

INTERIM  
IN-63-CR  
13CIT  
98117

# Guidance of Nonlinear Nonminimum-Phase Dynamic Systems

*Primary*

## Performance Report

### Principal Investigator

Dr. Santosh Devasia  
2252 Merrill Engineering Building  
Department of Mechanical Engineering  
University of Utah  
Salt Lake City, Utah 84112  
Tel: (801) 581 4613  
Email: santosh@me.utah.edu

### Administrative Contact for Grantee's Institution

Office of Sponsored Projects  
1471 Federal Way  
University of Utah  
Salt Lake City, Utah 84112

**Period:** 3/1/96 - 2/28/97

**Grant Number:** NAG 2-1042

## Summary

The research work has advanced the inversion-based guidance theory for:

- systems with non-hyperbolic internal dynamics;
- systems with parameter jumps; and
- systems where a redesign of the output trajectory is desired.

### Non-hyperbolic Internal Dynamics

A technique to achieve output tracking for nonminimum phase linear systems with non-hyperbolic and near non-hyperbolic internal dynamics was developed [1]. This approach integrated stable inversion techniques, that achieve exact-tracking, with approximation techniques, that modify the internal dynamics to achieve desirable performance. Such modification of the internal dynamics was used (a) to remove non-hyperbolicity which is an obstruction to applying stable inversion techniques and (b) to reduce large preactuation times needed to apply stable inversion for near non-hyperbolic cases. The method was applied to an example helicopter hover control problem with near non-hyperbolic internal dynamics for illustrating the trade-off between exact tracking and reduction of preactuation time.

Future work will extend these results to guidance of nonlinear nonhyperbolic systems.

### Systems with Parameter Jumps

The exact output tracking problem for systems with parameter jumps was considered [2]. Necessary and sufficient conditions were derived for the elimination of switching-introduced output transient. While previous works had studied this problem by developing a regulator that maintains exact tracking through parameter jumps (switches), such techniques are, however, only applicable to minimum-phase systems. In contrast, our approach is also applicable to nonminimum-phase systems and leads to bounded but possibly non-causal solutions. In addition, for the case when the reference trajectories are generated by an exosystem, we developed an exact-tracking controller which could be written in a feedback form. As in standard regulator theory, we also obtained a linear map from the states of the exosystem to the desired system state, which was defined via a matrix differential equation. The

constant solution of this differential equation provided asymptotic tracking, and coincided with the feedback law used in standard regulator theory. Future work will extend these results to nonlinear systems and generalize these results to hybrid systems.

## Output Trajectory Redesign

We studied the optimal redesign of output trajectory for linear invertible systems [3, 4]. The specified output trajectory uniquely determines the required input and state trajectories, that are found through inversion. These input-state trajectories exactly track the desired output, however, they might not meet acceptable performance requirements. For example, the required inputs might cause actuator-saturation during an exact tracking maneuver (as in the flight control of conventional take-off and landing aircraft). In such situations, a compromise is desired between the tracking requirement and other goals like reduction of internal vibrations and prevention of actuator saturation – i.e., the desired output trajectory needs to be redesigned.

We posed the trajectory redesign problem as an optimization of a general quadratic cost function, and solved it in the context of linear systems.

Future work will be aimed at extending these results to the nonlinear case and at developing algorithms for practical implementation.

## References

- [1] S. Devasia. Output tracking for systems with non-hyperbolic and near non-hyperbolic internal dynamics: Helicopter hover control. *Submitted to ACC 97, and AIAA J. of Guidance, Control, and Dynamics*, 1996.
- [2] S. Devasia, B. Paden, and C. Rossi. Exact-output tracking theory for systems with parameter jumps. *International Journal of Control*, Accepted, 1996.
- [3] S. Devasia. Optimal output-trajectory redesign for invertible systems. *AIAA J. of Guidance, Control, and Dynamics*, 19(5):1189–1191, 1996.
- [4] J.S. Dewey and S. Devasia. Experimental and theoretical results in output- trajectory redesign for flexible structures. *CDC 96, also submitted to ASME Journal of Dynamic Systems, Measurement and Control*, 1996.

# **Appendix**

## SANTOSH DEVASIA

Department of Mechanical Engineering  
U of Utah, SLC, UT 84112  
Email: santosh@me.utah.edu

(801) 581 4613 (Work)  
(801) 581 8692 (Fax)  
So. Sec. # [REDACTED]

### EDUCATION

- Doctor of Philosophy (December 1993)  
Mechanical Engineering, University of California, Santa Barbara (UCSB)  
Thesis: **Design, Dynamics and Control of Intelligent Structures**  
Major: Dynamic Systems, Controls and Robotics  
Minors: (1) Solid Mechanics and Structures (2) Mathematics
- Masters of Science (December 1990)  
Mechanical Engineering, University of California, Santa Barbara
- Bachelor of Technology (June 1988)  
Mechanical Engineering, Indian Institute of Technology, Kharagpur, India

### AWARDS AND HONORS

- The University of California Special Regents Fellowship (1988-92).
- Received the Deans Award for outstanding teaching assistant in Mechanical Engineering at UCSB (1992).
- Delco Electronics Corporation's Best Graduate Dissertation Award (1993).
- Controls and Decision Conference (CDC) Best Student Paper Finalist Award (1994)
- UCSB School of Engineering Dean's Fellowship (1993-94).
- The National Merit Scholarship for undergraduate study (1984-88).

### FUNDING

Nonlinear Inversion-Based High-Speed/High-Precision  
Tracking for Scanning-Probe-Based Nanotechnology (PI)  
NSF Design and Manufacturing Division, DMI-9612300, 9/1/96-8/30/99, \$180099  
This includes \$11500 Matching funds for equipment from U. of Utah.

Guidance of Nonlinear Nonminimum Phase Dynamic Systems (PI)  
NASA-Ames Research Center, NAG 2-1042, \$24967

Ultrasound Hyperthermia System for Cancer Therapy (Co-Investigator)  
NIH-53716(2). 7/1/95-3/31/2000, Five Years, Annual Funding of \$237815

### TEACHING EXPERIENCE

Assistant Professor, Mechanical Eng. Department, U of Utah (Aug. 1994-).

- Research and Teaching in Nonlinear Dynamics, Vibrations, Controls and Intelligent Structures areas. Presently supervising Six graduate students.
- Developed a new 3-quarter Mechatronics Course, and a Control System Design Course.

## JOURNAL PUBLICATIONS

- E.A. Bailey, A.W. Dutton, M. Mattingly, S. Devasia and R.B. Roemer  
"A Comparison of Reduced-Order Modeling Techniques for Application in Hyperthermia Control and Estimation" Submitted to International Journal of Hyperthermia, Oct., 1996.
- S. Devasia "Output Tracking for Systems with Non-Hyperbolic and Near Non-Hyperbolic Internal Dynamics: Helicopter Hover Control" Submitted to AIAA Journal of Guidance, Control, and Dynamics, Sept., 1996.
- M. Mattingly, E. A. Bailey, A. W. Dutton, R. B. Roemer, and S. Devasia  
"Reduced-order Modeling for Hyperthermia: An Extended Balanced-Realization-Based Approach," Submitted to IEEE Trans. on Biomedical Eng., Aug. 1996.
- J.S. Dewey, and S. Devasia "Experimental and Theoretical Results in Output-Trajectory Redesign for Flexible Structures," Submitted to ASME Journal of Dynamic Systems, Measurement, and Control, June, 1996.
- S. Devasia and B. Paden, "Exact Output Tracking for Nonlinear Time-Varying Systems," Submitted to IEEE Transactions on Automatic Control, June 1995.
- S. Devasia, B. Paden, and C. Rossi "Exact-Output Tracking Theory for Systems with Parameter Jumps," Accepted, International Journal of Control, August, 1996.
- S. Devasia, "Optimal Output-Trajectory Redesign for Invertible Systems," AIAA J. of Guidance, Control, and Dynamics, Vol. 19 (5), Sept.-Oct., pp. 1189-1191, 1996.
- S. Devasia, D. Chen, and B. Paden, " Nonlinear Inversion-Based Output Tracking" IEEE Transactions on Automatic Control, Vol. 41 (7), pp. 930-942, July, 1996.
- Ph. Martin, S. Devasia, and B. Paden, "A different look at output tracking: Control of a VTOL Aircraft," Automatica, Vol. 32 (1), pp. 101-107, January 1996.
- S. Devasia, and E. Bayo, " Inverse Dynamics: Simultaneous Trajectory Tracking and Vibration Reduction with Distributed Actuators," Journal of Dynamics and Control, Vol. 4, pp. 299-309, 1994.
- R. Ledesma, S. Devasia, and E. Bayo, " Inverse Dynamics of Spatial Open-Chain Flexible Manipulators with Lumped and Distributed Actuators," Journal of Robotics Systems, Vol. 11 (4), pp. 327-338, 1994.
- S. Devasia, and E. Bayo, "Redundant Actuators to Achieve Minimal Vibration Tracking of Flexible Multibodies: Theory and Application," Journal of Nonlinear Dynamics , Vol. 6, pp. 419-431, 1994.
- S. Devasia, T. Meressi, B. Paden, and E. Bayo, " Piezo-electric Actuator Design for Vibration Suppression: Placement and Sizing, " AIAA Journal of Guidance, Control and Dynamics , Vol. 16 (5), pp. 859-864, 1993.

NIS  
51-63  
98137  
308  
245963

# Output Tracking for Systems with Non-Hyperbolic and Near Non-Hyperbolic Internal Dynamics: Helicopter Hover Control \*

Santosh Devasia †

Assistant Professor

Dept. of Mechanical Engineering

U. of Utah, SLC, UT 84112, USA

email: santosh@me.utah.edu Tel: (801) 581 4613

Submitted to AIAA J. of Guidance, Control, and Dynamics

---

\*Work Supported by NASA Ames Research Center Grant NAG-2-1042

†AIAA Member

*Abstract* - A technique to achieve output tracking for nonminimum phase linear systems with non-hyperbolic and near non-hyperbolic internal dynamics is presented. This approach integrates stable inversion techniques, that achieve exact-tracking, with approximation techniques, that modify the internal dynamics to achieve desirable performance. Such modification of the internal dynamics is used (a) to remove non-hyperbolicity which is an obstruction to applying stable inversion techniques and (b) to reduce large preactuation times needed to apply stable inversion for near non-hyperbolic cases. The method is applied to an example helicopter hover control problem with near non-hyperbolic internal dynamics for illustrating the trade-off between exact tracking and reduction of preactuation time.

## 1 Introduction

Precision output tracking controllers are needed to meet increasingly stringent performance requirements in applications like flexible structures, aircraft and air traffic control, robotics, and manufacturing systems. While perfect tracking of minimum phase systems is relatively easy to achieve, output tracking of nonminimum phase systems tends to be more challenging due to fundamental limitations on transient tracking performance [1]. This *poor* transient performance has been mitigated by using preactuation in the stable-inversion based approaches for nonminimum phase systems [2, 3, 4, 5]. However, the required preactuation time (during which *most* of the preactuation control effort is required) is large if the zeros of a linear system that lie on the open right half of the complex plane are close to the imaginary axis. In the limiting case, with the zeros on the imaginary axis (non-hyperbolic internal dynamics), presently available inversion-based techniques fail because the preactuation time



needed becomes infinite. In this paper we present a design technique for output tracking of linear non-minimum phase systems, which have non-hyperbolic and near non-hyperbolic non-minimum phase internal dynamics. This technique is then applied to an example helicopter hover control and simulation results are presented.

Output tracking has a long history marked by the development of regulator theory for linear systems by Francis and Wonham [6] and generalized to the nonlinear case by Byrnes and Isidori [7]. These approaches asymptotically track an output from a class of exosystem-generated outputs. Further, the Byrnes-Isidori regulator has been extended in [8, 9] and computational issues have been studied in [10, 11]. Although nonlinear regulator design is computationally difficult, the linear regulator is easily designed by solving a manageable set of linear equations. A problem, however, with the regulator approach is that the exosystem states are often switched to describe the desired output – this leads to transient tracking-errors after the switching instants. Such switching caused transient errors can be avoided by using inversion-based approaches to output tracking [4, 12]. Thus, it is advantageous to use inversion-based output tracking when precision tracking of a particular output trajectory is required.

Inversion, which is key to our approach, was restricted to causal inverses of minimum phase systems in the early works by Silverman and Hirschorn (e.g., [13, 14]) because they lead to unbounded inverses in the nonminimum phase case. Di Benedetto and Lucibello [15] considered the inversion of time varying nonminimum phase systems with a choice of the system's initial conditions. Instead of choosing initial conditions, preactuation was used by noncausal stable inversion techniques developed in [2, 3, 4, 16, 5]. Such noncausal inverses, which require preactuation, have been successfully applied to the output tracking of flexible

structures [17, 18] and aircraft and air traffic control [19, 20]. However, the fundamental limitation of presently available inversion- schemes is that they fail if the internal dynamics is non-hyperbolic. Even when the internal dynamics is hyperbolic, if the right-half-plane zeros of the system are close to the imaginary axis (the near non-hyperbolic case) then the required preactuation time tends to become unacceptably large. In summary, output tracking remains a challenge for nonminimum phase systems with non-hyperbolic or near non-hyperbolic internal dynamics.

There are several approximation based output tracking techniques, where the central philosophy is to replace the internal dynamics with a dynamics that provides satisfactory behavior, and then to develop the controller based on the altered system [21, 22, 23]. The technique most relevant to this paper is developed by Gopalswamy and Hedrick [23] – where trajectory modifications are considered to stabilize the internal dynamics. This technique, however, requires hyperbolicity of the internal dynamics for computational purposes. The development of computational techniques for stable inversion (e.g., [16]) motivates the present integration of the stable inversion scheme with approximation techniques - especially for systems with non-hyperbolic internal dynamics where the existing stable-inversion techniques fail. However, instead of stabilizing the unstable internal dynamics, we only aim to modify the non-hyperbolic behavior with a *small* perturbation of the internal dynamics. Additionally, in nonminimum phase systems with near non-hyperbolic internal dynamics the present approach allows a tradeoff between the precision tracking requirement and the amount of preactuation time, needed to apply the stable inversion based output tracking technique.

The approximate inversion-based technique is developed in Section 2 and the technique is applied to a helicopter hover control example in Section 3, where simulation results are

discussed. Our conclusions are in Section 4.

## 2 Stable Inversion

### 2.1 Inversion-Based Output Tracking Scheme

Here we describe how the inversion approach is used to develop output tracking controllers.

Consider a linear system described by

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ y(t) &= C x(t) \end{aligned} \tag{1}$$

which has the same number of inputs as outputs,  $u(t), y(t) \in \mathbb{R}^p$ , and  $x(t) \in \mathbb{R}^n$ . Let  $y_d(\cdot)$  be the desired output trajectory to be tracked. Then in the inversion-based approach we, first, find a nominal input-state trajectory,  $[u_{ff}(\cdot), x_{ref}(\cdot)]$  that satisfies the system equations (1) and yields the desired output exactly, i.e.,

$$\left. \begin{aligned} \dot{x}_{ref}(t) &= A x_{ref}(t) + B u_{ff}(t) \\ y_d(t) &= C x_{ref}(t) \end{aligned} \right\} \forall t \in (-\infty, \infty) \tag{2}$$

and, second, we stabilize the exact-output yielding state trajectory,  $x_{ref}(\cdot)$ , by using state feedback (see Figure 1). Thus  $x(t) \rightarrow x_{ref}(t)$  and  $y(t) \rightarrow y_d(t)$  as  $t \rightarrow \infty$  and output tracking is achieved. It is noted that in this output tracking scheme, the reference state trajectory  $x_{ref}(\cdot)$  and the feedforward input  $u_{ff}(\cdot)$  are computed off-line. While stabilization of the reference state trajectory can be easily achieved through standard techniques [24] like state feedback of the form  $K[x(t) - x_{ref}(t)]$ , the main challenge is to find the inverse input-state trajectory  $[u_{ff}(\cdot), x_{ref}(\cdot)]$  – especially for systems with nonminimum phase dynamics. This

paper addresses the off-line computation of the inverse input-state trajectory for a given desired trajectory,  $y_d(\cdot)$ .

