Viscoplastic Model Development With an Eye Toward Characterization*

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A viscoplastic theory is developed that reduces analytically to creep theory under steady-state conditions. A viscoplastic model is constructed within this theoretical framework by defining material functions that have close ties to the physics of inelasticity. As a consequence, this model is easily characterized—only steady-state creep data, monotonic stress-strain curves, and saturated stress-strain hysteresis loops are required. The model is applied to the copper alloy NARloy Z.

1 Introduction

In mankind's enduring pursuit to go faster and further with greater economy and safety in its diverse variety of vehicles that travel across land and sea or through the air and space, we are taxing our materials to their utmost capabilities. Consequently, the need for accurate material models to describe the various physical properties of a given material is much more critical in the design and development of these vehicles than it has ever been, and this need can only be expected to continue to grow.

The analysis of metallic response for high temperature applications requires mathematical models capable of predicting accurately the short-term plastic strains, the long-term creep strains, and interactions between them. Viscoplastic models attempt to do that. Multiaxial, cyclic and nonisothermal histories are normal service conditions, not exceptional ones, all of which challenge the predictive capabilities of such models.

Prior to the advent of the computer, viscoplasticity was a theory in its infancy; however, over the past two decades substantial advancements have been made to the theory. Because of viscoplasticity's innate nature, which leads to systems of first-order, ordinary, differential equations that are nonlinear, coupled, and mathematically stiff, a unique mathematical structure (like that of elasticity) is not to be expected. Nevertheless, these past two decades have given the community a vast wealth of experience with a variety of evolution equations—what works, what does not, and in many cases, some physical insight as to why. Using this experience base, we have set out to develop a viscoplastic model whose predictive capabilities are in reasonable agreement with experiments. A special emphasis in this development process was that the resulting model must be characterized easily. The need to follow this requirement is vital. Experience has taught us that a model's ease of characterization without calibration via exotic experiments is often considered by many industrial users of

viscoplasticity to be of greater value than the model's ability to predict accurately a material's behavior (within reason). We therefore seek to strike a balance between accuracy and ease of characterization, using physics as our guidepost.

The paper begins with a brief overview of the theories of elasticity and creep. This is followed by a definition of viscoplastic flow and the introduction of the required internal state variables. The next section demonstrates how a viscoplastic theory can be constructed to reduce analytically to creep theory under steady-state conditions. This important section demonstrates how a bridge between these two theories can be built—a concept that is not prevalent in the viscoplastic literature. By building this bridge, the model not only has a stronger physical base, but it also reduces substantially the complexity of material characterization. A succinct description of the viscoplastic model is given, and for illustrative purposes, the copper alloy NARloy Z is modeled. This material finds applications where moderate strength is required under conditions of very high heat flux, e.g., it is used as the nozzle liner material in the main rocket engines of NASA's space shuttles where steep, rapidly applied, thermal gradients cause large localized strains.

2 Elasticity

The stress, σ_{ij}, is taken to be related to the infinitesimal strain, ϵ_{ij}, through the constitutive equations of an isotropic Hookean material, viz.

$$S_{ij} = 2\mu(\mathrm{E}_{ij} - \epsilon_{ij}^p) \quad \text{where} \quad \epsilon_{kk}^p = 0, \tag{1}$$

and

$$\sigma_{kk} = 3\kappa(\epsilon_{kk} - \alpha(T - T_0)\delta_{kk}), \tag{2}$$

which are characterized by the shear, μ, and bulk, κ, elastic moduli, and where

$$S_{ij} = \sigma_{ij} - 1/3\sigma_{kk}\delta_{ij} \quad \text{and} \quad \mathrm{E}_{ij} = \epsilon_{ij} - 1/3\epsilon_{kk}\delta_{ij} \tag{3}$$

denote the deviatoric stress and strain, respectively. The mean coefficient of thermal expansion, α, acts on the difference between the current temperature, T, and some reference temperature, T_0. The Kronecker delta, δ_{ij}, has the value 1 if $i = j$, otherwise it is 0. Repeated Latin indices are summed from 1

Table 1 Elastic constants for NARloy Z (Anonymous, 1986)

Constants	Units	Value
α	K^{-1}	16.5×10^{-6}
μ_0	MPa	52,000
μ_1	MPa/K	-14
ν	—	0.34

$$\mu = \mu_0 + \mu_1 T, \qquad T \text{ is in K}$$

to 3 in the usual manner. Equation (1) characterizes the deviatoric stress response, while Eq. (2) characterizes the hydrostatic stress response. The plastic strain, ϵ_{ij}^p, and thermal strain, $\alpha(T - T_0)\delta_{ij}$, are, in essence, eigenstrains that represent deviations from deviatoric and hydrostatic elastic behaviors, respectively.

