u_0(y,z,x_0), \tag{18}$$

that is of the same order as the subharmonic instability wave (see Equation (9)) that initially produces the interaction. However, the associated cross-flow velocities

$$v = \sigma\epsilon\left(\frac{\epsilon}{\sigma}\right)^{\frac{1}{3}} v_0, \quad w = \epsilon\left(\frac{\epsilon}{\sigma}\right)^{\frac{1}{3}} w_0 \tag{19}$$

turn out to be somewhat smaller than this. For a two-dimensional mean flow

$$\left(u_0, v_0, w_0\right) = Re\left(\bar{u}_0(y,x_0)e^{2iZ}, \bar{v}_0 e^{2iZ}, \bar{w}_0 e^{2iZ}\right) \tag{20}$$

In the remainder of the paper, we discuss the implications of the fundamental Equations (14) to (16).

5. The Resonant Diad Interaction

We begin by considering the case where the interacting modes are all of the same frequency. As already noted, this situation is relevant to two-dimensional supersonic boundary layers and to flows at any Mach number when sufficiently strong streamwise vortices are present.

The modal amplitude is now determined by Equation (14). Its kernel function K is relatively complicated when viscous effects are retained, as in Wu, Lee, and Cowley (1993), but in the inviscid limit originally considered by Goldstein and Choi (1989), Goldstein and Wundrow (1995), and Wundrow and Goldstein (1994), it is simply

$$K = \left(\bar{x}-x_1\right)\left[k_2\left(\bar{x}-x_1\right)^2 + k_3\left(\bar{x}-x_2\right)^2 - k_1\left(\bar{x}-x_2\right)\left(x_1-x_2\right)\right] , \tag{21}$$

where the constants k_1, k_2, k_3 depend on the topology of the mean flow as well as on the model structure of the upstream linear instability wave.

For a two-dimensional mean flow, k_1, k_2, k_3 depend only on the obliqueness angle

$$\theta = \tan^{-1}(2\bar{\beta}/\bar{\alpha}) \tag{22}$$

of the linear instability waves and are given by

$$k_1 = -\frac{1}{2}\tan^2\theta\cos^2 2\theta \quad \textit{and} \quad k_2 = k_3 = -\frac{1}{2}\tan^2\theta\cos 2\theta \tag{23}$$

K will then vanish when $\theta = \pi/4$, and the inviscid solution to (14) will develop a singularity at a finite downstream position (Goldstein and Choi, 1989), say $\tilde{x}_s$, at all other angles. $\tilde{A}$ therefore exhibits explosive growth as $\tilde{x} \to \tilde{x}_s$, and the local asymptotic behavior is then given by (Goldstein and Choi, 1989, and Shukhman, 1991)

$$\tilde{A} \sim \frac{a}{\left(\tilde{x}_s - \tilde{x}\right)^{3+i\psi}} \qquad \text{as } \tilde{x} \to \tilde{x}_s, \tag{24}$$

where the real parameters a and ψ are related to the original parameters $\tilde{\kappa}$ and $\tilde{\gamma}$ through quadratures.

Figure 4 is a plot of the scaled amplitude function $\tilde{A}$ versus the scaled streamwise coordinate $\tilde{x}$, as calculated numerically from Equations (14), (21), and (23) for $\tilde{\alpha} = 1.2$ and various values of θ. The curves show that the solution initially exhibits the linear growth and that the explosive growth occurs very suddenly once nonlinearity comes into play. The dashed curves are the local asymptotic expansions calculated from Equation (24). This latter result implies that the overall wave-number/growth-rate scaling is preserved right up to the singularity when $\sigma = 1$, which means that the overall asymptotic structure remains intact until the instability wave

figure) 4

amplitude becomes 0(1) everywhere in the flow, and that the motion is then governed by the full nonlinear Euler equations in the next stage of evolution.