## 2.2 The Internal Dynamics

In this subsection, it is shown that finding the inverse input-state trajectory is equivalent to finding bounded solutions to the system's internal dynamics. Let the linear system (1) have a well defined vector relative degree,  $r := [r_1, r_2, \dots, r_p]$ . Then the output's derivatives are given as [25]

$$\frac{d^{r_k}}{dt^{r_k}} y_k = C_k A^{r_k} x + C_k A^{r_k-1} B u \quad (3)$$

where  $C_k$  is the  $k^{\text{th}}$  row of  $C$ , and  $1 \leq k \leq p$ . In vector notation let equation (3) be rewritten as

$$y^{(r)}(t) = A_x x(t) + B_y u(t) \quad (4)$$

where

$$y^{(r)} := \left[ \frac{d^{r_1} y_1}{dt^{r_1}} \quad \frac{d^{r_2} y_2}{dt^{r_2}} \quad \dots \quad \frac{d^{r_p} y_p}{dt^{r_p}} \right]^T$$

$$A_x := \begin{bmatrix} C_1 A^{r_1} \\ C_2 A^{r_2} \\ \vdots \\ C_p A^{r_p} \end{bmatrix}; \quad B_y := \begin{bmatrix} C_1 A^{r_1-1} B \\ C_2 A^{r_2-1} B \\ \vdots \\ C_p A^{r_p-1} B \end{bmatrix},$$

and  $B_y$  is invertible because of the well-defined relative degree assumption. Equation (4) motivates the choice of the control law of the form

$$u_{ff}(t) = B_y^{-1} [y_d^{(r)}(t) - A_x x(t)] \quad (5)$$

for all  $t \in (-\infty, \infty)$ . Substituting this control law in equation (4), it is seen that exact tracking is maintained, i.e.,

$$y^{(r)}(t) = y_d^{(r)}(t).$$

To study the effect of this control law consider a change of coordinates  $T$  such that [25]

$$\begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} = T x(t)$$

where  $\zeta(t)$  consists of the output and its time-derivatives

$$\zeta(t) := \left[ y_1, \dot{y}_1, \dots, \frac{d^{r_1-1}}{dt^{r_1-1}} y_1, y_2, \dot{y}_2, \dots, \frac{d^{r_2-1}}{dt^{r_2-1}} y_2, \dots, y_p, \dot{y}_p, \dots, \frac{d^{r_p-1}}{dt^{r_p-1}} y_p \right]'$$

The system equation (1) can then be re-written in the new-coordinates as

$$\dot{\zeta}(t) = \hat{A}_1 \zeta + \hat{A}_2 \eta + \hat{B}_1 u \quad (6)$$

$$\dot{\eta}(t) = \hat{A}_3 \zeta + \hat{A}_4 \eta + \hat{B}_2 u \quad (7)$$

where

$$\hat{A} := T^{-1} A T := \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}; \quad \text{and} \quad \hat{B} := \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = T^{-1} B$$

In the new co-ordinates, the control law for maintaining exact tracking (Equation (5)) can be written as

$$u(t) = B_y^{-1} \left[ y_d^{(r)}(t) - A_\zeta \zeta_d(t) - A_\eta \eta(t) \right] \quad (8)$$

where

$$\begin{bmatrix} A_\zeta \\ A_\eta \end{bmatrix} := A_x T^{-1}.$$

Note that the desired  $\zeta(\cdot)$  is known when the desired output trajectory  $y_d(\cdot)$  and the output's time derivatives are specified. This desired  $\zeta(\cdot)$  is defined as  $\zeta_d(\cdot)$ . Since the control law was

chosen such that exact tracking is maintained,  $y^{(r)}(t) = y_d^{(r)}(t)$  we also have  $\dot{\zeta}(t) = \dot{\zeta}_d(t)$ , and Equations (6) and (7) become

$$\dot{\zeta}(t) = \dot{\zeta}_d(t) \quad (9)$$

$$\dot{\eta}(t) = \hat{A}_3\zeta + \hat{A}_4\eta + \hat{B}_2B_y^{-1} [y_d^{(r)}(t) - A_\zeta\zeta_d(t) - A_\eta\eta(t)]. \quad (10)$$

This is the inverse system, and in particular, Equation (10) is the internal dynamics. Solving the internal dynamics is key to finding the inverse input-state trajectories. If a bounded solution,  $\eta_d(\cdot)$ , to the internal dynamics (10) can be found then the feedforward input can be found through equation (8) as

$$u_{ff}(t) = B_y^{-1} [y_d^{(r)}(t) - A_\zeta\zeta_d(t) - A_\eta\eta_d(t)] \quad (11)$$

and the reference trajectory can be found as

$$x_{ref}(t) = T^{-1} \begin{bmatrix} \zeta_d(t) \\ \eta_d(t) \end{bmatrix}.$$

Thus a bounded solution to the internal dynamics (10) is required to find the inverse and to apply the output tracking scheme shown in Figure 1.

### 2.3 Modified Internal Dynamics

Standard inversion schemes [13, 14] that integrate (forward in time) the internal dynamics (10) lead to unbounded solutions since the internal dynamics is unstable for nonminimum phase systems. Noncausal inversion (e.g., [4]) leads to bounded but noncausal solution to the internal dynamics. Such stable-inversion techniques are, however, not applicable to systems with non-hyperbolic internal dynamics. In this subsection a compromise between

stable inversion and approximation based inversion schemes is proposed. The key is to modify the internal dynamics by giving up exact output tracking – enough to remove the non-hyperbolicity, and then to apply stable-inversion. The difference between the proposed technique and other approximation techniques is that the internal dynamics is perturbed only to remove the non-hyperbolicity, and not to stabilize the entire internal dynamics.

To change the non-hyperbolicity of the internal dynamics an extra term,  $v(t)$ , is added to the control law (8) as follows

$$u_{ff}(t) = B_y^{-1} \left[ y_d^{(r)}(t) - A_\zeta \zeta_d(t) - A_\eta \eta(t) + v(t) \right]. \quad (12)$$

With this modified control law the inverse system (Equations (9) and (10)) become

$$\frac{d}{dt} \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} = \hat{S} \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t) + G_v v(t) \quad (13)$$

where  $e_\zeta(t) := \zeta(t) - \zeta_d(t)$  is the error in the output and the output's derivatives,

$$\hat{S} = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & (\hat{A}_4 + \hat{B}_2 B_y^{-1} A_\eta) \end{bmatrix}, \quad G_v = \begin{bmatrix} \hat{B}_1 B_y^{-1} \\ \hat{B}_2 B_y^{-1} \end{bmatrix},$$

$$G_y = \begin{bmatrix} 0 & 0 \\ (\hat{A}_3 - \hat{B}_2 B_y^{-1} A_\zeta) & \hat{B}_2 B_y^{-1} \end{bmatrix}, \quad \text{and } Y_d(t) = \begin{bmatrix} \zeta_d(t) \\ y_d^{(r)}(t) \end{bmatrix}$$

Assuming that the original system  $(A, B)$  is controllable, we also have  $(\hat{S}, G_v)$  controllable and hence there exists a feedback of the form

$$v(t) = F \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \quad (14)$$

such that the modified inverse system (13) is hyperbolic – i.e., all poles on the imaginary axis are moved. Note that this change to an hyperbolic system can be achieved through arbitrarily

small  $F$  since non-hyperbolicity is not structurally stable property. The hyperbolic system

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} &= (S + G_v F) \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t) \\ &:= S \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t) \end{aligned} \quad (15)$$

is the modified inverse system. This modification of the internal dynamics can also be used to move unstable poles of the inverse system that may be close to the imaginary axis for reducing the required preactuation time. Next, stable inversion of the modified inverse system is carried out [4].

## 2.4 Computation of the Inverse

We begin by decoupling the modified internal dynamics (15) into stable ( $z_s$ ) and unstable ( $z_u$ ) subsystems. Since the modified internal dynamics is hyperbolic, there exists a decoupling transformation  $U$  such that the modified inverse system (15) can be written as

$$\begin{aligned} \dot{z}_s(t) &= S_s z_s(t) + G_s Y_d(t) \\ \dot{z}_u(t) &= S_u z_u(t) + G_u Y_d(t) \end{aligned} \quad (16)$$

where

$$z(t) := \begin{bmatrix} z_s(t) \\ z_u(t) \end{bmatrix} = U \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix}. \quad (17)$$

To find bounded solutions to the unstable inverse systems, the boundary conditions that  $z_s(-\infty) = 0$  and  $z_u(\infty) = 0$  are applied to Equation (16). This leads to unique bounded solutions to the modified internal dynamics by flowing the stable subsystem forward in time



and flowing the unstable system backward in time – as

$$z_{s,d}(t) = \int_{-\infty}^t e^{S_s(t-\tau)} G_u Y_d(\tau) d\tau \quad \forall t \in (-\infty, \infty), \quad (18)$$

$$z_{u,d}(t) = - \int_t^{\infty} e^{S_u(t-\tau)} G_u Y_d(\tau) d\tau \quad \forall t \in (-\infty, \infty).$$

This completes the technique. To summarize, the bounded solution (18) is used to find the reference state trajectory as  $x_{ref}(t) = T^{-1}U^{-1}z_d(t)$ , and to find the feedforward input,  $u_{ff}(\cdot)$  from equation (12). This inverse,  $[u_{ff}(\cdot), x_{ref}(\cdot)]$ , is then used in the control scheme shown in Figure 1 to obtain output tracking.

## 2.5 Preactuation Time and Unstable Poles of Inverse System

The connection between the amount of preactuation time required to apply the inversion-based feedforward input and the unstable poles of the modified inverse is established in the following Lemma. **Lemma** Let

1. the support of  $Y_d(\cdot)$  be contained in  $[t_0, \infty)$  for some  $t_0$ ,
2. all the unstable poles of internal dynamics represented by the eigenvalues of  $S_u$  lie to the right, in the complex plane, of the line  $\text{Re}(s) = \alpha$  for some positive  $\alpha$ , and
3.  $\|G_u Y_d(\cdot)\|_{\infty} < \beta$ .

Then there exists  $M$  such that  $\|u_{ff}(t)\|_{\infty} < M e^{\alpha(t-t_0)}$  for all time before the start of the maneuver,  $t < t_0$ .

**Proof:** From Condition 2 of the Lemma, there exists a positive constant  $M_{S_u}$  such that

$$\|e^{S_u(t-\tau)}\|_{\infty} < M_{S_u} e^{\alpha(t-\tau)} \quad \forall t < \tau. \quad (19)$$

Then for all  $t < t_0$ ,

$$\begin{aligned}
\|u_{ff}(t)\|_\infty &= \|B_y^{-1} [y_d^{(r)}(t) - A_\zeta \zeta_d(t) - A_\eta \eta(t) + v(t)]\|_\infty \text{ from Equation (12)} \\
&= \|B_y^{-1} [-A_\eta \eta(t) + v(t)]\|_\infty \text{ from Condition 1 of the Lemma} \\
&= \left\| \left( \begin{bmatrix} 0 & 0 \\ 0 & -B_y^{-1} A_\eta \end{bmatrix} + F \right) \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty \text{ from Equation (14)} \\
&:= \left\| A_U \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty \\
&\leq \|A_U\|_\infty \left\| \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty \\
&\leq \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \begin{bmatrix} z_s(t) \\ z_u(t) \end{bmatrix} \right\|_\infty \text{ from Equation (17)} \\
&= \|A_U\|_\infty \|U^{-1}\|_\infty \|z_u(t)\|_\infty \text{ since } z_s(t) = 0 \text{ for all } t < t_0 \\
&= \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \int_t^\infty e^{S_u(t-\tau)} G_u Y_d(\tau) d\tau \right\|_\infty \text{ from Equation (18)} \\
&= \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \int_{t_0}^\infty e^{S_u(t-\tau)} G_u Y_d(\tau) d\tau \right\|_\infty \text{ from Condition 1 of Lemma} \\
&< \beta M_{S_u} \|A_U\|_\infty \|U^{-1}\|_\infty \int_{t_0}^\infty e^{\alpha(t-\tau)} d\tau
\end{aligned}$$

from Equation (19) and Condition 3 of Lemma. Integrating the above expression we get

$$\begin{aligned}
\|u_{ff}(t)\|_\infty &< \frac{\beta M_{S_u}}{\alpha} \|A_U\|_\infty \|U^{-1}\|_\infty e^{\alpha(t-t_0)} \\
&:= M e^{\alpha(t-t_0)},
\end{aligned}$$

which concludes the proof. ■

The above Lemma states that the preactuation input tends to zero exponentially, as we go back in time from the start of the maneuver at  $t_0$ . The rate at which the preactuation

becomes zero can be increased by moving the unstable poles of the modified inverse away from the imaginary axis – at the expense of exact output tracking. The trade-off between exact-tracking of the desired output and reduction of the pre-actuation time is illustrated in the following example.

### 3 Example: Helicopter Hover Control

Here, we apply the output-tracking technique to the hover control of a Bell 205 Helicopter, which has near non-hyperbolic unstable internal dynamics. We consider one of the cases studied previously in [26], wherein the aircraft dynamics was trimmed at a nominal 5° pitch attitude, with a mid-range weight, a mid-position center of gravity and operating in-ground effect at near sea level. The linearized model is given as [26, 27]

$$\dot{x} = Ax + Bu \tag{20}$$

where

$$A = \begin{bmatrix} 0 & 0.03 & 0.18 & -0.01 & -0.42 & 0.08 & -9.81 & 0 \\ -0.10 & -0.39 & 0.09 & -0.10 & -0.72 & 0.68 & 0 & 0 \\ 0.01 & -0.01 & -0.19 & 0 & 0.23 & 0.04 & 0 & 0 \\ 0.02 & 0 & -0.41 & -0.05 & -0.27 & 0.27 & 0 & 9.81 \\ 0.03 & -0.02 & -0.88 & -0.04 & -0.57 & 0.14 & 0 & 0 \\ -0.01 & -0.02 & -0.06 & 0.07 & -0.32 & -0.71 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

$$B = \begin{bmatrix} 0.08 & 0.13 & 0 & 0 \\ -1.17 & 0.04 & 0 & 0.01 \\ 0 & -0.07 & 0 & 0.01 \\ -0.04 & 0 & 0.11 & 0.19 \\ -0.04 & 0 & 0.22 & 0.17 \\ 0.17 & 0 & 0.03 & -0.47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (22)$$

$$x = \begin{bmatrix} U \\ W \\ Q \\ V \\ P \\ R \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{roll rate} \\ \text{pitch attitude} \\ \text{roll attitude} \end{bmatrix}, \quad (23)$$

and

$$u = \begin{bmatrix} \delta_C \\ \delta_B \\ \delta_A \\ \delta_P \end{bmatrix} = \begin{bmatrix} \text{collective} \\ \text{longitudinal cyclic} \\ \text{lateral cyclic} \\ \text{tail rotor collective} \end{bmatrix}. \quad (24)$$

It is noted that the helicopter's actual dynamic behavior differs because of modeling errors like nonlinearities and unmodeled dynamics. In output tracking control schemes that depend on the model, such modeling errors need to be corrected through feedback in the control scheme (see Figure 1). In particular, modeling errors can be compensated by robust stabilization of the reference state-trajectory, see for example [26]. The goal of the present paper is to develop inversion-based feedforward and reference state trajectories for use in the control scheme shown in Figure 1. In the following, we apply the inversion technique to control the helicopters forward, lateral and vertical velocities and its yaw rate. The forward velocity and the yaw rate are to be kept at zero and the desired profiles of lateral and vertical velocities and accelerations are as shown in Figures 2 and 3.

### 3.1 Internal Dynamics

To find the internal dynamics we begin with a change in the co-ordinates. Let,  $\zeta$  be defined as the outputs

$$\zeta(t) := \begin{bmatrix} U(t) \\ W(t) \\ V(t) \\ R(t) \end{bmatrix} \quad (25)$$

and let  $\eta$  be the remaining states,

$$\eta(t) := \begin{bmatrix} Q(t) \\ \theta(t) \\ P(t) \\ \phi(t) \end{bmatrix} \quad (26)$$

In the  $(\zeta, \eta)$  co-ordinate system, given by

$$\begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t) \quad (27)$$

$$:= Tx(t),$$

the system equations can be re-written as

$$\begin{bmatrix} \dot{\zeta}(t) \\ \dot{\eta}(t) \end{bmatrix} = TAT^{-1} \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} + TBu(t)$$

$$:= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t). \quad (28)$$

Given a desired output trajectory and the desired output's time derivatives,  $[\zeta_d(\cdot), \dot{\zeta}_d(\cdot)]$ , the exact output tracking control law (see Section 2.4) is found from equation (28) as

$$u_{ff}(t) = B_1^{-1} [\dot{\zeta}_d(t) - A_1\zeta_d(t) - A_2\eta(t)] \quad (29)$$

With this control law the inverse system becomes (from equation (28))

$$\dot{\zeta}(t) = \dot{\zeta}_d(t) \quad (30)$$

$$\dot{\eta}(t) = A_4\eta(t) + A_3\zeta(t) + B_2u(t)$$

$$\begin{aligned}
&= [A_4 - B_2 B_1^{-1} A_2] \eta(t) + B_2 B_1^{-1} [\dot{\zeta}_d(t) - A_1 \zeta_d(t)] \\
&:= A_\eta \eta(t) + B_2 B_1^{-1} [\dot{\zeta}_d(t) - A_1 \zeta_d(t)].
\end{aligned} \tag{31}$$

The problem is solved by finding a bounded solution to the internal dynamics (31). However, the bounded solution found through stable-inversion is noncausal and could require a large preactuation time if the poles of the internal dynamics are unstable and lie close to the imaginary axis in the complex plane. For this particular example, there are two such complex conjugate poles near the imaginary axis,  $0.0425 \pm 4.3055i$ . We modify the exact-tracking control law (29) to shift these poles away from the imaginary axis to  $2 \pm 4.3055i$ . This is described next.