Young's modulus, E, and Poisson's ratio, ν, are the two elastic constants that are usually determined via experiment. The expressions,

$$\mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \kappa = \frac{E}{3(1 - 2\nu)}, \tag{4}$$

define their interdependence with the elastic moduli of Eqs. (1) and (2). Only two elastic moduli are independent for elastically isotropic materials. Values for the elastic constants of NARloy Z (typical composition: Cu-3%Ag-0.5%Zr) are given in Table 1.

3 Creep

The evolution of plastic strain which describes the classical theory of creep (Odqvist, 1974) is given by

$$\dot{\epsilon}_{ij}^p|_{ss} = 1/2 \| \dot{\epsilon}^p \|_{ss} \frac{S_{ij}}{\|S\|}, \tag{5}$$

with the subscript "ss" implying steady-state, and where $\|\dot{\epsilon}^p\|$ denotes the magnitude of plastic strain-rate, and $\|S\|$ denotes the magnitude of deviatoric stress. The subscript signifying steady state is not attached to S_{ij} because stress is a controllable external variable, whereas creep rate is a response variable. This equation states that an increment in creep strain accumulates in the current direction of the deviatoric stress. A dot is placed over a variable to signify its time rate-of-change.

The norms, or magnitudes, pertaining to the deviatoric tensors of this paper are defined by

$$\|I\| = \sqrt{2 I_{ij} I_{ij}} \quad \text{and} \quad \|J\| = \sqrt{1/2 J_{ij} J_{ij}}, \tag{6}$$

where I_{ij} is any deviatoric "strain-like" tensor, and J_{ij} is any deviatoric "stress-like" tensor. These are the norms of von Mises (1913), where the coefficients under the radical signs scale the theory for shear.

In the theory of creep, $\|\dot{\epsilon}^p\|_{ss}$ is described by a kinetic equation, i.e., an equation of state. Zener and Hollomon (1944) determined that such a kinetic equation can, to a good approximation, be decomposed into a product of two functions; in particular, at steady state

$$\| \dot{\epsilon}^p \|_{ss} = \vartheta[T] Z_{ss}\left[\frac{\|S\|}{C} \right] \geq 0, \tag{7}$$

where $\vartheta > 0$ is a thermal function, $Z \geq 0$ is the Zener parameter, and $C > 0$ is a strength parameter that normalizes the stress. The Zener parameter is a temperature normalized measure of the plastic strain-rate. Square brackets, [·], are used throughout this paper to denote "function of," and are therefore kept logically separate from parentheses, (·), which are used for mathematical groupings.

In the physical description of the thermal function, ϑ, there is a parameter called the activation energy, Q, which—for creep at low stresses and elevated temperatures—is associated with self-diffusion where the rate-controlling mechanism for deformation is dislocation climb (Sherby and Weertman, 1979).

At higher stresses and/or more moderate temperatures, the rate-controlling mechanism changes from diffusion-controlled dislocation climb to obstacle-controlled dislocation glide (Kocks et al., 1975). Along with this change in the deformation mechanism, there occurs a change in the activation energy (Sherby and Burke, 1968). Miller (1976) approximates the observed temperature dependence of the activation energy for steady-state flow with a linear function for temperatures below some threshold temperature, T_t, while for temperatures above this threshold the activation energy is kept constant, in accordance with the experimental observations of Dorn (1954). Because it is the free energy (not the activation energy) that drives the kinetics of plastic deformation (Kocks et al., 1975), Miller integrated his linear function for the activation energy and obtained the following Arrhenius-like expression for the thermal function,

$$\vartheta = \begin{cases} \exp\left[\dfrac{-Q}{kT} \right] & \text{when } T_t \leq T < T_m \\[2ex] \exp\left[\dfrac{-Q}{kT_t}\left(\ln\left[\dfrac{T_t}{T} \right] + 1 \right) \right] & \text{when } 0 < T \leq T_t \end{cases} \tag{8}$$

where k is the universal gas constant (8.314 J/mole·K). The applicability of this relationship is discussed elsewhere (Freed et al., 1992). The transition temperature, T_t, between these two domains in activation energy is not unique; it is known to depend on the strain-rates used to make the measurements for activation energy. An increase in strain-rate increases the transition temperature (Sherby and Burke, 1968). For the vast majority of engineering applications, a transition temperature of $T_t \approx 1/2 T_m$ seems appropriate for f.c.c. metals, and is used in our characterization of NARloy Z.