However, the growth-rate amplitude scaling is not preserved in the long wavelength limit $\sigma \to 0$ (corresponding to, say, the weak streamwise vortex amplification mechanism of Goldstein and Wundrow, 1995). In this case, the critical layer expands to fill the inviscid wall layer that surrounds the critical layer, causing the flow to become fully nonlinear while the instability amplitudes are still small. The next stage of evolution is then characterized by a three-layer structure and is governed by the three-dimensional, unsteady "triple deck" equations, but without the viscous terms (Goldstein and Lee, 1992). This does not, however, imply that the relevant scaling is the usual triple-deck scaling in this stage.

Wu, Lee, and Cowley (1993) showed that explosive growth also occurs in the general viscous case for two-dimensional mean flows and that the local asymptotic behavior in the vicinity of the singularity is still given by (24). However, they also showed that (as in Goldstein and Leib, 1989; and Leib, 1991) there is a certain range of parameters where explosive growth does not occur when the viscous parameter λ exceeds a certain (usually very large) value. The instability wave will then reach a peak amplitude at some fixed streamwise location and subsequently undergo viscous decay downstream of that point.

6. The Resonance and Triad Interaction

Now suppose that the scaled subharmonic amplitude A is very small and remains that way during the entire resonant interaction. Notice that this includes the case $A = 0(\epsilon/\sigma)^{\frac{1}{3}}$ where the subharmonic mode has the same amplitude scaling as the fundamental (Goldstein and Lee, 1992).

The last term can be neglected on the right-hand side of Equation (15), which then becomes

$$\frac{d\tilde{A}}{d\tilde{x}} = \tilde{\kappa}\tilde{A} + i\int_{-\infty}^{\tilde{x}} K_1 \tilde{A}_0(x_1)\tilde{A}^*(2x_1 - \tilde{x})dx_1, \tag{25}$$

while the fundamental-mode amplitude Equation (16) reduces to the linear growth-rate equation

$$\frac{d\tilde{A}_0}{d\tilde{x}} = \tilde{\kappa}_0\tilde{A}_0, \tag{26}$$

which merely reflects the fact that there is no backreaction of the subharmonic mode on the fundamental. It may seem rather surprising that this occurs even when the subharmonic-mode amplitude is much larger than that of the fundamental, but the critical-layer velocity jump that would produce backreaction at this level turns out to be identically zero in this case. It is worth noting that the backreaction term would have to be quadratic in the subharmonic-mode amplitudes if this reaction occurred at the equi-amplitude stage.

Since the second member of the subharmonic-mode Equation (25) is now linear in $\tilde{A}$, we refer to it as the parametric resonance term. Its kernel function is given by Goldstein and Lee (1993)

$$K_1 = \frac{3}{2}\left(\tilde{x} - x_1\right)^2 e^{-(2/3)\bar{\lambda}\left(\tilde{x}-x_1\right)^3}, \tag{27}$$

when the mean flow is two-dimensional. Here $\bar{\lambda}$ is a suitably renormalized parameter corresponding to the original viscous parameter λ.

Goldstein and Lee (1992) give an analytical solution to (25) through (27) for the inviscid limit $\bar{\lambda} = 0$, and Wundrow, Hultgren, and Goldstein (1994) extend it to the viscous case where $\bar{\lambda} = 0(1)$. These solutions show that

$$\tilde{A} \sim c_0 e^{(i/2)\arg i\bar{A}_0} e^{\tilde{\kappa}_{0r}\tilde{x}/5} e^{\int^{\tilde{x}_0}\left(\bar{A}_0/4\right)^{1/4} d\tilde{x}} \quad \text{as } \tilde{x} \to \infty, \tag{28}$$

provided that the shifting of the coordinate $\tilde{x}$ is correct to $0(\sigma)$ in the long wavelength limit where $\sigma << 1$ and $\tilde{\kappa} = (4/5)\tilde{\kappa}_0$. Here, $\tilde{x}_0$ is a shifted coordinate corresponding to $\tilde{x}$, c_0 is a real constant, and $\tilde{A}_0$ is given by Equation (26).