### 3.2 Modified Inverse System

Following the approach described in Sections 2.3, we modify the internal dynamics by adding a term  $v(t)$  to the control law (29) to obtain

$$u_{ff}(t) = B_1^{-1} [\dot{\zeta}_d(t) - A_1 \zeta_d(t) - A_2 \eta_d(t)] + v(t). \tag{32}$$

Substituting this control law into Equations 30 and 31, the modified inverse system is obtained as

$$\begin{aligned}
\begin{bmatrix} \dot{e}_\zeta(t) \\ \dot{\eta}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ A_3 & A_\eta \end{bmatrix} \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v(t) + \\
&\quad \begin{bmatrix} 0 & 0 \\ (A_3 - B_2 B_1^{-1} A_1) & B_2 B_1^{-1} \end{bmatrix} \begin{bmatrix} \zeta_d(t) \\ \dot{\zeta}_d(t) \end{bmatrix}
\end{aligned}$$

$$:= \hat{S} \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_v v(t) + G_y Y_d(t). \quad (33)$$

where  $e_\zeta(t) := \zeta(t) - \zeta_d(t)$ . The poles of the inverse system can be moved to any desired location by using the control  $v(t)$  because  $(\hat{S}, G_v)$  is controllable. However, such modifications, aimed at reducing preactuation time, will also lead to a loss of precision in output tracking. This tradeoff between the reduction of preactuation time and the loss of precision in tracking is illustrated through simulation.

### 3.3 Simulation Results and Discussion

Two sets of simulations were performed. First, stable inversion was applied to the original system which lead to exact output-tracking inverse input-state trajectories. Second, simulations were performed when the unstable poles of the inverse system are moved from  $0.0425 \pm 4.3055i$  to  $2 \pm 4.3055i$  for reducing the amount of preactuation time required. Further, the inverse system also has four poles at the origin - corresponding to four pure integrators for  $(e_\zeta(\cdot))$  dynamics - which were moved to  $-1, -2, -3$  and  $-4$  for stability of the numerical integration scheme.

Figures 2 and 3 show the desired output trajectories for the lateral and vertical motions (corresponding to unit displacements in the two directions), while the forward velocity and yaw rate were to be maintained at zero – the maneuver starts at time  $t = 0$ . Figures 4 and 5 show the output trajectories achieved by the inverse state-trajectory,  $x_{ref}(\cdot)$ , which is to be used as a reference trajectory in the feedback scheme shown in Figure 1. The corresponding feedforward inputs are shown in figures 6 and 7. Note here that the feedforward inputs



are non-zero before the start of the maneuver, i.e., time  $t < 0$ , and hence preactuation is required.

Figures 4 and 5 show that exact-output tracking reference state trajectories can be found, even when the internal dynamics is unstable, through the stable inversion approach. The stable-inversion technique yields bounded solutions to the unstable internal dynamics, i.e., the pitch and roll motions are bounded as shown in Figure 8. However, the feedforward input found through exact-inversion requires substantial preactuation time as shown in Figures 6 and 7, i.e., the preactuation remains non-zero for a significant time before the start of the maneuver at  $t = 0$ . Figure 9 shows that about 30 seconds of preactuation is needed to apply the inverse of the original system for output tracking – in contrast, modification of the internal dynamics reduces the preactuation needed from 30 seconds to 1 second (see Figure 9). As seen in Figures 4 and 5 the output trajectories are still tracked well by the modified inverse. Further, this substantial reduction in preactuation time is achieved with similar control efforts and with similar roll and pitch motions (see Figures 6-8). Thus the approach presented here, allows a trade-off between precision tracking and the amount of preactuation that is acceptable. Future work will generalize the results to nonlinear nonminimum phase systems with non-hyperbolic internal dynamics.

## 4 Conclusions

A technique to achieve output tracking for nonminimum phase linear systems with non-hyperbolic and near non-hyperbolic internal dynamics was presented. This approach is an integration of the stable inversion technique that aims at exact tracking with the approxi-

mation approach that modifies the internal dynamics to achieve desirable performance. The method was applied to an example helicopter hover control problem to illustrate the effects of modifying the internal dynamics. It was shown that substantial reduction in preactuation time is possible by giving up some of the precision in tracking – thus making the stable inversion approach viable for practical application.

## References

- [1] L. Qui, and E. J. Davison. Performance limitations of non-minimum phase systems in the servomechanism problem. *Automatica*, 29, March:337–349, 1993.
- [2] E. Bayo. A finite-element approach to control the end-point motion of single-link flexible robot. *J. of Robotic Systems*, 4,(1):63–75, 1987.
- [3] B. Paden and D. Chen. A state-space condition for the invertibility of nonlinear nonminimum-phase systems. *Advances in Robust and Nonlinear Control Systems, DSC-ASME*, 43:37–41, 1992.
- [4] S. Devasia, D. Chen, and B. Paden. Nonlinear inversion-based output tracking. *IEEE Transactions on Automatic Control*, 41(7):930–943, 1996.
- [5] L.R. Hunt, G. Meyer, and R. Su. Noncausal inverses for linear systems. *IEEE Transactions on Automatic Control*, 41(4):608–611, 1996.
- [6] B.A. Francis and W. M. Wonham. The internal model principle of control theory. *Automatica*, 12:457–465, 1976.

- [7] A. Isidori and C. I. Byrnes. Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2):131–140, 1990.
- [8] B. Castillo. Output regulation of nonlinear systems with more inputs than outputs. *International Journal of Control*, 57,(6):1343–1356, 1993.
- [9] J. Huang. Output regulation of nonlinear systems with nonhyperbolic zero dynamics. *IEEE Transactions on Automatic Control*, 40(8):1497–1500, 1995.
- [10] J. Huang and W. J. Rugh. An approximation method for the nonlinear servomechanism problem. *IEEE Transactions on Automatic Control*, Sept:1395–1398, 1992.
- [11] A. J. Krener. *The Construction of Optimal Linear and Nonlinear Regulators. In Systems, Models and Feedback: Theory and Application. A. Isidori and T.J. Tarn (editors).* Birkhauser, Boston, 1992.
- [12] S. Devasia, B. Paden, and C. Rossi. Exact-output tracking theory for systems with parameter jumps. *International Journal of Control*, Accepted, 1996.
- [13] L. M. Silverman. Inversion of multivariable linear systems. *IEEE Transactions on Automatic Control*, 14(3):270–276, 1969.
- [14] R. M. Hirschorn. Invertibility of multivariable nonlinear control system. *IEEE Transactions on Automatic Control*, 24(6):855–865, 1979.
- [15] M. D. Di Benedetto and P. Lucibello. Inversion of nonlinear time-varying systems. *IEEE Trans. Automatic Control*, 38:1259–1264, 1993.

- [16] S. Devasia and B. Paden. Exact output tracking for nonlinear time-varying systems. *Proceedings of the 33rd CDC, Lake Buena Vista, Florida*, pages 2346–2355, 1994.
- [17] B. Paden, D. Chen, R. Ledesma, and E. Bayo. Exponentially stable tracking control for multi-joint flexible manipulators. *ASME Journal of Dynamic Systems, Measurement and Control*, 115:53–59, 1993.
- [18] S. Devasia and E. Bayo. Redundant actuators to achieve minimal vibration trajectory tracking of flexible multibodies: Theory and application. *Journal of Nonlinear Dynamics*, 6:419–431, 1994.
- [19] P. Martin, S. Devasia, and B. Paden. A different look at output tracking: Control of a vtol aircraft. *Automatica*, 32(1):101–107, 1996.
- [20] G. Meyer, L. R. Hunt, and R. Su. Nonlinear system guidance in the presence of transmission zero dynamics. *NASA Technical Memorandum 4661*, January, 1995.
- [21] R. Gurumoorthy and S. R. Sanders. Controlling non-minimum phase nonlinear systems - the inverted pendulum on a cart example. *Proceedings of the American Control Conference*, pages 680–685, 1993.
- [22] Z. Xu and J. Hauser, "Higher order approximate feedback linearization about a manifold for multi-input systems" a, *IEEE Transactions on Automatic Control*. Vol. 40 No.5, pp. 833-840, 1995.
- [23] S. Gopalswamy and J.K. Hedrick. Tracking nonlinear non-minimum phase systems using sliding control. *International Journal of Control*, 57.5:1141–1158, 1993.

- [24] Hassan K. Khalil. *Nonlinear Systems*. Macmillan, 1991.
- [25] A. Isidori. *Nonlinear Control Systems: An Introduction*. Springer-Verlag, 1989.
- [26] J. K. Pieper, S. Baillie, and K. R. Goheen. Linear quadratic optimal model-following control of a helicopter in hover. *Optimal Control Applications and Methods*, 17:123–140, 1996.
- [27] R. K. Heffley, W. F. Jewell, J. M. Lehman, and R. A. Van Winkel. A compilation of helicopter handling qualities data. *NASA Contract Report 3144 under Grant NAS2-9344*, 1, 1979.

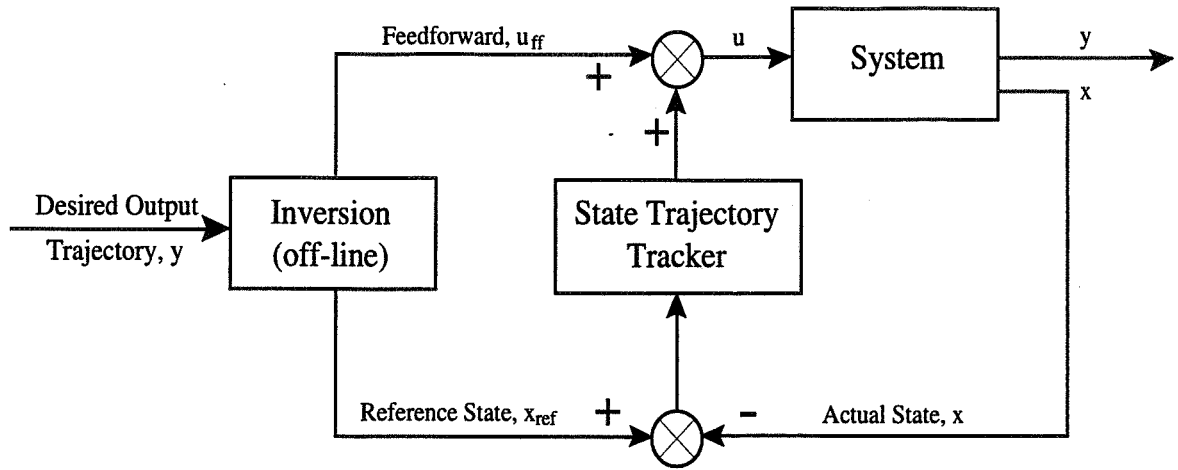


Figure 1. The Control Scheme

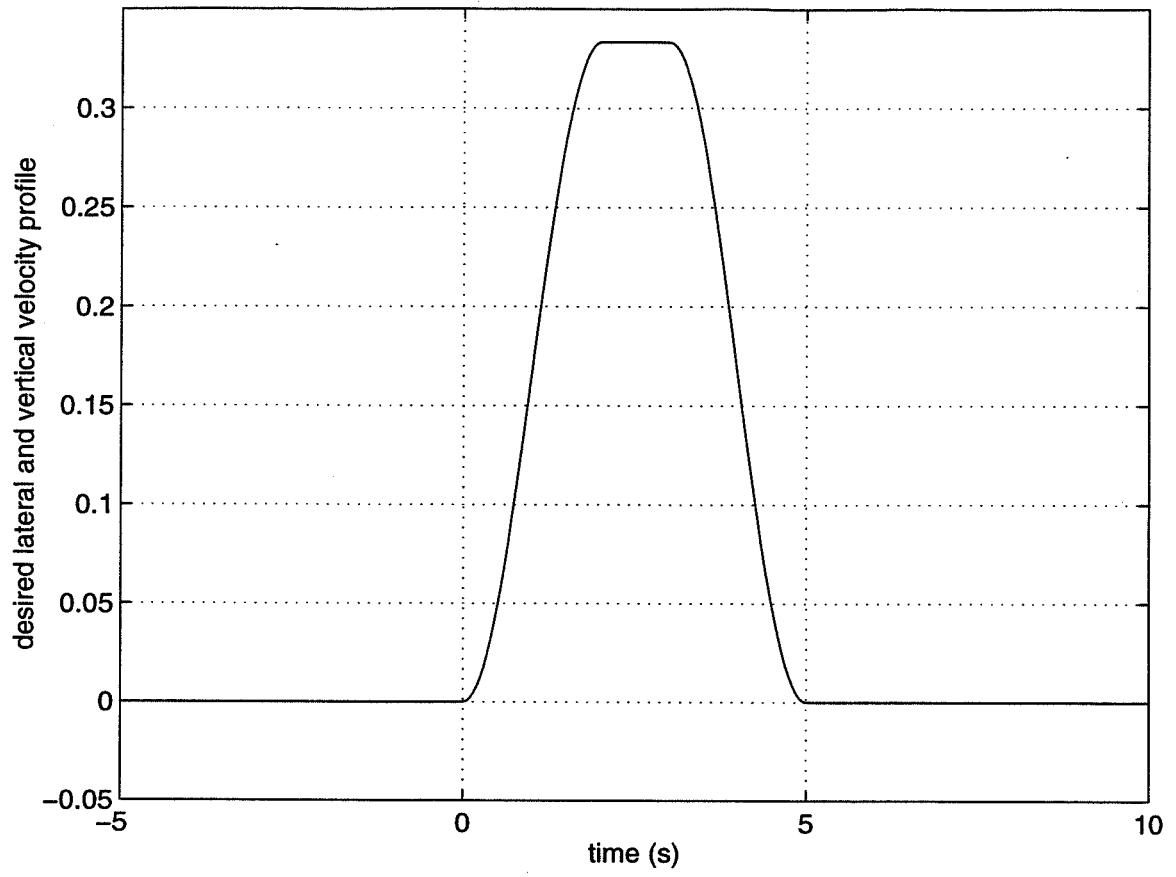


Figure 2. Desired Lateral and Vertical Velocity Profile.

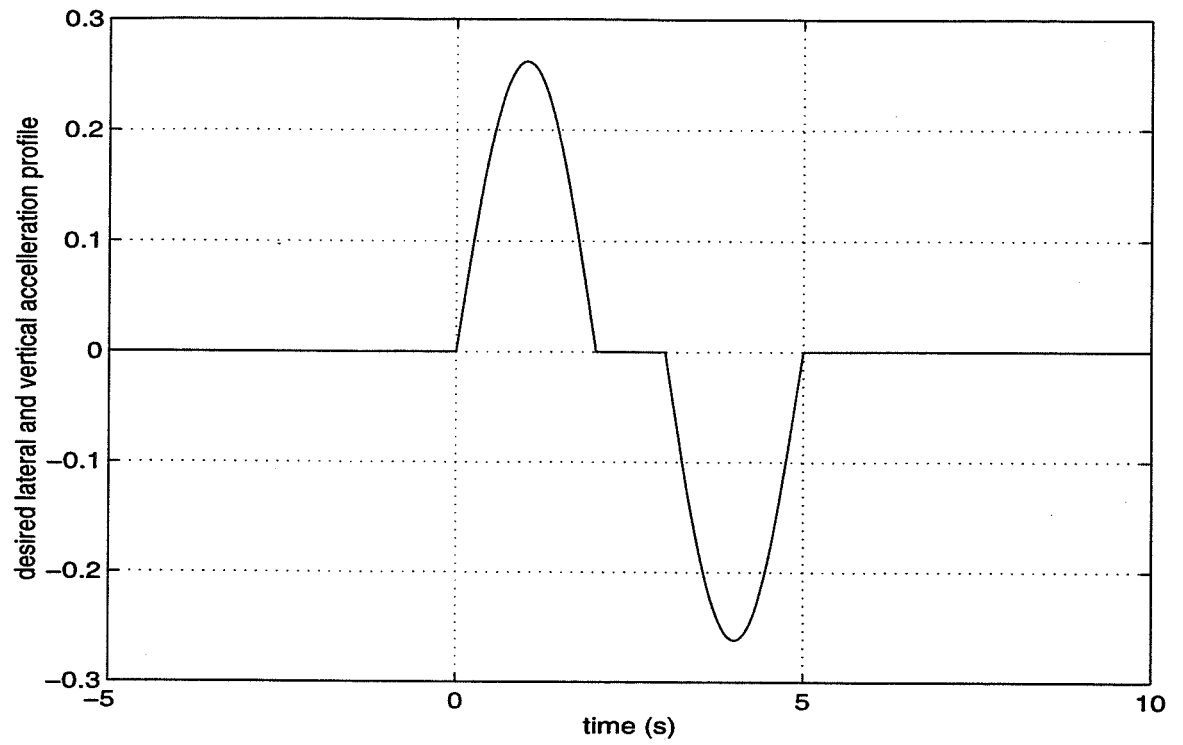


Figure 3. Desired Lateral and Vertical Acceleration Profile.