When the mechanism for deformation changes from diffusion-controlled dislocation climb to obstacle-controlled dislocation glide, the creep response changes from power-law to exponential behavior (Ashby, 1972). Following the approach of Miller (1976), we adopt Garofalo's (1963) empirical expression for the steady-state Zener parameter, i.e.,

$$Z_{ss} = A \sinh^n\left[\frac{\|S\|}{C} \right], \tag{9}$$

where $A > 0$, $C > 0$ and $n > 0$ are the material constants. For stress states below power-law breakdown, i.e., when $\|S\| < C$, the steady-state Zener parameter of Garofalo reduces to the power-law relationship

$$Z_{ss} = A\left(\frac{\|S\|}{C} \right)^n, \tag{10}$$

thereby designating dislocation climb as the rate-controlling mechanism. (Note: A, C, and n are independent in Eq. (9) but not in Eq. (10).) Similarly, when the stress exceeds powerlaw breakdown, i.e., when $\|S\| > C$, Garofalo's Zener parameter reduces to the exponential relationship

$$Z_{ss} = A' \exp\left[\frac{\|S\|}{C'} \right], \tag{11}$$

where $A' = A/2^n$ and $C' = C/n$, thereby designating dislocation glide as the rate-controlling mechanism. The ability of Eqs. (7)-(9) to correlate the stationary creep-rate data of NARloy Z is demonstrated in Fig. 1. The material constants obtained from this correlation are given in Table 2. Because none of these data lie within the power-law domain, the exponential creep equation, Eq. (11), was used to determine values for A' and C' leading to the straight line fit shown in the log/linear plot of Fig. 1(a), where $A' = 5 \times 10^{17}$ s^{-1} and $C' = 3.5$ MPa for the predefined values of $Q = 450,000$ J/mole (Lewis, 1970) and $T_t = 400°$C (assumed).[1] Taking $n = 4$ (assumed), the values

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[1] This seems to be an excessively large value for Q, but it is the only experimentally determined value currently available to us.
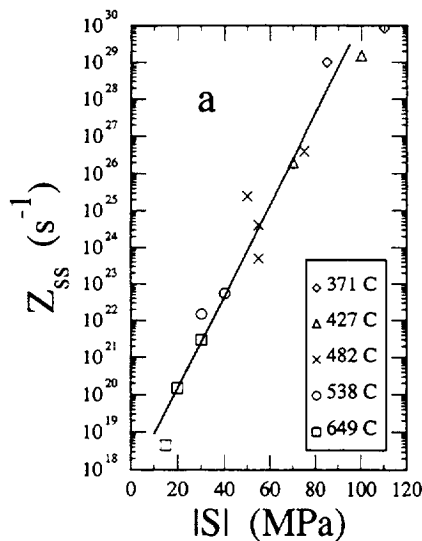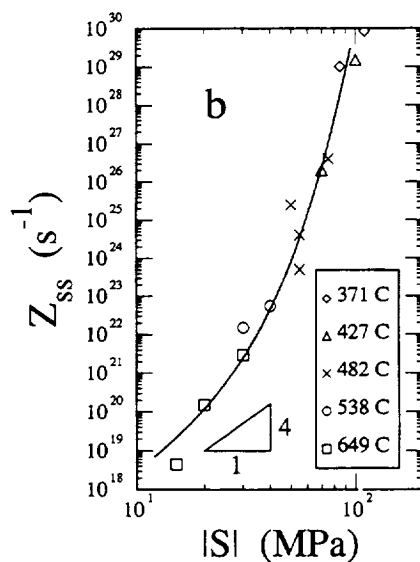
Fig. 1(a)

Fig. 1(b)

Fig. 1 Steady state creep behavior of NARloy Z. Data are from Lewis (1970) with the reported material composition: Cu-2.89%Ag-0.22%Zr.

for A' and C' have been converted to those of A and C that are reported in Table 2. The result is the curved line presented in the log/log plot of Fig. 1(b). We note that the value of C for NARloy Z, i.e., 14 MPa, obtained with this choice for n, i.e., 4, is in agreement with the value of C for Cu, i.e., 13 MPa, reported in Freed and Walker (1993a).

This continuum representation for creep is well established. Our viscoplastic model reduces analytically to this creep model under steady-state conditions. Hence, the material constants that characterize this creep model also appear in our viscoplastic model, which simplifies substantially its characterization process.

4 Viscoplastic Flow

A general mathematical structure for viscoplasticity (Freed et al., 1991) may admit up to three kinds of internal state variables; they are: (i) the (scalar-valued) drag strength, $D > 0$; (ii) the (scalar-valued) yield stress, $Y \geq 0$; and (iii) the (deviatoric tensor-valued) back stress, B_{ij}. The drag strength and yield stress account for isotropic hardening effects, while the back stress accounts for kinematic (flow-induced anisotropic) hardening effects.

Table 2 Steady-state creep constants for NARloy Z

Constant	Units	Value
A	s^{-1}	8×10^{18}
C	MPa	14
n	—	4
Q	J/mol.	450,000
T_m	K	~1350
$T_t = 1/2 T_m$		

Prager's (1949) constitutive relation is used to describe the evolution of viscoplastic flow, i.e.,

$$\dot{\epsilon}^p_{ij} = 1/2 \| \dot{\epsilon}^p \| \frac{S_{ij} - B_{ij}}{\| S - B \|}. \tag{12}$$

This particular choice for the flow law implies that a nested set of flow surfaces exists; they are surfaces of constant plastic strain-rate when evaluated under isothermal conditions. This constitutes a set of ellipsoids in deviatoric stress space that are centered on the back stress.