Notice that K_1 (as given by Equation (27)) becomes highly concentrated around $\tilde{x} = x_1$ in the strongly viscous limit,

$$\bar{\lambda} \to \infty, \text{ with } \hat{\kappa} \equiv \bar{\lambda}^{1/3}\tilde{\kappa} = 0(1). \tag{29}$$

Equation (25) therefore reduces to the ordinary differential equation

$$\frac{d\hat{A}}{d\tilde{x}} = \hat{\kappa}\hat{A} + \frac{3i}{4}\hat{A}_0\hat{A}^*, \tag{30}$$

where

$$\hat{x} = \bar{\lambda}^{-1/3}\tilde{x},\ \hat{A} = \bar{\lambda}^{-1/3}\tilde{A},\ \text{and}\ \hat{A}_0(\tilde{x}) = \bar{\lambda}^{-2/3}\tilde{A}_0. \tag{31}$$

The limit (29) corresponds to (among other things) the flat plate or Blasius boundary layer, i.e., the flow in which the resonant-triad interaction was first analyzed by Craik (1971). In fact, Equation (30) is within a constant factor of the equation obtained by Craik (1971), who used conventional Stuart-Watson-Landau theory (Stuart, 1960; Watson, 1960; and Landau and Lifshitz, 1987) together with finite Reynolds-number-type arguments to derive his result. However, the corresponding limiting form of the general fundamental-mode amplitude Equation (16) is still the linear Equation (26) and not the nonlinear plane wave equation obtained by Craik (1971).

Equations (26), (30), and (31) imply that (Craik, 1971; and Wundrow, Hultgren, and Goldstein, 1994)

$$\hat{A} \sim \hat{C}_0 e^{i\pi/4} e^{\hat{\kappa}\hat{x} + \frac{3}{4\kappa_{0r}} e^{\kappa_{0r}\hat{x}}} \quad \text{as } \hat{x} \to \infty, \tag{32}$$

where $\hat{C}_0$ is a real constant: We have chosen the origin of the $\tilde{x}$ coordinates so that

$$\hat{A}_0 = e^{\hat{\kappa}_{0r}\hat{x}} \tag{33}$$

and, for simplicity, we assume that $\hat{\kappa}$ is real.

Notice that Equation (32) does not reduce to the limiting form of Equation (28) as $\bar{\lambda} \to \infty$, which means that the limits $\bar{\lambda} \to \infty$ and $\tilde{x} \to \infty$ cannot be interchanged and, consequently, that there must be some intermediate solution that connects the asymptotic solutions (28) and (32). In fact, Wundrow, Hultgren, and Goldstein (1994) show that the approximation (30) becomes invalid when $\hat{\kappa}\hat{x} = 0(\ell n\bar{\lambda}^{2/3})$ and that the nonequilibrium effects become of the same order as the viscous effects for larger values of $\hat{x}$, at which point the flow begins to evolve on the faster scale

$$\bar{x} = \bar{\lambda}^{1/3}\left(\tilde{x} - \frac{1}{\kappa_{0r}}\ell n\bar{\lambda}^{2/3}\right) \tag{34}$$

and the oblique mode amplitude is then determined by the fully nonequilibrium Equation (25), but with $\hat{A}_0(\hat{x})$ treated as a slowly varying function of $\bar{x}$ and the linear growth term $\hat{\kappa}\hat{A}$ treated as a higher order effect (Goldstein, 1994).

This means that the critical-layer dynamics can eventually be controlled by non-equilibrium (or growth rate) effects, even in the Blasius boundary layer (where the initial linear critical layer is of the viscous or equilibrium type), and that the uniformly valid solution for the instability-wave amplitude will then be determined by the nonequilibrium Equation (25) and not by the viscous-limit Equation (30).

6. Concluding Remarks

In most boundary layer flows, it is the three-dimensional instability waves that ultimately exhibit the most rapid growth--either directly from the initial linear stage or indirectly through an intermediate parametric resonance stage. The cubic self-interaction between the three-dimensional instability waves of the same frequency is one of the first strictly nonlinear interactions to come into play as the instability waves evolve downstream in such flows. This interaction can have a dominant effect on the subsequent instability-wave development--producing a local singularity (and consequently explosive growth) at a finite downstream position in the inviscid limit and sometimes producing viscous decay when viscosity is present (Goldstein, 1994; and Wu, Lee, and Cowley, 1993).