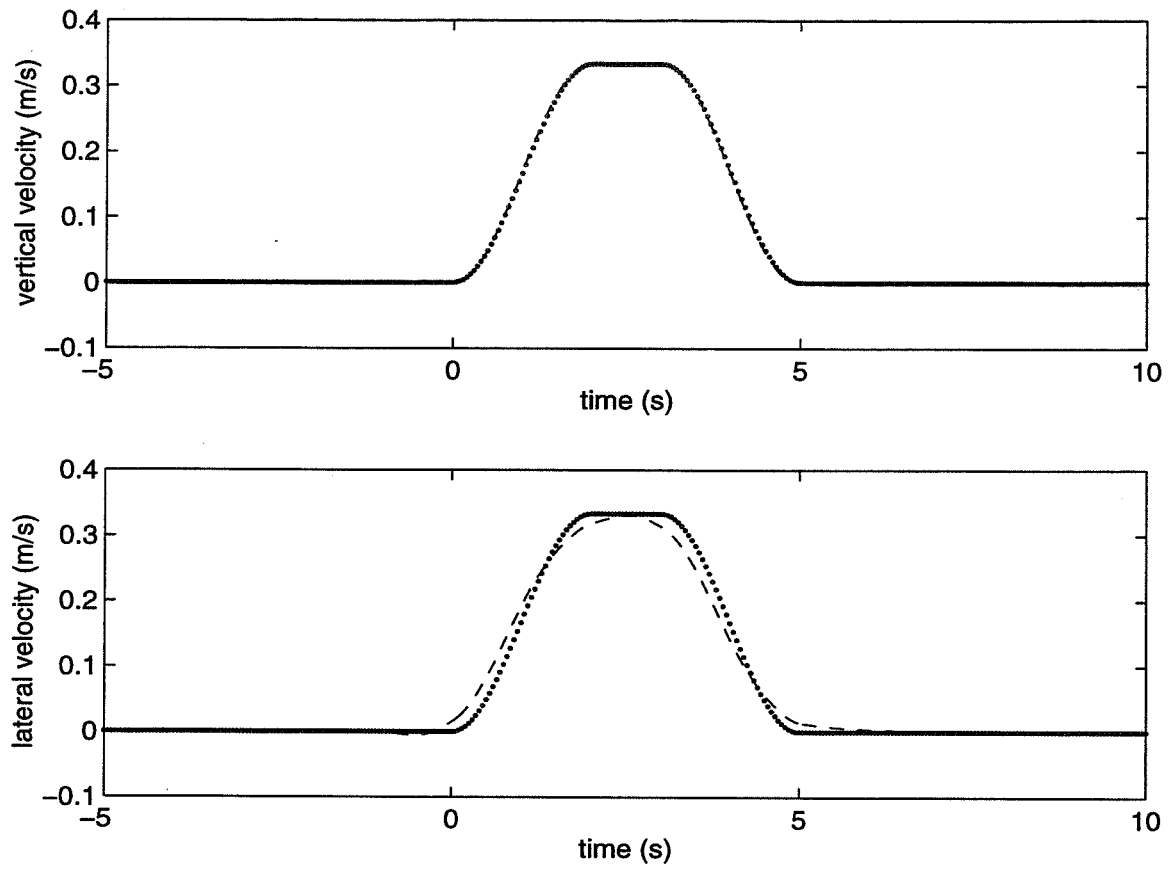


Figure 4. Lateral and Vertical Velocity achieved by the Inverse Reference State Trajectory. The dotted line is for the exact-tracking case without modification of the internal dynamics and the dashed line is with modification.

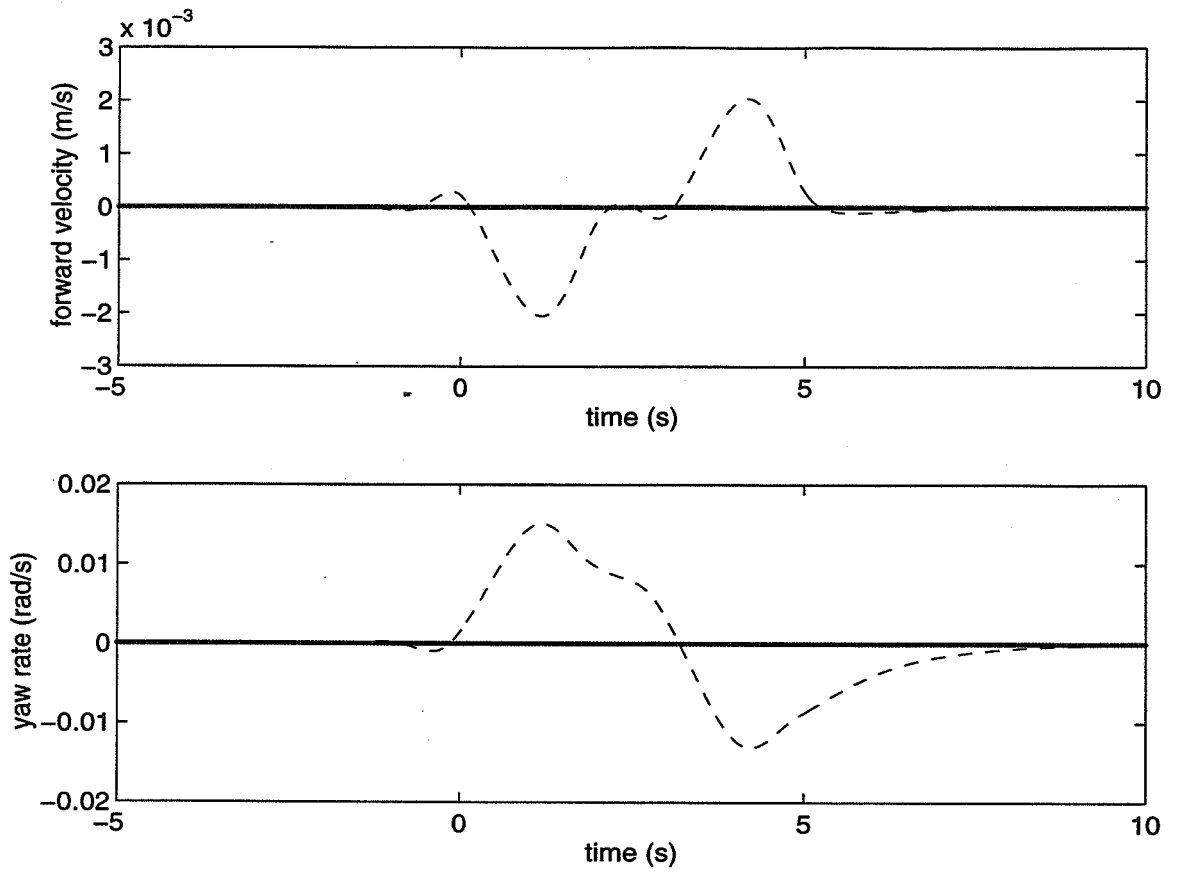


Figure 5. Forward Velocity and Yaw Rate achieved by the Inverse Reference State Trajectory – the desired value is zero. The dotted line is for the exact-tracking case without modification of the internal dynamics and the dashed line is with modification.

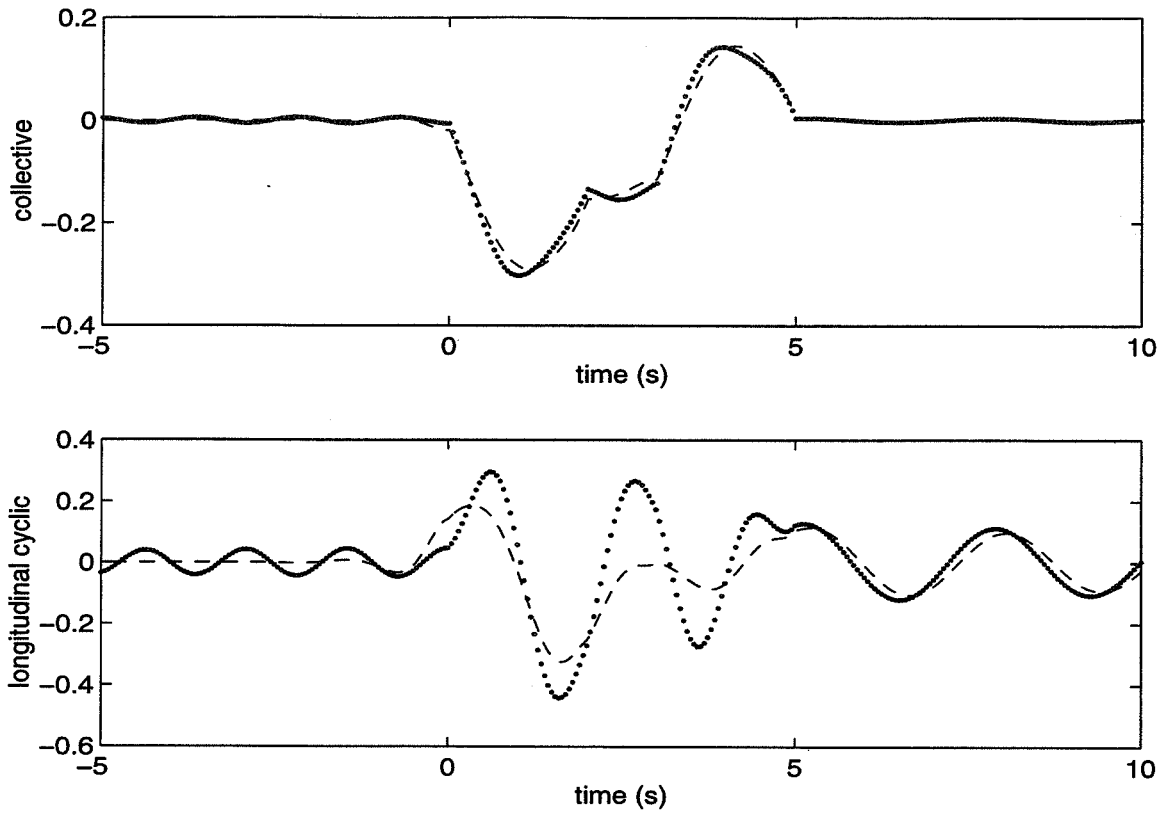


Figure 6. Feedforward Inputs. Dotted line is without modification of internal dynamics and the dashed line is with modification.

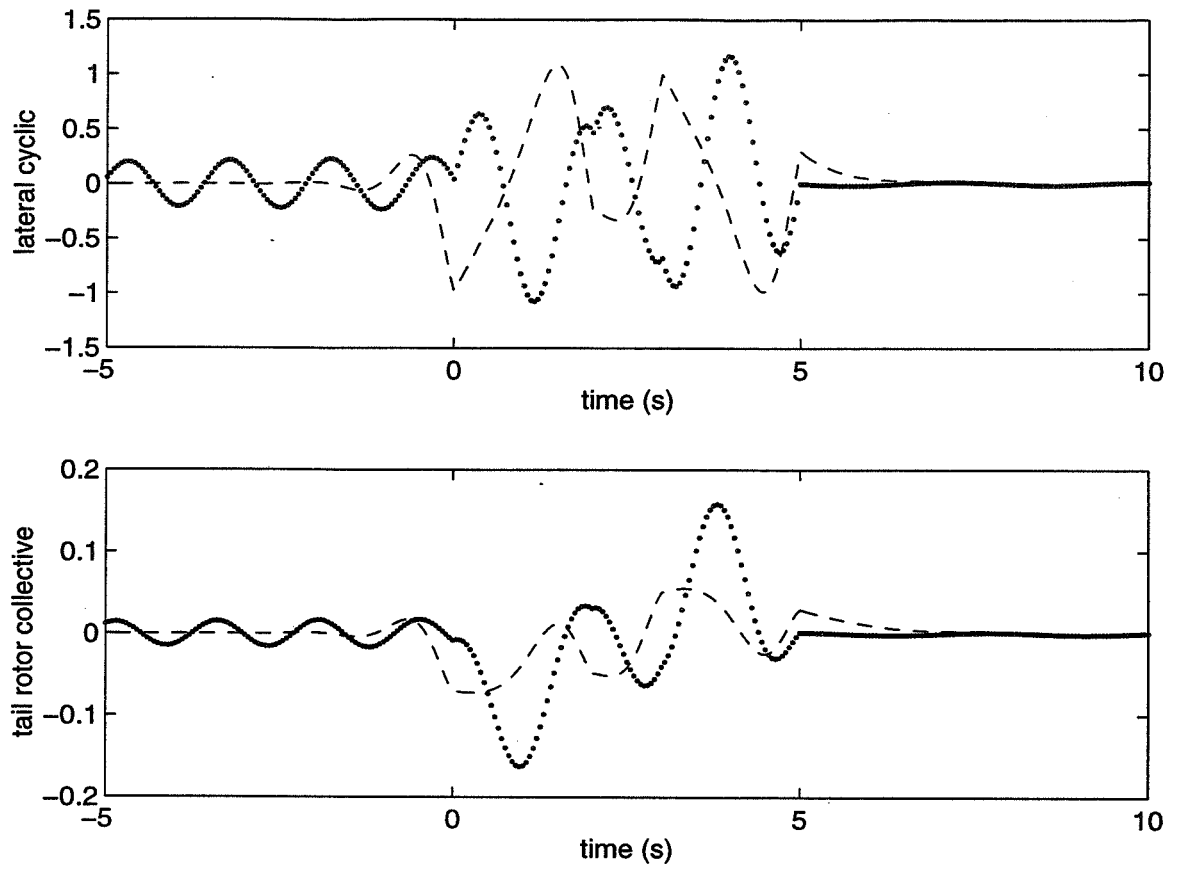


Figure 7. Feedforward Inputs. Dotted line is without modification of internal dynamics and the dashed line is with modification.

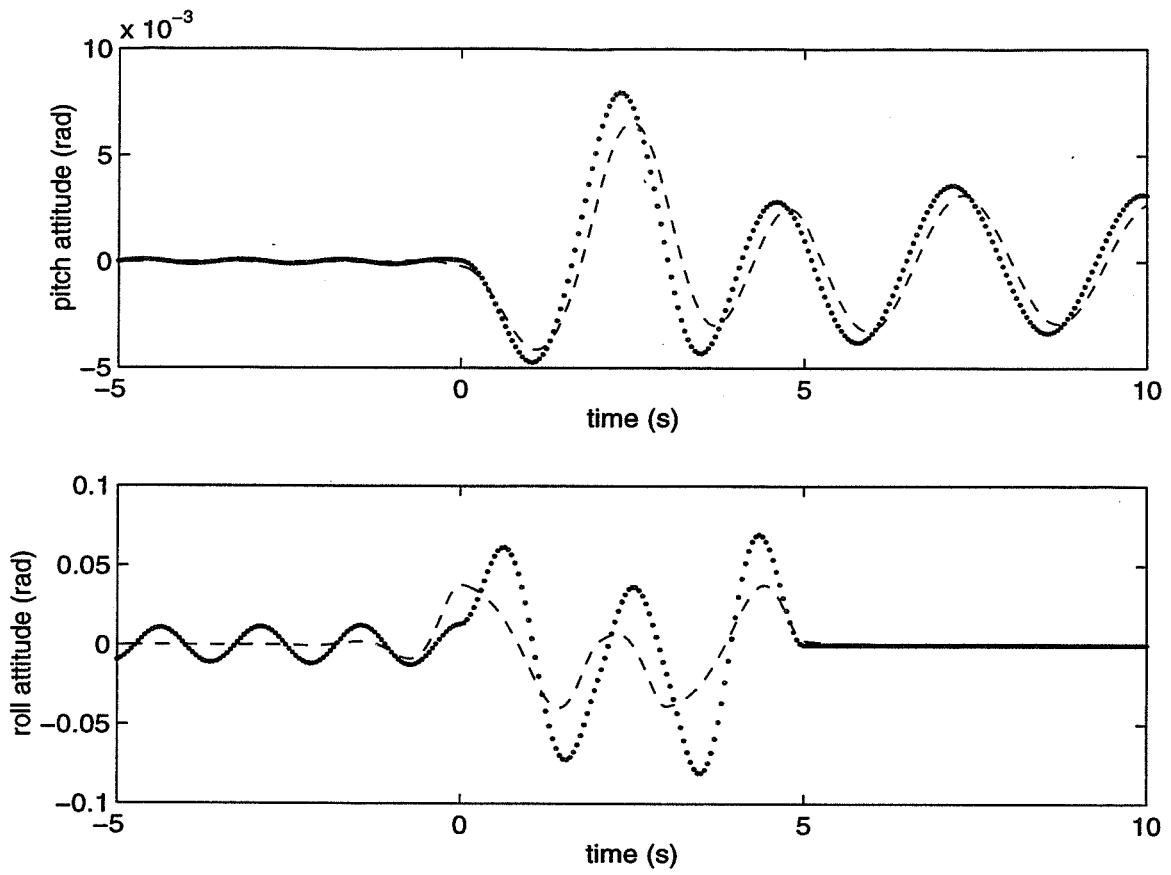


Figure 8. Internal Dynamics. Dotted line is without modification of internal dynamics and the dashed line is with modification.