The kinetics of viscoplasticity are taken to be described by a Zener and Hollomon (1944) type decomposition of state (Freed et al., 1991), viz.

$$\| \dot{\epsilon}^p \| = \vartheta[T] Z \left[\left\langle \frac{\| S - B \| - Y}{D} \right\rangle \right] \geq 0, \tag{13}$$

where the Macauley bracket, $\langle (\| S - B \| - Y)/D \rangle$, has either a value of 0 whenever $\| S - B \| < Y$ (defining the elastic domain), or a value of $(\| S - B \| - Y)/D$ whenever $\| S - B \| > Y$ (defining the viscoplastic domain), with $\| S - B \| = Y$ establishing the yield surface. Many viscoplastic models have no distinct yield surface, i.e., they set Y to 0. The distinguishing feature between viscoplasticity (a rate-dependent theory) and plasticity (a rate-independent theory) is that viscoplasticity admits states both inside and outside of the yield surface (governed by a kinetic equation of state); whereas, plasticity admits only states that are inside and on the yield surface (governed by a consistency condition), but not outside of it. As a consequence, the plastic strain-rate is continuous as one moves from the elastic domain across the yield surface and into the inelastic domain of viscoplastic response; whereas, the accumulation of plastic strain is discontinuous as one moves from the elastic domain onto the yield surface in plasticity. The elastic domain of many viscoplastic models is shrunk to a point, as they do not admit a yield surface.

The internal state variables—B_{ij}, D and Y—are described by evolution equations that are functions of state. The back stress evolves rapidly when compared with the rates of evolution for the drag strength and yield stress, which is a source of mathematical stiffness in the governing equations of viscoplasticity. The evolution of the back stress accounts for the change in material stiffness that is observed during the transition from elastic to plastic behavior, while the evolutions of the drag strength and yield stress account for the more gradual work hardening processes that are caused by the overall accumulation of plastic deformation. The internal variables are considered to evolve phenomenologically through competitive processes associated with strain hardening, strain-induced dynamic recovery, and time-induced thermal recovery. Their specific functional forms are presented later in Section 6, whose derivations are given in the conference proceedings' version of this paper, i.e., Freed and Walker (1993b).

5 Creep $\Rightarrow$ Viscoplasticity

In the process of going from creep theory to viscoplasticity, one must remove the steady-state constraint that is present in creep, and thereby extend the domain of admissible states to include transient behavior. In other words, viscoplasticity is capable of modeling both primary and secondary creep be-

havior. The modeling of transient behavior is done through the introduction of internal state variables. Although the purpose of viscoplasticity is to model rate-dependent transient behavior, it is not unreasonable to also require that it reduces to creep theory under steady-state conditions. An important objective in our development of a viscoplastic theory is that it reduces *analytically* to creep theory when at steady state. Not only is this a realistic requirement, but it also strengthens the physics of the theory, and it simplifies greatly the process of model characterization—about half of our viscoplastic material constants come from correlating stationary creep-rate data alone.

In order for a viscoplastic theory to reduce analytically to creep theory when at steady-state (i.e., when $\mathbf{B} = 0$, $\dot{D} = 0$, and $\dot{Y} = 0$ for $\|\dot{\epsilon}^p\| > 0$) two conditions must be satisfied. First, the back stress must be coaxial with the stress at steady state so that the directions of plastic strain-rate defined by Eqs. (5) and (12) are also coaxial at steady state. And second, it is necessary that the kinetics of viscoplasticity, Eq. (13), reduce analytically to the kinetics of creep, Eq. (7), under steady-state conditions. The evolution law for back stress given in Eq. (28) satisfies this first constraint. To satisfy the second constraint, one must first hypothesize a relationship between the steady-state and transient Zener parameters, and then hypothesize another one between the internal and external variables, when at steady state (Freed and Walker, 1990). We therefore suppose that

$$Z \equiv Z_{ss}\left[\left\langle\frac{\|\mathbf{S} - \mathbf{B}\| - Y}{D}\right\rangle\right], \tag{14}$$

in support of Eq. (13). This relationship implies that the transient Zener parameter, Z, has the same functional form as the steady-state Zener parameter, Z_{ss}, but with a different argument; in particular, and in accordance with Eq. (9), we take

$$Z = A \sinh^n\left[\left\langle\frac{\|\mathbf{S} - \mathbf{B}\| - Y}{D}\right\rangle\right], \tag{15}$$

which is similar in form to the kinetics of Miller's (1976) viscoplastic model, but with a yield stress and without a power acting on the Macauley bracket.