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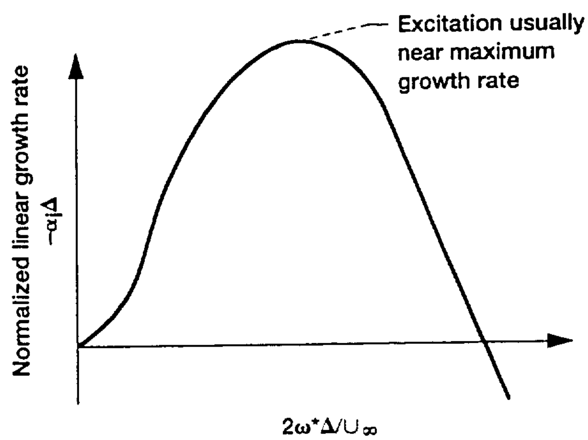

Figure 1.—Typical linear growth rate curve.

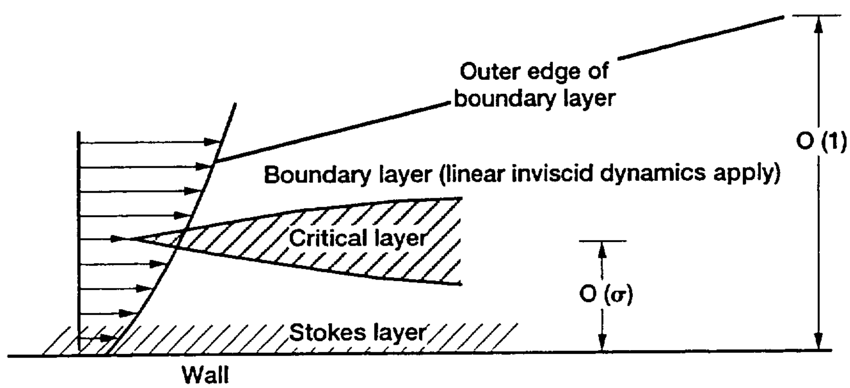

Figure 2.—Asymptotic structure of flow.

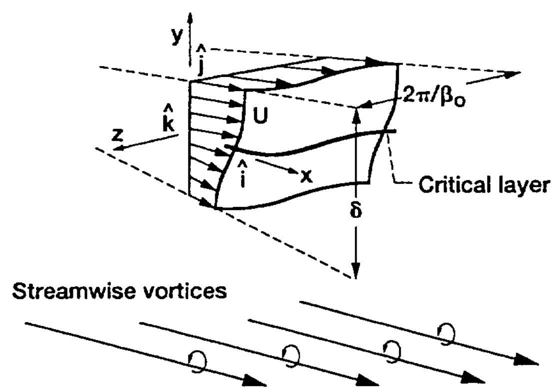

Figure 3.—Flow outside the critical layer.

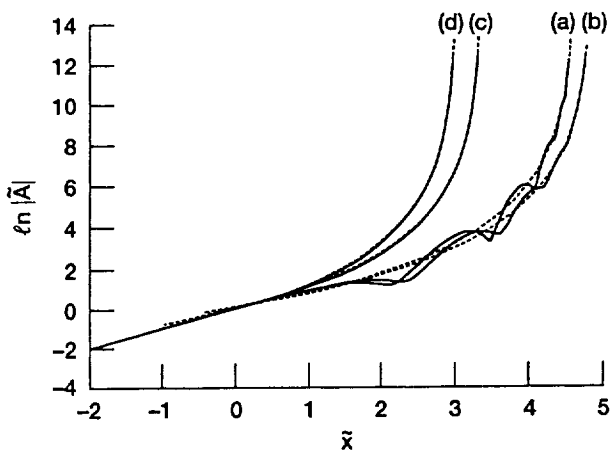

Figure 4.—Scaled amplitude vs. the scaled streamwise coordinate: (a) θ = 15°; (b) θ = 30°; (c) θ = 60°; (d) θ = 75°. Solid lines: numerical solutions; dashed lines: local asymptotic solutions (from Wu, Lee, and Cowley (1993)).