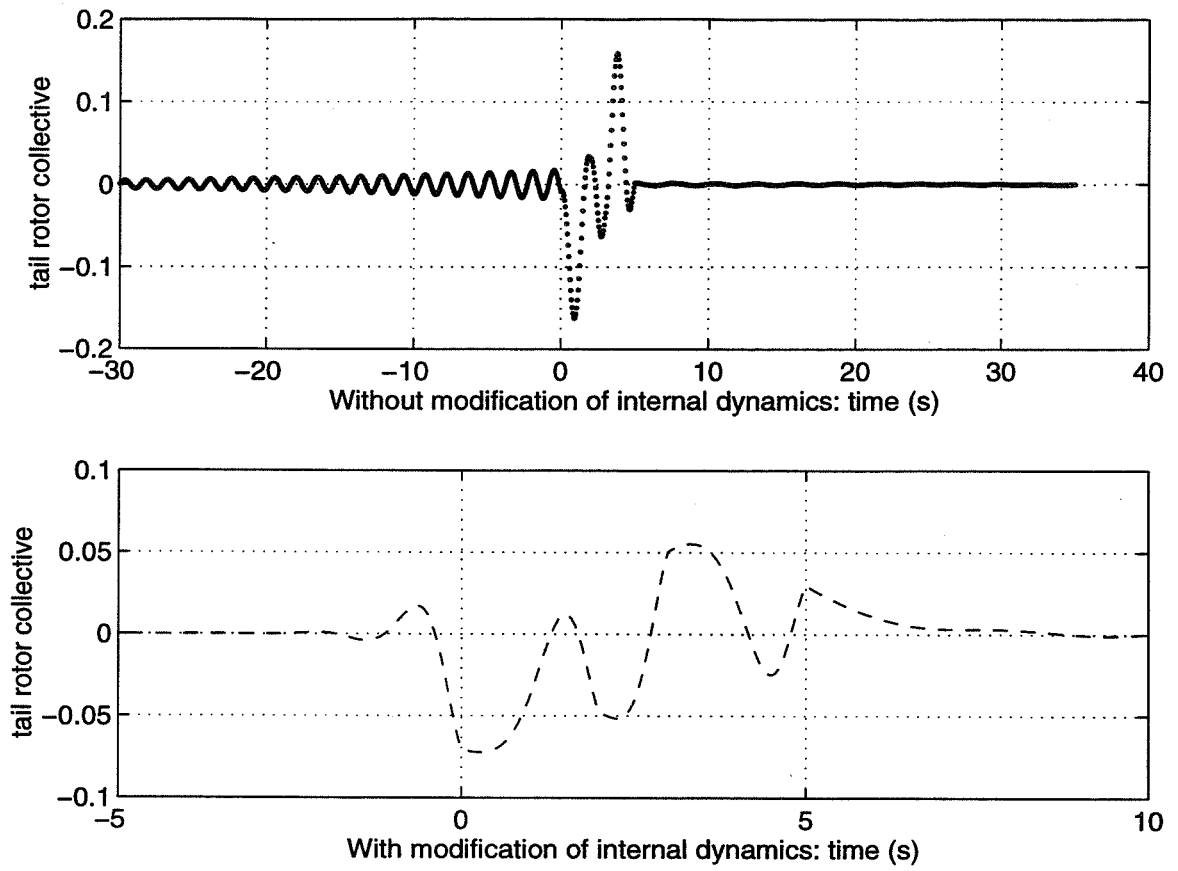


Figure 9. Comparison of Required Preactuation in the Feedforward.

NIS

52-63

98138

245964

221

# Exact-Output Tracking Theory for Systems with Parameter Jumps \*

Santosh Devasia\*, Brad Paden†, Carlo Rossi‡

\*Dept. of Mechanical Engineering

U. of Utah, Salt Lake City, UT 84112, USA

e-mail : santosh@me.utah.edu Tel: (801) 581 4613

†Dept. of Mechanical Engineering

U. of California at Santa Barbara, CA 93106, USA

‡DEIS - Università di Bologna

Viale Risorgimento 2, 401362 Bologna - Italy

*Abstract* - In this paper we consider the exact output tracking problem for systems with parameter jumps. Necessary and sufficient conditions are derived for the elimination of switching-introduced output transient. Previous works have studied this problem by developing a regulator that maintains exact tracking through parameter jumps (switches). Such techniques are, however, only applicable to minimum-phase systems. In contrast, our approach is applicable to nonminimum-phase systems and obtains bounded but possibly non-causal solutions. If the reference trajectories are generated by an exosystem, then we develop an exact-tracking controller in a feedback form. As in standard regulator theory, we obtain a linear map from the states of the exosystem to the desired system state which is defined via a matrix differential equation. The constant solution of this differential equation provides asymptotic tracking, and coincides with the feedback law used in standard regulator theory. The obtained results are applied to a simple flexible manipulator with jumps in the pay-load mass.

## 1. INTRODUCTION

We study the exact-output tracking of systems, which are described by

$$\begin{cases} \dot{x}(t) = A[k(t)]x(t) + B[k(t)]u(t) \\ y(t) = C[k(t)]x(t) \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ , with same number of inputs as outputs  $u(t), y(t) \in \mathbb{R}^p$ . The system matrices  $A(k), B(k), C(k)$  are constant over time intervals  $I_k$ , where  $k$  belongs to a finite index set  $\mathcal{K} \triangleq [0, \dots, N]$ , and the parameter change (switch) at times  $t = t_1, t_2, \dots, t_N$ , (See Figure 1). Here, the switching time are known, in contrast to systems where the switches may be signal driven.

For constant linear systems, the asymptotic output-tracking problems for constant linear systems has received much attention in the past. In particular, the regulator theory (*Francis (1977); Basile and Marro (1992); Wonham (1985)*) provides a general framework in which the asymptotic output tracking can be solved when the reference trajectory is generated through a linear exosystem. In the presence of switches in the system, one technique for achieving output regulation is to switch the regulator. Note that regulation can be recovered between two consecutive switches (due to asymptotic properties) – especially if the switching occurs far apart in time. However, this technique also tends to induce transients in the output during the switches.



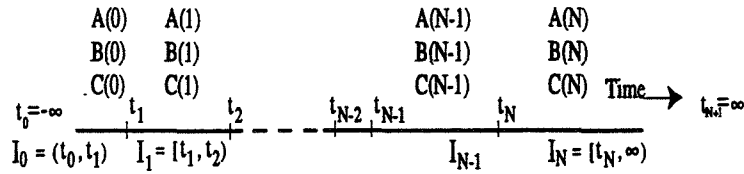


Figure 1: *The Switching Times*

In order to eliminate these *switching-caused* transients, a regulation scheme that maintains exact trajectory tracking across system switches must be used. This fairly new problem has been studied in (*Marro and Piazzi (1993)*), for minimum-phase systems. In this work, a feedforward action is used in conjunction with the feedback defined by the regulator in order to cancel the output transients across the switches.

In the present paper we propose an alternative approach for exact output-tracking of switched-systems, which is also applicable to nonminimum-phase systems. In the nonminimum-phase case, a bounded non-causal solution is obtained (*Devasia et al. (1996)*) that requires the preknowledge of the reference trajectory and of all the switching times. Such exact tracking schemes based on noncausal schemes is useful in problems like aircraft guidance (*Meyer et al. (1995)* and *Hunt et al. (1996)*).

We present necessary and sufficient conditions for the solvability of the inversion problem for linear systems with switches; the inverse is used to track the desired output. We consider two kinds of desired output trajectories: (a) a single pre-specified trajectory or (b) belonging to a class of outputs generated by a given linear exosystem, that could undergo parameter changes as well. In this latter case, we obtain the solution in a time-varying feedback form, where the feedback matrix satisfies a matrix ordinary differential equation. The equilibrium solution of this differential equation solves the asymptotic output tracking problem, and coincides with the feedback matrix resulting from the standard regulator. This establishes an interesting connection between our approach and the traditional regulator theory.

The paper is organized as follows: in Section 2 the exact tracking of a single prescribed output trajectory is considered, and necessary and sufficient conditions are presented. A geometric version of the obtained conditions is

also provided. In Section 3, the case of reference trajectory obtained through a linear exosystem will be treated. The conditions of the previous section when rearranged establish a close relationship with the traditional theory of output regulation. Section 4 focuses on the additional problem of stability of the closed loop system. Finally, section 5 presents the application of the developed theory to a simple nonminimum-phase switched system, given by a flexible beam subjected to step variation of the pay-load mass. Conclusions end the paper.

## 2. TRACKING A PRESCRIBED OUTPUT TRAJECTORY

Below we formulate the exact tracking problem for a prescribed output trajectory, and establish necessary and sufficient conditions for its solvability. Geometric interpretations of these conditions are also provided.

### *The Inversion Problem*

Given a desired output trajectory  $y_d$ , find a pair of state and input trajectories  $x_d, u_d$  such that

1.  $x_d, u_d$  satisfy the system equation (1):

$$\dot{x}_d(t) = A[k(t)]x_d(t) + B[k(t)]u_d(t); \quad \forall t \in (-\infty, \infty), \quad (2)$$

2. exact output tracking is achieved (even across switches):

$$y_d(t) = C[k(t)]x_d(t); \quad \forall t \in (-\infty, \infty),$$

3. and the inputs and state are bounded:

$$\|x_d(\cdot)\|_\infty < \infty,$$

$$\|u_d(\cdot)\|_\infty < \infty.$$

### *Using the Inverse for Exact-Output Tracking*

The existence of an inverse  $(u_d, x_d)$  implies that there are input-state trajectories that yield the desired output – exact output tracking is easily achieved by stabilizing the desired state trajectory. For example, use  $u_d$  as feedforward and use the error  $x - x_d$  for feedback (see Figure 2). Stabilization is not the central issue in this paper, and any scheme for feedback design can

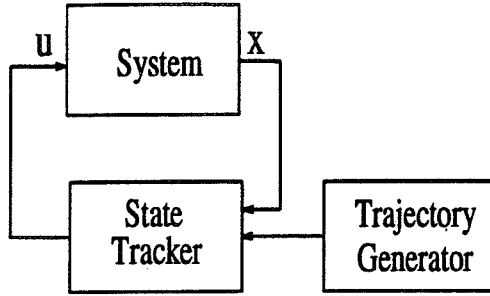


Figure 2: *The Control Scheme*

be used. For example, given  $(A(k), B(k))$  controllable for all  $k$ , the system may be stabilized through pole placement with all the closed loop poles in the same locations for all  $k$ .

Note that typically, the initial conditions of the system are different from the initial conditions of the desired state trajectory leading to initial transient errors typical of all tracking controllers. However, once the desired level of tracking is achieved (due to exponential reduction in error) our technique will maintain tracking across parameter switches. In contrast, switching standard regulators when the system parameters change will cause transient errors at the switching instants – tracking will not be maintained across switches.

#### *Exact-Tracking Maintaining Input*

We will assume the following:

**Assumption 1** *System  $A(k), B(k), C(k)$  has well defined vector relative degree (Isidori (1989)) for each  $k \in \mathcal{K}$ .*

Then we can find a co-ordinate transformation  $\hat{Q}_k$  such that (Isidori (1989))

$$\begin{bmatrix} Y_k(t) \\ z_k(t) \end{bmatrix} = \hat{Q}_k x(t) = \begin{bmatrix} C_k^* \\ Z_k \end{bmatrix} x(t)$$

where

$$Y_k(t) = [y_1, \dot{y}_1, \dots, \frac{d^{(r_{k,1}-1)}}{dt^{(r_{k,1}-1)}} y_1, y_2, \dot{y}_2, \dots, \frac{d^{(r_{k,2}-1)}}{dt^{(r_{k,2}-1)}} y_2, \dots, y_p, \dot{y}_p, \dots, \frac{d^{(r_{k,p}-1)}}{dt^{(r_{k,p}-1)}} y_p]' ,$$

and  $r_k = [r_{k,1}, r_{k,2}, \dots, r_{k,p}]$  is the systems vector relative degree for  $t_k \leq t \leq t_{k+1}$  (where we assume  $t_0 = -\infty$ ). Here,  $C_k^*$  maps the system states into

the outputs and its time derivatives and  $Z_k$  maps the states into the internal dynamics,  $z_k$ .

Note that a necessary condition for exact output tracking in the interval  $I_k$  is that the system state at time  $t_k$  is such that

$$Y_k(t_k) = Y_{k,d}(t_k)$$

In addition, to maintain exact tracking we need to ensure that

$$\dot{Y}_k(t) = \dot{Y}_{k,d}(t) \quad \forall t \in [t_k, t_{k+1})$$

Let

$$y_d^{(r_k)} = \left[ \frac{d^{(r_{k,1})}}{dt^{(r_{k,1})}} y_{d,1}, \frac{d^{(r_{k,2})}}{dt^{(r_{k,2})}} y_{d,2}, \dots, \frac{d^{(r_{k,p})}}{dt^{(r_{k,p})}} y_{d,p} \right]',$$

then we can find the following unique control law (provided assumption 1 is satisfied) (*Isidori (1989)*)

$$u_d(t) = F_k x(t) + G_k y_d^{(r_k)}(t) \quad (3)$$

such that the time derivatives of the output is the same as that of the desired output trajectory  $y_d$  (this is also a necessary condition for exact output tracking). This exact-tracking control is completely determined by the state  $x(t_k)$ , and by the desired output along with its derivatives up to the order  $r_k$ .

Substituting control law (3) in (2) we obtain for  $t \in I_k$ :

$$\dot{x}(t) = A_y(k)x(t) + B_y(k)y_d^{(r_k)}(t) \quad (4)$$

where  $A_y(k) = A(k) + B(k)F_k$ ,  $B_y(k) = B(k)G_k$ . In the transformed coordinates the system equations for  $t \in I_k$  are of the form

$$\begin{aligned} \dot{Y}_k(t) &= \dot{Y}_{k,d}(t) \\ \dot{z}_k(t) &= A_z(k)z_k(t) + A_{z,y}(k)Y_k(t) + B_z(k)y_d^{(r_k)}(t) \end{aligned} \quad (5)$$

Our objective is to define under which conditions it is possible to define a feasible state trajectory  $x_d(t)$  such that exact trajectory tracking is preserved through all the time intervals. There are two main hurdles. Firstly, the existence of at least a state trajectory which maintains exact output tracking has to be determined. This depends on *compatibility* of the desired output

with the given system. Secondly, the state trajectories have to be bounded. In systems with unstable internal-dynamics (nonminimum phase systems) generic solutions tend to be unbounded. In this case, we need to establish additional conditions for the existence of bounded solutions to the internal dynamics.

In this paper, we restrict ourselves to the case where (a)  $Y_{k,d}$ , the output along with its time derivatives, has a compact support  $[T_i, T_f] \in (-\infty, \infty)$ , (b) the switching occurs within this compact set, and (c) the internal-dynamics is hyperbolic. More formally

**Assumption 2** *The desired output trajectory  $y_d(\cdot)$  and its time derivatives are bounded and have compact support  $S(y_d) = [T_i, T_f]$ . The switching in the system parameters are at fixed times  $t_k \in (T_i, T_f)$  for every  $k \in [1, \dots, N]$ .*

**Assumption 3** *System (1) has hyperbolic internal dynamics, i.e. the eigenvalues of  $A_z(k)$  have non-zero real parts (no centers) for  $k = 0$  and  $k = N$ . This is equivalent to requiring that the original system (1) have no zeros which lie on the imaginary axis (Isidori (1989)) for  $k = 0$  and  $k = N$ .*

The last assumption implies the existence of transformations  $Q_0, Q_N$  such that system state can be partitioned into

$$\begin{bmatrix} Y_k(t) \\ z_{sk}(t) \\ z_{uk}(t) \end{bmatrix} = Q_k x(t) = \begin{bmatrix} C_k^* \\ Z_{sk} \\ Z_{uk} \end{bmatrix} x(t) \quad (6)$$

where  $z_{sk}$  and  $z_{uk}$  are the coordinates for the stable and the unstable subspaces of the system's internal-dynamics.

#### Notations

Towards establishing conditions for the existence of solutions to the exact tracking problem for a prescribed output, we first study the dynamic evolution of the system for a given initial condition.

Given an initial condition in an interval  $I_k$  the system's evolution for  $t_k \leq t \leq t_{k+1}$  is described by

$$x_d(t) = e^{A_y(k)(t-t_k)} x_d(t_k) + \int_{t_k}^t e^{A_y(k)(t-\tau)} B_y(k) y_d^{(r_k)}(\tau) d\tau.$$

In a more compact form

$$x_d(t) = \Phi_k(t, t_k)x_d(t_k) + h_k(t, t_k)$$

where

$$\Phi_k(t, t_k) = e^{A_y(k)(t-t_k)}$$

and

$$h_k(t, t_k) = \int_{t_k}^t e^{A_y(k)(t-\tau)} B_y(k) y_d^{(\tau_k)}(\tau) d\tau.$$

The above equations describe the flow in an interval where the system doesn't undergo switches. In order to obtain a representation of the system state in terms of an initial state that does not belong to the same interval, we define flow compositions as follows:

$$\Psi_{k,i}(t, t_i) = \Phi_k(t, t_k) \circ \Phi_{k-1}(t_k, t_{k-1}) \circ \dots \circ \Phi_i(t_{i+1}, t_i)$$

$$H_{k,i}(t, t_i) = h_k(t, t_k) + \sum_{j=i}^{k-1} \Psi_{k-1,j}(t_{k-1}, t_j) h_j(t_{j+1}, t_j)$$

where

$$\Psi_{i,i}(t, t_i) = \Phi_i(t, t_i)$$

and

$$H_{i,i}(t, t_i) = h_i(t, t_i)$$

The system evolution for an initial condition  $x(T_i)$  can be rewritten as:

$$x(t) = \Psi_{k,0}(t, T_i)x(T_i) + H_{k,0}(t, T_i) \quad (7)$$

*Necessary and Sufficient Conditions*

We first formally state the result.