Furthermore, we shall suppose that

$$\|\mathbf{B}\|_{ss} = f\iota_{ss}[\|\mathbf{S}\|]\|\mathbf{S}\|, \quad D_{ss} = D_0 + \delta\|\mathbf{S}\|$$

$$\text{and} \quad Y_{ss} = (1 - f)\iota_{ss}[\|\mathbf{S}\|]\|\mathbf{S}\|, \tag{16}$$

in support of experimental evidence, where $\iota_{ss} > 0$ and $\delta > 0$ are the steady-state fractions of applied stress that are associated with the internal stress (i.e., the back and yield stresses) and the drag strength, respectively, such that $1/2 < \iota_{ss} < 1$. The parameter f partitions the internal stress between isotropic and kinematic contributions, such that $0 < f < 1$. The fact the drag strength is taken to be proportional to the saturation stress is a consequence of the fact that the drag strength represents the material's innate strength to resist plastic flow, i.e., D is a strength parameter—not a stress parameter. We take the internal stress to be a nonlinear function of the applied stress at saturation because that is what the experimental data of Argon and Takeuchi (1981) and Čadek (1987) suggest. A similar hypothesis to that of Eq. (16) is given in Freed and Walker (1993a) for the case where the internal stress is composed of two back stresses with no yield stress.

Because the applied stress and the back stress must be coaxial at steady state, as discussed above, it follows that

$$\|\mathbf{S} - \mathbf{B}\|_{ss} = \|\mathbf{S}\| - \|\mathbf{B}\|_{ss}. \tag{17}$$

Therefore, upon equating the arguments of the Zener parameters in Eqs. (7) and (14), while utilizing Eqs. (16) and (17), one obtains the result
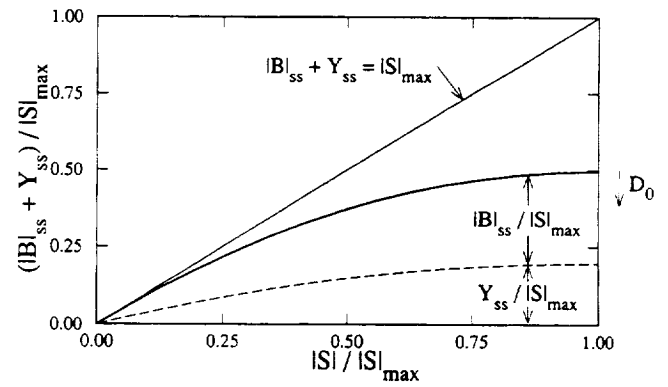
Fig. 2 Schematic of internal stress (back plus yield stresses) at steady state versus applied stress

$$\iota_{ss} = \frac{C - D_0 - \delta\|\mathbf{S}\|}{C}$$

$$= \frac{C - D_0 + \sqrt{(C - D_0)^2 - 4\delta C(\|\mathbf{B}\|_{ss} + Y_{ss})}}{2C}. \tag{18}$$

If one uses Eq. (16) and writes $\|\mathbf{B}\|_{ss} + Y_{ss} = \iota_{ss}[\|\mathbf{S}\|]\|\mathbf{S}\|$, then from Eq. (18) one determines that $\|\mathbf{B}\|_{ss} + Y_{ss} = (C - D_0 - \delta\|\mathbf{S}\|)\|\mathbf{S}\|/C$. Because $\partial(\|\mathbf{B}\|_{ss} + Y_{ss})/\partial\|\mathbf{S}\| = 0$ establishes the maximum state of internal stress, one is lead to the result

$$\|\mathbf{S}\|_{max} = \frac{C - D_0}{2\delta}. \tag{19}$$

Substituting this relationship back into Eqs. (16) and (18) gives additional upper bounds for: the back stress,

$$\|\mathbf{B}\|_{max} = f\frac{(C - D_0)^2}{4\delta C}, \tag{20}$$

the drag strength,

$$D_{max} = 1/2(C + D_0), \tag{21}$$

and the yield stress,

$$Y_{max} = (1 - f)\frac{(C - D_0)^2}{4\delta C}. \tag{22}$$

Similar bounds are given in Freed and Walker (1993a) for the case where the internal stress is composed of two back stresses and no yield stress. *It is a remarkable fact that one can bound the stress and internal state variables without specifying anything about how these internal state variables evolve.*

Restricting ι_{ss} to be real valued, and considering ι_{min} to be associated with the maximum attainable magnitude of internal stress, one finds on approaching the limit of zero stress that the ratio of internal stress to applied stress at steady state is at its maximum, i.e.,

$$\lim_{\|\mathbf{S}\|\to 0} \iota_{ss} \equiv \iota_{max} = \frac{C - D_0}{C} \approx 1, \tag{23}$$

which is in reasonable agreement with Argon and Takeuchi's (1981) and Čadek's (1987) experimental observations. Approaching the limit of maximum stress, this ratio attains its minimum, i.e.,

$$\lim_{\|\mathbf{S}\|\to\|\mathbf{S}\|_{max}} \iota_{ss} \equiv \iota_{min} = \frac{C - D_0}{2C} \approx 1/2, \tag{24}$$

which is in reasonable agreement with Lowe and Miller's (1983) and Argon and Bhattacharya's (1987) experimental observations. A schematic of the steady-state internal stress versus the applied stress—as predicted by Eqs. (16) and (18) with typical values of $D_0 = C/100$ and $f = 0.6$—is presented in Fig. 2. The trends depicted therein are in qualitative agreement with the experimental results referenced above.

Table 3 Additional viscoplastic constants for NARloy Z

Constants	Units	Value
δ	—	0.035
f	—	0.65
b	MPa	11
ℓ	—	6.4
$D_0 = C/100$		

To be physically meaningful, $\|\mathbf{B}\| \geq 0$, $D > 0$, and $Y \geq 0$. Furthermore, their steady-state values ought to increase monotonically with increasing stress (Freed and Walker, 1990). This is verified easily for our hypothesis, Eqs. ((14), (16), and (17)), as long as $0 \leq \|\mathbf{B}\| \leq \|\mathbf{B}\|_{max}$, $D_0 \leq D \leq D_{max}$ and $0 \leq Y \leq Y_{max}$.

6 The Model

A succinct description of our viscoplastic model is given below. The stress is acquired through the constitutive equations

$$S_{ij} = 2\mu(E_{ij} - \epsilon_{ij}^p) \quad \text{and} \quad \sigma_{kk} = 3\kappa(\epsilon_{kk} - \alpha(T - T_0)\delta_{kk}). \quad (25)$$

The flow equation and kinetics that describe plastic straining are given by

$$\dot{\epsilon}_{ij}^p = 1/2 \|\dot{\epsilon}^p\| \frac{S_{ij} - B_{ij}}{\|\mathbf{S} - \mathbf{B}\|} \quad \text{and} \quad \|\dot{\epsilon}^p\| = \vartheta Z, \quad (26)$$

respectively, with the von Mises norm of effective stress being defined by

$$\|\mathbf{S} - \mathbf{B}\| = \sqrt{1/2 (S_{ij} - B_{ij})(S_{ij} - B_{ij})}. \quad (27)$$

The evolutions of back stress and drag strength are given by

$$\dot{B}_{ij} = 2H\left(\dot{\epsilon}_{ij}^p - \frac{B_{ij}}{2L} \|\dot{\epsilon}^p\|\right)$$

$$\text{and} \quad \dot{D} = h(\|\dot{\epsilon}^p\| - \Lambda \|\dot{\epsilon}^p\| - \vartheta r), \quad (28)$$

respectively, such that $D_0 \leq D \leq D_{max}$, while the yield stress is related through the state function

$$Y = (1 - f)\frac{(D - D_0)(C - D)}{\delta C}, \quad (29)$$

which is not an evolution equation. Associated with these relationships are the material functions:

$$\vartheta = \begin{cases} \exp\left[\dfrac{-Q}{kT}\right] & \text{when } T_t \leq T < T_m \\[2ex] \exp\left[\dfrac{-Q}{kT_t}\left(\ln\left[\dfrac{T_t}{T}\right] + 1\right)\right] & \text{when } 0 < T \leq T_t, \end{cases} \quad (30)$$

$$Z = A \sinh^n\left[\left\langle\frac{\|\mathbf{S} - \mathbf{B}\| - Y}{D}\right\rangle\right], \quad (31)$$

$$H = (0.1 + 0.9\xi)\mu \quad \text{and} \quad L = f\frac{(D - D_0)(C - D)}{\delta C}, \quad (32)$$

$$\Lambda = \ell\xi \quad \text{and} \quad r = A \sinh^n\left[\frac{D - D_0}{\delta C}\right], \quad (33)$$

with

$$\xi = \sqrt{\frac{1}{8}\left(\frac{S_{ij} - B_{ij}}{\|\mathbf{S} - \mathbf{B}\|} - \frac{B_{ij}}{L}\right)\left(\frac{S_{ij} - B_{ij}}{\|\mathbf{S} - \mathbf{B}\|} - \frac{B_{ij}}{L}\right)}. \quad (34)$$

Restricting the drag strength to be bound by the interval $D_0 \leq D \leq D_{max}$ restricts automatically the remaining variables: $0 \leq \|\mathbf{S}\| \leq \|\mathbf{S}\|_{max}$, $0 \leq \|\mathbf{B}\| \leq \|\mathbf{B}\|_{max}$ and $0 \leq Y \leq Y_{max}$. The development of the evolution equations for the back stress and drag strength, along with the derivations of their associated material functions, are given in the conference proceedings' version of this paper (Freed and Walker, 1993b). Also found therein is a detailed discussion of how one goes about char-
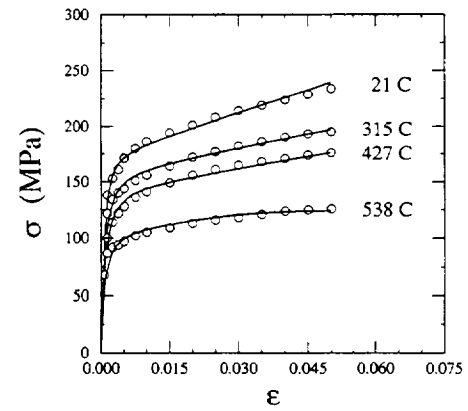