**Lemma 1** *Under assumptions 1-3, the exact output tracking problem is solvable with bounded solution if and only if the system of equations:*

$$Y_{dk}(t_k) = C_k^* Q_k \Psi_{k-1,0}(t_k, T_i) Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix} + C_k^* Q_k H_{k-1,0}(t_k, T_i) \quad \forall k = 1, \dots, N \quad (8)$$

$$0 = Z_{uk}Q_N \left( \Psi_{N,0}(T_f, T_i)Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix} + H_{N,0}(T_f, T_i) \right) \quad (9)$$

admits solution in  $z_{u,0}(T_i)$ .

*Proof.* System trajectories outside  $[T_i, T_f]$ , the compact support of  $Y_d$ , are bounded if and only if  $z_{uN}$ , the unstable component of the internal-dynamics, is zero at the end of the motion  $T_f$  and similarly the stable component  $z_{s0}$  is zero before time  $T_i$ . Formally:

$$x(T_i) = Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix}$$

$$x(T_f) = Q_N^{-1} \begin{bmatrix} 0 \\ z_{s,N}(T_f) \\ 0 \end{bmatrix}$$

Substituting the previous expressions in (7) computed at  $t = T_f$

$$x(T_f) = \Psi_{N,0}(T_f, T_i)x(T_i) + H_{N,0}(T_f, T_i)$$

gives (9). In addition, exact tracking in every interval  $I_k$  is possible if it is possible to find state trajectories that are continuous and such that  $C_k^*x(t_k) = Y_{dk}(t_k)$ . By using again (7) that gives the state at  $t = t_k$  as a function of the initial state and the constraint on  $x(T_i)$ , (8) easily follows. ■

Equation (8) will be referred in the following as *compatibility conditions*, while equation (9) will be referred to as *stability condition*. The compatibility conditions ensure that  $Y_d$  does not jump across switches (else unbounded inputs will be required)! The stability condition ensures that the autonomous system dynamics for  $t \rightarrow \pm\infty$  is bounded.

The algebraic conditions expressed by lemma 1 can also be interpreted in a geometric coordinate-free framework. To this end, let

$$\mathcal{L}_k = \{x : Y_k = Y_{d,k}(t_k)\} \quad k \in [0, \dots, N]$$

represents the set of the admissible system states at time  $t_k$  in order to achieve exact tracking in the time interval  $t_k \leq t < t_{k+1}$ .  $\mathcal{L}_k$  is in general

a linear variety in the state space, that reduces to a linear subspace (the system internal dynamics) for  $k = 0$ .

A necessary condition for achieving exact tracking when  $t < T_i$  is  $x_d(T_i) \in \mathcal{L}_0$ . Further, to maintain a bounded solution for all  $t < T_i$  it is necessary that the initial state belong to the unstable subspace of the system internal dynamics  $x_d(T_i) \in \mathcal{L}_{u,0}$ .

Note that every  $x_d(T_i)$  determines a unique  $x_d(t_1)$ , given by

$$x_d(t_1) = \Phi_0(t_1, T_i)x_d(T_i) + h_0(t_1, T_i)$$

Hence, we can define the image of the subspace  $\mathcal{L}_{u,0}$  as:

$$\Phi_0(t_1, t_0) \circ \mathcal{L}_{u,0} = \{x : x = \Phi_0(t_1, t_0)y + h_0(t_1, t_0); y \in \mathcal{L}_{u,0}\}$$

which represents the linear variety composed by the points reachable at time  $t_1$  with the constraint of  $y(t) = y_d(t)$  for all  $t \in [t_0, t_1)$ .

To maintain exact tracking in the next interval  $t \in [t_1, t_2)$ , it is necessary that  $x_d(t_1) \in \mathcal{L}_1$ . The compatibility condition at time  $t = t_1$  states that

$$x_d(t_1) \in \mathcal{L}_1 \cap \Phi_0(t_1, T_i) \circ \mathcal{L}_{u,0}$$

which is possible if and only if the linear variety:

$$\mathcal{S}_1 = \mathcal{L}_1 \cap \Phi_0(t_1, T_i) \circ \mathcal{L}_{u,0}$$

is not empty, i.e. if and only if  $\mathcal{L}_1$  intersects the image of  $\mathcal{L}_{u,0}$  under the system flow. The same procedure can be repeated for the switching time  $t = t_2$ . Starting from  $\mathcal{S}_1$ , we can flow forward in time. In order to achieve exact tracking in the interval  $[t_2, t_3)$ , the image of  $\mathcal{S}_1$  must intersect  $\mathcal{L}_2$ , i.e. the set

$$\mathcal{S}_2 = \mathcal{L}_2 \cap \Phi_1(t_2, t_1) \circ \mathcal{S}_1$$

must be not empty and more generally, the exact tracking in  $t \in [T_i, t_{k+1})$  is possible if and only if the image of  $\mathcal{S}_{k-1}$  under the system flow  $\Phi_{k-1}(t_k, t_{k-1})$  intersects  $\mathcal{L}_k$ , i.e. the set:

$$\mathcal{S}_k = \mathcal{L}_k \cap \Phi_{k-1}(t_k, t_{k-1}) \circ \mathcal{S}_{k-1} \quad (10)$$

is not empty for every  $k = 1, \dots, N$ . However, in order to obtain a bounded solution for  $t > T_f$ , the final state at time  $t = T_f$  must belong to the stable subspace of the system internal dynamics  $\mathcal{L}_{s,N}$ . Let

$$\mathcal{S}_{T_f} \triangleq \mathcal{L}_{s,N} \cap \Phi_N(T_f, t_N) \circ \mathcal{S}_N.$$

Hence, we have proved an analogue of lemma 1 in geometric terms



**Lemma 2** *The exact output tracking problem is solvable if and only if  $\mathcal{S}_{T_f}$  is non-empty, i.e.,*

$$\mathcal{S}_{T_f} \neq \emptyset. \quad (11)$$

Since the flow of a linear system is a homeomorphism, the dimension of a linear variety and that of its image are equal, and hence:

$$\dim(\tilde{\mathcal{S}}_i) \geq \dim(\mathcal{S}_j); \quad j \geq i$$

This means that at each iteration (10) the set of possible solutions could reduce at each  $K$  and that no solution is possible if it becomes the empty set for some  $k$  - i.e., it is empty for every  $j \geq k$ .

*Switched systems with invariant internal-dynamics subspace*

We present below a particular case in which the given conditions considerably simplify. This exemplifies the obtained results and will be illustrated with an example in section 5. Let

**Assumption 4** *System (1) has constant relative degree  $r = r_k$  for every  $k$  and matrix  $C_k^* = C^*$  is constant for every  $k$ .*

It follows from the previous assumption that the coordinates outside the internal dynamics  $Y_k$  are the same for every  $k$ , and thus the internal dynamics subspace is the same for every  $k$ . Note that the stable and unstable subspaces may still be different, and may switch around, but are constrained to belong to the same subspace!

Since  $Y_k = C^*x$ , the continuity of  $x$  implies that the compatibility conditions are always satisfied. Formally.

**Lemma 3** *If assumption 4 is satisfied, then the compatibility conditions are satisfied for every smooth enough ( $C^r$ ) desired output trajectory  $y_d(t)$ .*

In addition,  $\Phi_k(t_{k+1}, t_k) \circ \mathcal{S}_k \subset \mathcal{L}_{k+1}$  implies that

$$\mathcal{S}_{k+1} = \Phi_k(t_{k+1}, t_k) \circ \mathcal{S}_k$$

is nonempty since  $0 \in \mathcal{S}_0$ . Further  $\dim(\mathcal{S}_n) = \dim(\mathcal{L}_{0,u})$ .

The additional condition for boundedness of solutions for  $t > T_f$  is met if and only if  $\mathcal{S}_N$  intersects  $\mathcal{L}_{s,N}$ . The linear variety  $\mathcal{S}_N$  can be expressed as

$$\mathcal{S}_N = \text{Im}(S_N) + v$$

while the stable subspace of the internal dynamics can be expressed as

$$\mathcal{L}_{s,N} = \text{Im}(L_{s,N}).$$

It immediately follows from the previous considerations:

**Lemma 4** *If assumptions 1-4 hold, then the exact output tracking problem with bounded solution is solvable if*

$$\text{rank} \begin{bmatrix} S_N & L_{N,s} \end{bmatrix} = n_z \quad (12)$$

where  $n_z$  is the dimension of the system internal dynamics at  $t = T_f$ .

As a last remark to this section, note that if not only the internal dynamics subspace remains constant across the switchings as ensured by Assumption 4, but also its stable and unstable subspaces do, then condition (12) is always satisfied and the problem has a solution for every admissible  $y_d(t)$ . Moreover, this solution is unique for every given  $y_d(t)$ . This implies that the exact-tracking problem for a system without switches is always solvable.

### 3. $y_d$ GIVEN THROUGH AN EXOSYSTEM

In this section we consider how the previous results can be specified when the reference trajectory is not completely general but it is generated through a known linear exosystem, given by:

$$\begin{cases} \dot{x}_e = A_e(k)x_e \\ y_d = C_e(k)x_e \end{cases} \quad (13)$$

First we solve the tracking problem when the initial state of the exosystem (at time  $T_i$ ) is known in advance, hence the obtained result will be valid for the particular reference trajectory determined by the initial condition. Later, we will extend the approach to the case of unknown initial conditions. In

this case, we look for asymptotic tracking of the reference trajectories for arbitrary initial conditions of the exosystem.

We begin by studying the case where the state of the exosystem  $x_e(T_i)$  is known. The system equation (4) becomes

$$\dot{x}_d(t) = A_y(k)x(t) + B_y(k)C_e^*(k)x_e(t), \quad (14)$$

with  $y_d^{(r_k)}(t) \triangleq C_e^*(k)x_e(t)$  since all the time derivatives of the output can be written in terms of  $x_e$  by using equation 13.

The system evolution can then be rewritten as

$$x(t) = \Psi_{k,0}(t, T_i)x(T_i) + \hat{H}_{k,0}(t, T_i)x_e(T_i) \quad (15)$$

where

$$\begin{aligned} \hat{H}_{k,i}(t, t_i) &= \hat{h}_k(t, t_k) + \sum_{j=i}^{k-1} \Psi_{k-1,j}(t_{k-1}, t_j) \hat{h}_j(t_{j+1}, t_j), \\ \hat{h}_k(t, t_k) &= \int_{t_k}^t e^{A_y(k)(t-\tau)} B_y(k) C_e^*(k) \hat{\Psi}_{k,0}(\tau, T_i) d\tau. \end{aligned}$$

and  $\hat{\Psi}_{k,0}(\tau, T_i)$  is the evolution of the exosystem (analogous to equation 7). A solution to the exact tracking problem exists if lemma 2 is satisfied. The compatibility condition is satisfied for all initial conditions  $x_d(T_i) \in \mathcal{L}_0$  if  $C_e^* = C^*$ . The stability condition becomes

$$0 = s_1 z_{u,0}(T_i) + s_2 x_e(T_i). \quad (16)$$

where

$$\begin{aligned} s_1 &= Z_{uN} \Psi_{N,0}(T_f, T_i) Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ I z_{u,0}(T_i) \end{bmatrix} \\ s_2 &= Z_{uN} \hat{H}_{N,0}(T_f, T_i) \end{aligned}$$

In what is to follow, we will assume that the above equation has a unique solution (iff  $s_1$  is invertible). This yields a one to one relationship between the plant's state and the exosystem as follows

$$\begin{aligned} x_d(T_i) &= Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,T_i}(T_i) \end{bmatrix} \\ &= Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ -s_1^{-1} s_2 x_e(T_i) \end{bmatrix} \triangleq G(T_i) x_e(T_i) \end{aligned}$$

What is interesting is that we can also write the desired exact tracking state trajectory in terms of the exosystem state. Substituting the above expression in equation (15) we get

$$x_d(t) = G(t)x_e(t)$$

where

$$G(t) \triangleq \left[ \Psi_{k,T_i}(t, T_i)G(T_i) + \hat{H}_{k,T_i}(t, T_i)C_e^* \right] \hat{\Psi}_{k,0}^{-1}(N, T_i)x_e(t) \quad (17)$$

It may be verified that  $G(t)$  satisfies the differential equation

$$\dot{G}_k(t) = A_y(k)G_k(t) - G_k^T(t)A_e(k) + C_e^*.$$

In the case of no switching a constant solution always exists for the above Lyapunov equation provided the eigenvalues of the exosystem  $A_e$  are different from the zeros of the plant eigenvalues of  $A_y$ . The above equation also provides a control strategy when the exosystem states are not known. We estimate the state of the exosystem as  $\hat{x}_e$  and regulate the trajectory  $\hat{x}_d = G(t)\hat{x}_e$ . The stability of such a controller is studied in the next section.

#### 4. STABILIZATION

If the state of the exosystem  $x_e$  is not known, then we could estimate it as  $\hat{x}_e$  with  $\|x_e(t) - \hat{x}_e(t)\|_2 \leq K_e e^{\alpha_e t} \|x_e(0) - \hat{x}_e(0)\|_2$ . We use  $\hat{x}_d(t) = G(t)\hat{x}_e(t)$  as the estimated desired state trajectory, and stabilize this trajectory by using the control scheme shown in Figure 3. Note that the feedforward used (equation 3) is completely specified in terms of the exosystem's state-estimate as follows

$$\begin{aligned} u_d &= F_{k,1}x_d(t) + F_{k,2}y_d^{(r_k)}(t) \\ &= F_{k,1}G(t)\hat{x}_e(t) + F_{k,2}C_e^*\hat{x}_e(t) \\ &\triangleq F_k\hat{x}_e(t). \end{aligned}$$

The state equations are of the form

$$\dot{x} = A_k x + B_k (F_k \hat{x}_e + K(x - G(t)\hat{x}_e)) .$$

We require that the system in each interval is either stable or stabilizable. For simplicity, we assume that  $A(k) + B(k)K$  is Hurwitz for all  $k$  - the

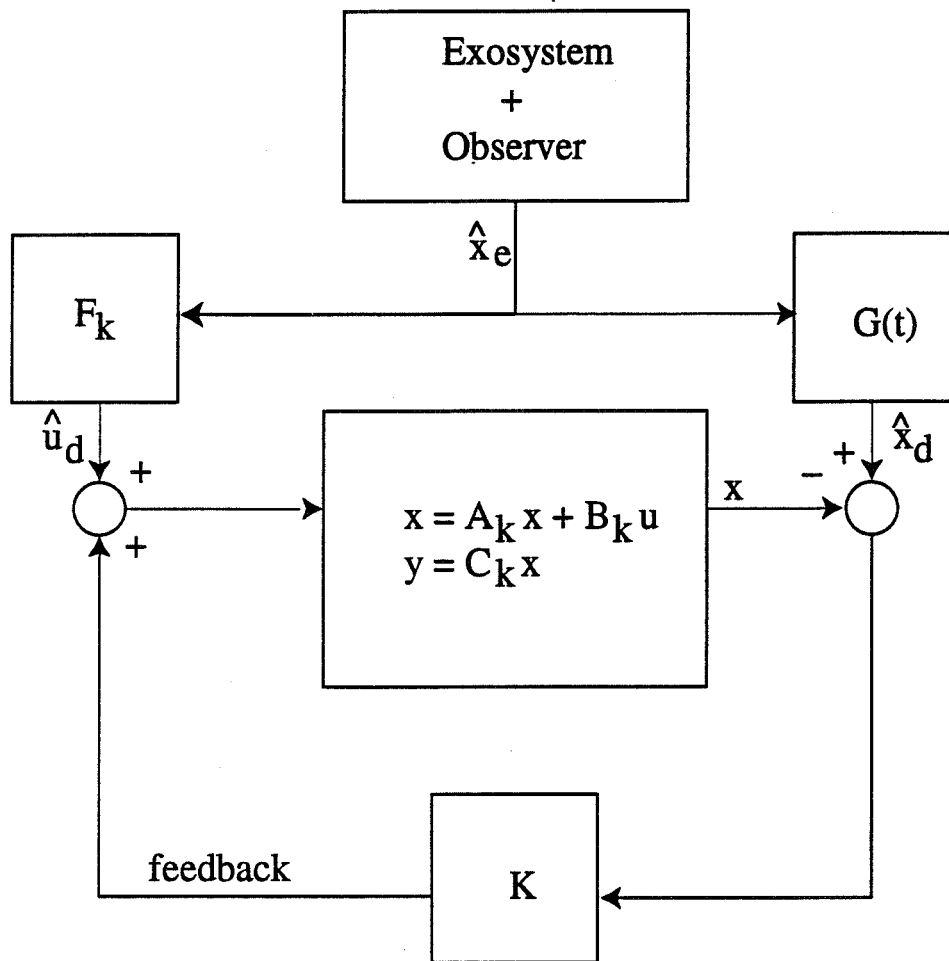


Figure 3: *Trajectory Tracking with Exosystem*

arguments remain valid if the system is stabilized through any other feedback control scheme.