Fig. 3 Tensile stress-strain curves for NARloy Z. Data are from the Anonymous (1986). Experiment (o) versus theory (—). $\dot{\epsilon} = 0.001$ s^{-1} (assumed), $D|_{t=0} = 2.02$ MPa (for $T = 21$ and 538°C) and $D|_{t=0} = 2.64$ MPa (for $T = 315$ and 427°C).
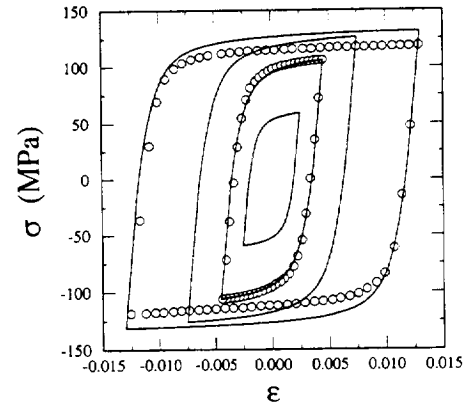

Fig. 4 Saturated hysteresis loops for NARloy Z. Data are from Conway et al. (1975). Experiment (o) versus theory (—). $\dot{\epsilon} = 0.004$ s^{-1} and $T = 538$°C.

acterizing this particular model. Values for the remaining transient constants of our viscoplastic model are given in Table 3 for NARloy Z.

For unaxial loading histories in tension and compression, the above governing equations hold with the following alterations:

$$\sigma = E(\epsilon - \epsilon^p - \alpha(T - T_0)), \quad \dot{\epsilon}^p = \text{sgn}[\sigma - \beta]\frac{\|\dot{\epsilon}^p\|}{\sqrt{3}}$$

$$\text{and} \quad \dot{\beta} = 3H\left(\dot{\epsilon}^p - \frac{\beta}{3L}\|\dot{\epsilon}^p\|\right), \quad (35)$$

given that $\sigma = \sigma_{11} = 3/2S_{11}$, $\beta = \beta_{11} = 3/2B_{11}$, $\epsilon = \epsilon_{11}$ and $\epsilon^p = \epsilon_{11}^p$. As for the material functions, the above equations apply with the following alterations:

$$Z = A \sinh^n\left[\left\langle\frac{|\sigma - \beta| - \sqrt{3}Y}{\sqrt{3}D}\right\rangle\right]$$

$$\text{and} \quad \xi = 1/2\left|\text{sgn}[\sigma - \beta]1 - \frac{\beta}{\sqrt{3}L}\right|. \quad (36)$$

The ability of the model to correlate (not predict) monotonic and cyclic material behavior is presented in Figs. 3 and 4, respectively, for NARloy Z. Data for this material are sparse, thereby not permitting a more detailed assessment of the model's predictive capability. In part, this demonstrates a design objective in our development of this model—the capability to characterize the model from a sparse data set.

This model is not perfect, and certainly not ideal, but hopefully it represents another step in that direction. It is a sim-

plified continuum description of complex microscopic phenomena; nevertheless, its development has been guided by the physics of these phenomena. There are several known deficiencies associated with this model. They are: the predicted, transient, rate dependence, which is extrapolated from steady-state dependence, does not always match experimentally observed rate dependence, for example, the rate dependence exhibited during a stress relaxation experiment (Freed and Walker, 1993a); predicted transient behavior in the region of transition between the domains of power-law and exponential behaviors, which is also taken from steady-state behavior, does not always agree with experimental observations (Loh, 1993); and the well-known fact of excessive, predicted, ratchetting behavior, which is a consequence of our using the Armstrong and Frederick (1966) evolution equation for back stress (Freed and Walker, 1993c).

7 Closure

By designing the development of our viscoplastic model in such a manner that it reduces analytically to a creep model under steady-state conditions, we have incorporated essential physics into our model, and we have also simplified greatly the process that one must go through in order to completely characterize a material with this model. In this sense, we have developed a viscoplastic model with an eye towards its characterization. This has particular merit because parameter estimation of a viscoplastic model is, in general, a very complex process that all too often prohibits its use in applications. A model's relative ease of characterization without the need for exotic experiments is often considered by many industrial users of viscoplasticity to be of greater value than the model's ability to predict accurately a material's behavior (within reason). Our model was developed with this fact in mind, where we have sought to strike a balance between accuracy and ease of characterization using physics as our guidepost.

References

Anonymous, 1986, *Materials Properties Manual*, Vol. III, 4th ed., Rockwell International, Rocketdyne Division, Canoga Park, CA.

Argon, A. S., and Bhattacharya, A. K., 1987, "Primary Creep in Nickel: Experiments and Theory," *Acta Metall.*, Vol. 35, pp. 1499–1514.