The desired trajectory satisfies

$$\dot{x}_d = A_k x_d + B_k F_k x_e.$$

Let  $e := x - x_d$ . Then the difference of the last two equations yields

$$\dot{e} = (A_k + B_k K)e + (B_k F_k + B_k K G(t))(\hat{x}_e - x_e)$$

The exponential stability of the error dynamics system follows from the following lemma

**Lemma 5** *Given the system  $\dot{e} = Ae + v(t)$  where  $K_1 e^{\alpha_1 t} < \|e^{-At}\|_2 < K_2 e^{\alpha_2 t}$  and  $\|v(t)\|_2 < K_3 e^{-\alpha_3 t}$ , with  $K_1, K_2, K_3$  positive, then  $e = 0$  is an exponentially stable equilibrium point provided  $\alpha_3 > \alpha_2 > 0$*

**Proof:** Using the variation of constants formula

$$\begin{aligned} \|e^{-At}e(t)\|_2 &\leq \|e(0) + \int_0^t e^{-A\tau}v(\tau)d\tau\|_2 \\ &\leq \|e(0)\|_2 + \|\int_0^t e^{-A\tau}v(\tau)d\tau\|_2 \\ &\leq \|e(0)\|_2 + \int_0^t \|e^{-A\tau}v(\tau)\|_2 d\tau \\ &\leq \|e(0)\|_2 + K_2 K_3 \int_0^t e^{\alpha_2 - \alpha_3} d\tau \\ &\leq \|e(0)\|_2 + K_2 K_3 \frac{1}{\alpha_3 - \alpha_2} \end{aligned}$$

provided  $\alpha_3 > \alpha_2$ . Hence

$$K_1 \|e^{\alpha_1 t}e(t)\|_2 \leq \|e^{-At}e(t)\|_2 \leq \|e(0)\|_2 + K_2 K_3 \frac{1}{\alpha_3 - \alpha_2}$$

the rhs being a constant. Therefore, for all  $\epsilon > 0$  there exists a positive constant  $K$  such that

$$\|e(t)\|_2 < K e^{-(\alpha_1 - \epsilon)t}$$

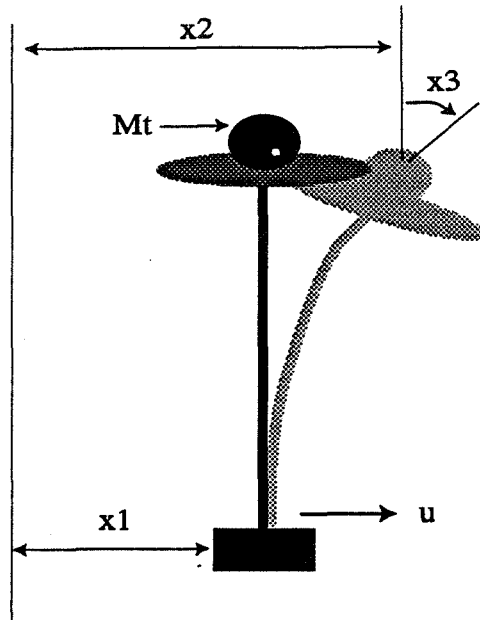


Figure 4: *Example*

## 5. EXAMPLE

Consider the flexible structure, cantilevered at the base and free at the top, shown in figure 4. It is modeled (with Finite Element Method) with a single flexural element. The degrees of freedom are the translational motion at the base ( $x_1$ ), and the top ( $x_2$ ), and the rotation at the top,  $x_3$ . Input is a translational force at the base and the system output is  $x_2$ . The structure is loaded with a mass  $m_t$ , which is changed at several instances. The flexural element has the following properties: mass 420; length 1; elastic modulus 1; and cross sectional area moment of inertia 1. The objective is to maintain the top of the structure along a prescribed trajectory so as to facilitate the transfer of the load. The equations of motion can be described by

$$M\ddot{x} + Sx = \hat{B}u$$

where

$$M = \begin{bmatrix} 156 & 54 & -13 \\ 54 & 156 + m_t & -22 \\ -13 & -22 & 4 \end{bmatrix} \quad S = \begin{bmatrix} 12 & -12 & 6 \\ -12 & 12 & -6 \\ 6 & -6 & 4 \end{bmatrix}$$

$x = [x_1 \ x_2 \ x_3]'$ , and  $\hat{B} = [1 \ 0 \ 0]'$ .

In the standard form  $x = [x', \dot{x}]'$  (abuse of notation) we get the dynamics as

$$\begin{aligned}\dot{x} &= A_k x + B_k u \\ y_k &= C_k x\end{aligned}$$

where

$$A_k = \begin{bmatrix} 0 & I \\ -M^{-1}S & 0 \end{bmatrix}; \quad B_k = \begin{bmatrix} 0 \\ M^{-1}\hat{B} \end{bmatrix}; \quad C_k = [0 \ 1 \ 0 \ 0 \ 0 \ 0];$$

The desired trajectory is generated by an exosystem of the form  $\dot{x}_e = A_e x_e$ , where

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and the desired output is  $y_d = [0 \ 0 \ 0 \ 1]x_e \triangleq C_e x_e$ . We also switch the exosystem to  $A_e = 0^{4 \times 4}$  at the initial and final times  $T_i = 0$  and  $T_f = 2 * \pi$ . The first two states of the exosystem form an oscillator and the second state is then integrated twice to obtain the desired output. Note that  $(A_e, C_e)$  is observable. Hence the exosystem states can be estimated. In our simulations we ensure that the output trajectories have a compact support  $[0, 2\pi]$  by choosing initial conditions of the exosystem of the form  $x_e(T_i) = [0; *; 0; 0]$ .

We also switch the mass  $m_t$  on the structure (see figure 4) to take values  $m_t = 0 \ \forall t \in [0, \pi/2]$ ,  $m_t = 100 \ \forall t \in (\pi/2, 1.5\pi]$ , and  $m_t = 10 \ \forall t \in (1.5\pi, 2\pi]$ , which denotes the jumps in the system. Note that  $C_k^*$  remains constant through the switches and hence the compatibility conditions are always satisfied. As illustrated in Section 3, the map  $G(0) : x_e(T_i) \rightarrow x_d(T_i)$  is given as

$$G(0) = \begin{bmatrix} -4.6715 & -3.3239 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 6.9406 & 2.6672 & 0 & 0 \\ -0.5935 & -1.4710 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2.6851 & -0.8211 & 0 & 0 \end{bmatrix}$$

As an example we simulate the forward dynamics with the initial condition for the exosystem as  $[0 \ 1 \ 0 \ 0]'$ . The corresponding initial system state for



exact output tracking is  $x_d(T_i) = [-3.3239 \ 0 \ 2.6672 \ -1.471 \ 0 \ -.8211]'$ . The simulation results are shown in Figure 5, where the exact tracking state trajectory is shown – this desired state trajectory yields the desired output with an error of  $10^{-6}$  for a motion of 2 units. This error is believed to be due to the numerical integration schemes. Further the initial conditions are large and unrealistic. The initial conditions of the system are typically not the same as the initial conditions of the desired state trajectory - this results in initial transient errors. If the system is stabilized then these errors decay exponentially, even if the system dynamics has switches. This ability to maintain tracking is a major advantage of our approach. Further, pre-actuation techniques to achieve these initial conditions (with output-error maintained at zero) has been developed in (*Devasia et al. (1996)*) and we expect to integrate the two approaches in the future.

## CONCLUSIONS

The problem of achieving exact output tracking for linear systems that present jumps in parameter's values has been analyzed. We established necessary and sufficient conditions for the existence of exact output-tracking bounded state trajectories. When the reference trajectory is generated through an exosystem, the feedforward action needed to maintain exact tracking can be written as a time-varying feedback. Further, this time-varying feedback is related to a map from the state of the exosystem to the desired system state. The map is linear and is shown to satisfy an ordinary differential equation. For the case of systems without switches the theory presented reduces to the standard regulator theory. We also showed that the desired trajectory can be stabilized and presented the simulation results for an example flexible structure with *switchingmass*.

Future work will attempt to remove the requirement of compact support for the output. There is also a need to address the tracking problem for systems whose internal-dynamics may not be hyperbolic.

## ACKNOWLEDGMENT

This work was supported through National Science Foundation grant MSS-9216690, NASA Ames Research Center Grant NAG 2-1042, and by the Astro Aerospace Corporation

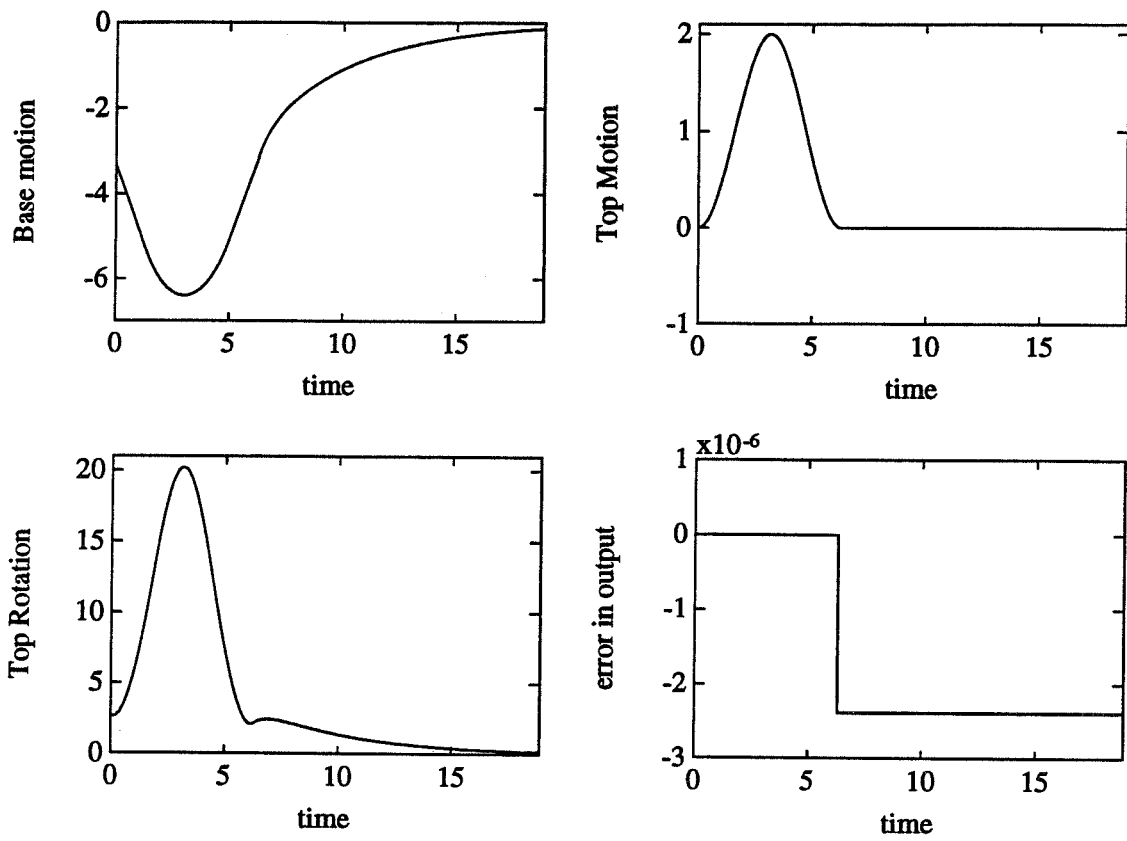


Figure 5: *Simulation results*

## References

- G. Basile and G. Marro. *Controlled and Conditioned Invariants in Linear System Theory*. Prentice-Hall, 1992.
- S. Devasia, D. Chen, and B. Paden. Nonlinear inversion-based output tracking. *IEEE Transactions on Automatic Control*, 41(7):930–943, 1996.
- B.A. Francis. The linear multivariable regulator problem. *SIAM94 J. Control and Optimization*, 15:486–505, 1977.
- L.R. Hunt, G. Meyer, and R. Su. Noncausal inverses for linear systems. *IEEE Transactions on Automatic Control*, 41(4):608–611, 1996.
- A. Isidori. *Nonlinear Control Systems: An Introduction*. Springer-Verlag, 1989.
- G. Marro and A. Piazzzi. Regulation without transients under large parameter jumps. *Proceedings of the 12th IFAC World Congress*, 4:23–26, 1993.
- G. Meyer, L.R. Hunt, and R. Su. Nonlinear system guidance. *Proceedings IEEE Conference on Decision and Control, New Orleans, Louisiana*, pages 590–595, 1995.
- W.M. Wonham. *Linear Multivariable Control: a Geometric Approach*. Springer-Verlag, 1985.

NIS

53-63

© W 014620

98139

045965

40

# Optimal Output Trajectory Redesign for Invertible Systems

S. Devasia

Reprinted from

## Journal of Guidance, Control, and Dynamics

Volume 19, Number 5, Pages 1189-1191



*A publication of the*  
American Institute of Aeronautics and Astronautics, Inc.  
1801 Alexander Bell Drive, Suite 500  
Reston, VA 22091

# Optimal Output Trajectory Redesign for Invertible Systems

S. Devasia\*

University of Utah, Salt Lake City, Utah 84112

## I. Introduction

GIVEN a desired output trajectory, inversion-based techniques find input-state trajectories required to exactly track the output.<sup>1,2</sup> These inversion-based techniques have been successfully applied to the endpoint tracking control of multijoint flexible manipulators in Ref. 3 and to aircraft control in Ref. 4. The specified output trajectory uniquely determines the required input and state trajectories that are found through inversion. These input-state trajectories exactly track the desired output; however, they might not meet acceptable performance requirements. For example, during slewing maneuvers of flexible structures, the structural deformations, which depend on the required state trajectories, may be unacceptably large. Further, the required inputs might cause actuator saturation during an exact tracking maneuver, for example, in the flight control of conventional takeoff and landing aircraft.<sup>5</sup> In such situations, a compromise is desired between the tracking requirement and other goals such as reduction of internal vibrations and prevention of actuator saturation; the desired output trajectory needs to be redesigned.

Here, we pose the trajectory redesign problem as an optimization of a general quadratic cost function and solve it in the context of linear systems. The solution is obtained as an off-line prefilter of the desired output trajectory. An advantage of our technique is that the prefilter is independent of the particular trajectory. The prefilter can therefore be precomputed, which is a major advantage over other optimization approaches (see Ref. 6 for further references).

Previous works have addressed the issue of preshaping inputs to minimize residual and in-maneuver vibrations for flexible structures; see, for example, Refs. 6 and 7. Since the command preshaping is computed offline, in Ref. 8, the use of noncausal prefilters has been suggested—such noncausality is allowable since the command preshaping is computed off-line. Further, minimization of optimal quadratic cost functions has also been previously used to preshape command inputs for disturbance rejection in Ref. 9. All of these approaches are applicable when the inputs to the systems are known a priori. Typically, outputs (not inputs) are specified in tracking problems, and hence the input trajectories have to be computed. The inputs to the system are, however, difficult to determine for non-minimum phase systems like flexible structures. One approach to solve this problem is to 1) choose a tracking controller (the desired output trajectory is now an input to the closed-loop system) and 2) redesign this input to the closed-loop system. Thus, we effectively perform output redesign.<sup>6</sup> These redesigns are, however, dependent on the choice of the tracking controllers.<sup>10</sup> Thus, the controller optimization and trajectory redesign problems become coupled; this coupled optimization is still an open problem. In contrast, we decouple the trajectory redesign problem from the choice of feedback-based tracking controller. It is noted that our approach remains valid when a particular tracking controller is chosen. In addition, the formulation of our problem not only allows for the minimization of residual vibrations as in available techniques<sup>6</sup> but also allows for the optimal reduction of vibrations during the maneuver, e.g., the altitude control of flexible spacecraft.<sup>9</sup> We begin by formulating the optimal output trajectory redesign problem and then solve it in the context of general linear systems. This theory is then applied to an example flexible structure, and simulation results are provided.

Received April 22, 1996; revision received May 30, 1996; accepted for publication May 30, 1996. Copyright © 1996 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

\*Assistant Professor, Department of Mechanical Engineering, Member AIAA.