Argon, A. S., and Takeuchi, S., 1981, "Internal Stresses in Power-Law Creep," *Acta Metall.*, Vol. 29, pp. 1877–1884.

Armstrong, P. J., and Frederick, C. O., 1966, "A Mathematical Representation of the Multiaxial Bauschinger Effect," Central Electric Generating Board, Berkeley Nuclear Laboratories, Report RD/B/N731.

Ashby, M. F., 1972, "A First Report on Deformation-Mechanism Maps," *Acta Metall.*, Vol. 20, pp. 887–897.

Čadek, J., 1987, "The Back Stress Concept in Power Law Creep of Metals: A Review," *Mater. Sci. Engrg.*, Vol. 94, pp. 79–92.

Conway, J. B., Stentz, R. H., and Berling, J. T., 1975, "High-Temperature, Low-Cycle Fatigue of Copper-Base Alloys for Rocket Nozzles: Part I—Data Summary for Materials Tested in Prior Programs," NASA CR-134908, November.

Dorn, J. E., 1954, "Some Fundamental Experiments on High Temperature Creep," *J. Mech. Phys. Solids*, Vol. 3, pp. 85–116.

Freed, A. D., Chaboche, J.-L., and Walker, K. P., 1991, "A Viscoplastic Theory with Thermodynamic Considerations," *Acta Mech.*, Vol. 90, pp. 155–174.

Freed, A. D., Raj, S. V., and Walker, K. P., 1992, "Stress Versus Temperature Dependence of Activation Energies for Creep," ASME JOURNAL OF ENGINEERING MATERIALS AND TECHNOLOGY, Vol. 114, pp. 46–50.

Freed, A. D., and Walker, K. P., 1990, "Steady-State and Transient Zener Parameters in Viscoplasticity: Drag Strength versus Yield Strength," (NASA TM-102487) *Appl. Mech. Rev.*, Vol. 43, No. 5, Pt. 2, pp. S328–S337.

Freed, A. D., and Walker, K. P., 1993a, "Viscoplasticity With Creep and Plasticity Bounds," *Int. J. Plast.*, Vol. 9, pp. 213–242.

Freed, A. D., and Walker, K. P., 1993b, "Viscoplastic Model Development With an Eye Towards Characterization," *Material Parameter Estimation for Modern Constitutive Equations*, L. A. Bertram et al., eds., ASME, New York, MD-Vol. 43, AMD-Vol. 168, pp. 71–88.

Freed, A. D., and Walker, K. P., 1993c, "Ratchetting Behavior in Viscoplasticity: A Technical Note," *Eur. J. Mech., A/Solids*, Vol. 12, pp. 191–196.

Garofalo, F., 1963, "An Empirical Relation Defining the Stress Dependence of Minimum Creep Rate in Metals," *Trans. AIME*, Vol. 227, pp. 351–356.

Kocks, U. F., Argon, A. S., and Ashby, M. F., 1975, "Thermodynamics and Kinetics of Slip," *Progress in Materials Science*, B. Chalmers et al., eds., Pergamon Press, Oxford, Vol. 19.

Lewis, J. R., 1970, "Creep Behavior of NARloy Z," Rockwell International, Rocketdyne Division, Canoga Park, CA, Report MA-SSE-70-902, July.

Loh, D. H., 1993, Personnal Communication, Rocketdyne Division of Rockwell Int., Canoga Park, CA.

Lowe, T. C., and Miller, A. K., 1983, "The Nature of Directional Strain Softening," *Scripta Metall.*, Vol. 17, pp. 1177–1182.

Miller, A., 1976, "An Inelastic Constitutive Model for Monotonic, Cyclic, and Creep Deformation," ASME JOURNAL OF ENGINEERING MATERIALS AND TECHNOLOGY, Vol. 98, pp. 97–105.

von Mises, R., 1913, "Mechanik der festen Körper im plastisch-deformablen Zustand," *Nachr. Ges. Wiss., Goettingen, Math. Phys. Kl.*, pp. 582–592.

Odqvist, F. K. G., 1974, *Mathematical Theory of Creep and Creep Rupture*, 2nd ed., Oxford University Press, London.

Prager, W., 1949, "Recent Developments in the Mathematical Theory of Plasticity," *J. Appl. Phys.*, Vol. 20, pp. 235–241.

Sherby, O. D., and Burke, P. M., 1968, "Mechanical Behavior of Crystalline Solids at Elevated Temperature," *Progress in Materials Science*, B. Chalmers and W. Hume-Rothery, eds., Pergamon Press, Oxford, Vol. 13, pp. 325–390.

Sherby, O. D., and Weertman, J., 1979, "Diffusion-Controlled Dislocation Creep: A Defense," *Acta Metall.*, Vol. 27, pp. 387–400.

Zener, C., and Hollomon, J. H., 1944, "Effect of Strain Rate Upon Plastic Flow of Steel," *J. Appl. Phys.*, Vol. 15, pp. 22–32.