## II. Problem Formulation

### System Inversion for Exact Tracking

Consider a square system described by

$$\dot{x} = Ax + Bu; \quad y = Cx$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$ , and  $y \in \mathbb{R}^p$ . The inversion approach<sup>2</sup> finds a bounded input-state trajectory that satisfies the preceding system equations and yields the exact desired output, i.e.,

$$\dot{x}_{\text{ref}} = Ax_{\text{ref}} + Bu_{\text{ff}}; \quad y_d = Cx_{\text{ref}}$$

The inverse input-state trajectories can be described in terms of Fourier transforms as<sup>1,11</sup>

$$\begin{aligned} u_{\text{ff}}(j\omega) &= [C(j\omega I - A)^{-1}B]^{-1}y_d(j\omega) = G_y^{-1}(j\omega)y_d(j\omega) \\ x_{\text{ref}}(j\omega) &= [(j\omega I - A)^{-1}B]u_{\text{ff}}(j\omega) = G_x(j\omega)u_{\text{ff}}(j\omega) \end{aligned} \quad (1)$$

This Fourier-based inversion approach has been extended to nonlinear time-varying nonminimum phase systems in Ref. 2; however, we restrict our discussion to linear time-invariant systems.

*Remark.* We note two results. One, an inverse exists if the output and a certain number of its time derivatives are bounded. The number of time derivatives of the output that needs to be specified for an inverse to exist is well defined and depends on the relative degree of the system.<sup>2,12</sup> Second, for linear hyperbolic systems, if the inverse exists, then it is unique.<sup>1,2</sup>

### Performance Criterion

Trajectory redesign seeks a compromise between the goal of tracking the desired trajectory and other goals such as reducing the magnitude of input and vibrations. We formulate this redesign problem as the minimization of a quadratic cost function of the type

$$\int_{-\infty}^{\infty} \{u(t)^T R u(t) + x(t)^T Q_x x(t) + [y(t) - y_d(t)]^T Q_y [y(t) - y_d(t)]\} dt$$

where  $R$ ,  $Q_x$ , and  $Q_y$  represent the weight on control input, states, and the error in output tracking, respectively.

Using Parseval's theorem, we rewrite our optimization problem in frequency domain as the minimization of the cost function

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \{u(j\omega)^* R u(j\omega) + x(j\omega)^* Q_x x(j\omega) \\ &+ [y(j\omega) - y_d(j\omega)]^* Q_y [y(j\omega) - y_d(j\omega)]\} d\omega \end{aligned} \quad (2)$$

where the superscript \* denotes complex conjugate transpose.

### Optimal Redesign of the Output

Our main result is given by the following lemma, which shows that the optimal output trajectory redesign can be described as a prefilter, which maps desired output trajectories  $y_d$  to the redesigned output trajectory  $y_{\text{opt}}$ . This prefilter  $G_f$  does not depend on the particular choice of desired trajectory and hence can be precomputed.

*Lemma.* The modified output trajectory  $y_{\text{opt}}$  is given by  $y_{\text{opt}}(j\omega) = G_f(j\omega)y_d(j\omega)$ , where

$$\begin{aligned} G_f(j\omega) &= 1 - G_y [R + G_x^* Q_x G_x + G_y^* Q_y G_y]^{-1} \\ &\times [R + G_x^* Q_x G_x] G_y^{-1} \end{aligned}$$

The modified input trajectory  $u_{\text{opt}}$  is given by  $u_{\text{opt}}(j\omega) = u_{\text{ff}}(j\omega) + v_{\text{opt}}(j\omega)$ , where  $v_{\text{opt}}(j\omega) = G_v(j\omega)y_d(j\omega)$  and

$$\begin{aligned} G_v(j\omega) &= -(R + G_x^* Q_x G_x + G_y^* Q_y G_y)^{-1} \\ &\times (R + G_x^* Q_x G_x) G_y^{-1} \end{aligned} \quad (3)$$

Note that the dependence on  $j\omega$  is not explicitly written for compactness.

*Proof.* Without loss of generality, we rewrite the input  $u$  as the sum of the feedforward input,  $G_y^{-1}y_d$ , found from inversion of the desired trajectory, and an arbitrary  $v$ :

$$u(j\omega) = u_{\text{ff}}(j\omega) + v(j\omega) = G_y^{-1}(j\omega)y_d(j\omega) + v(j\omega) \quad (4)$$

Substituting  $x(j\omega) = G_x(j\omega)u(j\omega)$  and  $y(j\omega) = G_y(j\omega)u(j\omega)$  along with the preceding Eq. (4) for  $u$  into the cost function given by Eq. (2), we obtain

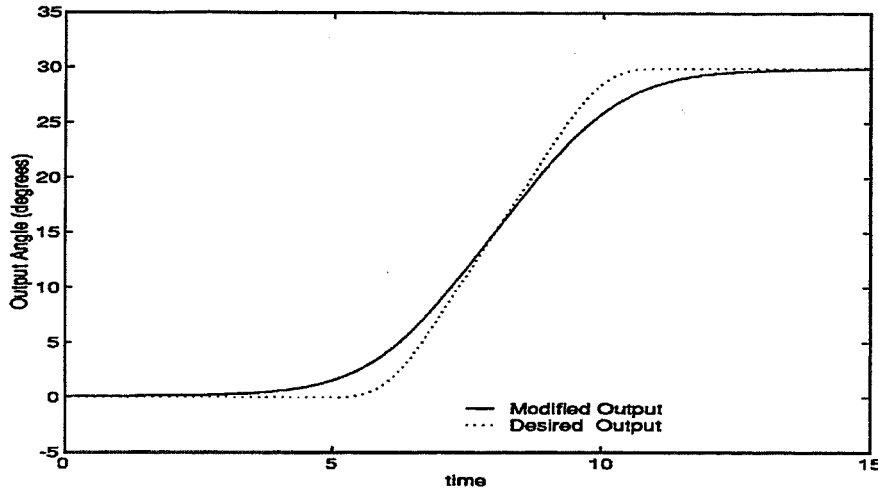


Fig. 1 Output redesign.

$$\begin{aligned}
 J = \int_{-\infty}^{\infty} & \{ [v + (R + G_x^* Q_x G_x + G_y^* Q_y G_y)^{-1} \\
 & \times (R + G_x^* Q_x G_x) G_y^{-1} y_d]^* * (R + G_x^* Q_x G_x + G_y^* Q_y G_y) \\
 & \times [v + (R + G_x^* Q_x G_x + G_y^* Q_y G_y)^{-1} \\
 & \times (R + G_x^* Q_x G_x) G_y^{-1} y_d] + (G_y^{-1} y_d)^* [(R + G_x^* Q_x G_x) \\
 & + (R + G_x^* Q_x G_x)^* * (R + G_x^* Q_x G_x + G_y^* Q_y G_y)^{-1} \\
 & \times (R + G_x^* Q_x G_x)] (G_y^{-1} y_d) \} dw
 \end{aligned}$$

Note that the cost function is quadratic in  $v$ . Therefore, the cost function is minimized by setting this quadratic term to zero, i.e., choosing  $v(j\omega) = v_{\text{opt}}(j\omega) = G_v(j\omega)y_d(j\omega)$ , where  $G_v$  is defined by Eq. (3) in the lemma. The choice of  $v = v_{\text{opt}}$  defines the optimal input  $u_{\text{opt}}$  through Eq. (4) as

$$u_{\text{opt}}(j\omega) = [G_y^{-1}(j\omega) + G_v(j\omega)]y_d(j\omega) \quad (5)$$

The result follows from  $y_{\text{opt}}(j\omega) = G_y(j\omega)u_{\text{opt}}(j\omega) = [1 + G_y(j\omega)G_v(j\omega)]y_d(j\omega)$ .  $\square$

### III. Example

Consider a flexible structure consisting of two freely rotating disks connected by a thin shaft. A motor is attached between the connecting shaft and one of the disks. Input to the system is torque  $\tau$  provided by a dc motor, and the outputs are the angular rotations of the two disks  $\theta_1$  and  $\theta_2$ . These angular rotations are measured using potentiometers. The transfer function of an experimental system, which includes the rigid-body mode and one flexible mode, was obtained using a HP3562A Dynamic Signal Analyzer as

$$\begin{aligned}
 \frac{\theta_1}{\tau} &= \frac{1.8139s^2 + 0.3077s + 6.1041}{s^4 + 0.2765s^3 + 6.1041s^2} \\
 \frac{\theta_2}{\tau} &= \frac{0.27s^2 - 0.1187s + 6.1041}{s^4 + 0.2765s^3 + 6.1041s^2}
 \end{aligned} \quad (6)$$

With the state vector  $x$  chosen as  $x = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T$ , the system equations can be represented in state-space form as  $\dot{x} = Ax + Bu$ , i.e.,

$$\begin{aligned}
 \frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.1555 & -0.1640 & 3.1555 & 0.3845 \\ 0 & 0 & 0 & 1 \\ 2.8956 & -0.0899 & -2.8956 & -0.1124 \end{bmatrix} \\
 &\times \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8139 \\ 0 \\ 0.27 \end{bmatrix} \tau
 \end{aligned}$$

with  $y = \theta_2 = [0 \ 1 \ 0 \ 0]x$ . The control objective is to track the angular rotation  $\theta_2$  of the disk that is farthest away from the motor (see Fig. 1 for the desired output trajectory).

The relative degree of a single-input/single-output linear system is the number of zeros at infinity.<sup>12</sup> For our system, with the torque as input and  $\theta_2$  as output, the transfer function has four poles and two finite zeros [see Eq. (6)]. Thus, the number of zeros at infinity are two, and hence the relative degree is two. This implies that the second derivative of the desired output, i.e., the desired angular acceleration profile of the output, uniquely determines the required input-state trajectory and the resulting structural vibration,  $\theta_1 - \theta_2$ .<sup>2</sup> If the internal vibrations are to be reduced, then we have to relax the exact tracking requirement. Similarly, to reduce the required input amplitudes we have to compromise exact tracking. This tradeoff can be represented as the minimization of a general quadratic cost function (Sec. II) of the form

$$\begin{aligned}
 \int_{-\infty}^{\infty} & \{ u(t)^T R u(t) + x(t)^T Q_x x(t) \\
 & + [y(t) - y_d(t)]^T Q_y [y(t) - y_d(t)] \} dt
 \end{aligned}$$

where  $R = r$ ,  $Q_y = q_y$ , and

$$Q_x = q_x \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The scalars  $r$ ,  $q_x$ , and  $q_y$  represent the relative weight on the reduction of inputs, structural vibrations, and tracking errors, respectively. To reduce the vibrations and control inputs, we choose  $r = 1$ ,  $q_x = 5000$ , and  $q_y = 1$  in our simulations. Figure 1 shows the modification for a desired trajectory—about 10% of the final slew angle. The maximum magnitude of the required input, however, is reduced by 60%, and the corresponding structural vibration,  $\theta_1 - \theta_2$ , is reduced by 20% (compared with results from exact tracking of the initial desired trajectory).

### IV. Conclusion

We formulated and solved the trajectory redesign problem in the context of linear invertible systems, including nonminimum phase systems. Thus, we provide a systematic approach to an optimal tradeoff between tracking desired trajectory and other goals such as vibration reduction and reduction of required inputs. The approach was applied to an example flexible structure, and simulation results were presented. Future work will address trajectory redesign for nonlinear systems.

### Acknowledgment

This work was funded through NASA Ames Research Center Grant NAG 2-1042.

### References

- <sup>1</sup>Bayo, E., "A Finite-Element Approach to Control the End-Point Motion of Single-Link Flexible Robot," *Journal of Robotic Systems*, Vol. 4, No. 1, 1987, pp. 63-75.
- <sup>2</sup>Devasia, S., and Paden, B., "Exact Output Tracking for Nonlinear Time-Varying Systems," *Proceedings of the 33rd IEEE Conference on Decision and Control* (Lake Buena Vista, FL), IEEE Control Systems Society, New York, 1994, pp. 2346-2355.
- <sup>3</sup>Paden, B., Chen, D., Ledesma, R., and Bayo, E., "Exponentially Stable Tracking Control for Multi-Joint Flexible Manipulators," *Journal of Dynamic Systems, Measurement and Control*, Vol. 115, March 1993, pp. 53-59.
- <sup>4</sup>Martin, P., Devasia, S., and Paden, B., "A Different Look at Output Tracking: Control of a VTOL Aircraft," *Proceedings of the 33rd IEEE Conference on Decision and Control* (Lake Buena Vista, FL), IEEE Control Systems Society, New York, 1994, pp. 2376-2381.
- <sup>5</sup>Tomlin, C., Lygeros, J., and Shastri, S., "Output Tracking for a Nonminimum Phase Dynamic CTOL Aircraft Model," *Proceedings of the 34th IEEE Conference on Decision and Control* (New Orleans, LA), IEEE Control Systems Society, New York, 1995, pp. 1867-1872.
- <sup>6</sup>Singer, N. C., and Seering, W. P., "Preshaping Command Inputs to Reduce System Vibration," *Journal of Dynamic Systems, Measurement and Control*, Vol. 112, March 1990, pp. 76-82.
- <sup>7</sup>Smith, O. J. M., *Feedback Control Systems*, McGraw-Hill, New York, 1958.
- <sup>8</sup>Singer, N. C., and Seering, W. P., "Using Acausal Shaping Techniques to Reduce Robot Vibration," *Proceedings of the IEEE International Conference on Robotics and Automation*, Vol. 3, IEEE Control Systems Society, New York, 1988, pp. 1434-1439.
- <sup>9</sup>Chun, H. M., Turner, J. D., and Juang, J., "Disturbance-Accommodating Tracking Maneuvers of Flexible Spacecraft," *Journal of the Astronautical Sciences*, Vol. 33, No. 2, 1985, pp. 197-216.
- <sup>10</sup>Cook, G., "Discussion: Preshaping Command Inputs to Reduce System Vibration," *Journal of Dynamic Systems, Measurement and Control*, Vol. 115, June 1993, pp. 309, 310.
- <sup>11</sup>Kwon, D., and Book, W. J., "An Inverse Dynamic Method Yielding Flexible Manipulator State Trajectories," *Proceedings of the American Control Conference*, American Automatic Control Council, New York, 1990, pp. 186-193.
- <sup>12</sup>Isidori, A., *Nonlinear Control Systems: An Introduction*, Springer-Verlag, New York, 1989.

# Experimental and Theoretical Results in Output-Trajectory Redesign for Flexible Structures \*

NIS  
54-63

2/10/86

98140

045966

61

J. S. Dewey and S. Devasia  
Mechanical Engineering Department  
University of Utah, SLC, UT 84112  
CDC 1996

## Abstract

In this paper we study the optimal redesign of output trajectory for linear invertible systems. This is particularly important for tracking control of flexible structures because the input-state trajectories that achieve the required output may cause excessive vibrations in the structure. A trade-off is then required between tracking and vibrations reduction. We pose and solve this problem as the minimization of a quadratic cost function. The theory is developed and applied to the output tracking of a flexible structure and experimental results are presented.

## 1 Introduction

Large space structures, like manipulators used to assemble the space station, are lightweight, and hence flexible. The structural flexibility results in elastic vibrations. These vibrations are caused not only by exogenous perturbations but also arise during maneuvers like slewing. Hence the output tracking control for flexible structures is a formidable task. Recent works have solved the output tracking problem, for example, given a desired output trajectory, inversion-based techniques find

input-state trajectories that exactly track the output [1, 2, 3, 4]. These inversion-based techniques have been successfully applied to the control of multi-joint flexible manipulators in [5, 6], and to aircraft control in [7, 8, 9]. Note that the specified output trajectory uniquely determines the required input and state trajectories. Therefore, although these input-state trajectories exactly track the desired output, they might not meet performance requirements in flexible structures. For example, during slewing maneuvers of a flexible manipulator, the structural deformations - determined by the inverse state trajectories - may be unacceptably large. In such situations, a compromise is desired between the tracking requirement and the other goals like the reduction of internal vibrations and prevention of actuator saturation - the output trajectory needs to be redesigned.

Here, we pose the trajectory redesign problem as an optimization of a general quadratic cost function, and solve it in the context of linear systems. The solution is obtained as an off-line prefilter of the desired output trajectory. Thus the redesigned output trajectory can be obtained through a simple convolution similar to the convolutions prevalent in the command shaping approaches, see for example [10, 11]. An advantage of our technique is that the pre-filter is independent of the partic-

\*This work was supported by NASA-Ames Research Center Grant No. NAG 2-1042



ular trajectory. The pre-filter can therefore be pre-computed which is a major advantage over other optimization approaches (see [11] for references).

Previous works have addressed the issue of pre-shaping inputs, rather than desired outputs, to minimize residual and in-maneuver vibrations for flexible structures – see for example [11, 12, 13, 14, 15, 16]. Note that in these problems, the command preshaping is computed off-line, and hence noncausal pre-filters can be used [10]. Further, minimization of optimal quadratic cost functions have also been previously used to preshape command inputs for disturbance rejection in [17]. A common problem with all these approaches is that they are applicable when the inputs to the systems are known *a priori*. In tracking problems, however, outputs are usually specified and not the inputs: input trajectories have to be computed from the desired outputs. For nonminimum phase systems (flexible structures with non-collocated inputs and outputs) the inputs to the system are difficult to determine. One approach to solve this problem is to: (1) choose a tracking controller – the desired output trajectory is now an input to the closed loop system; and (2) redesign this *input* to the closed loop system. Thus, effectively, we perform output redesign [11]. These redesigns are, however, dependent on the choice of the tracking controllers [18]. Thus the controller optimization and trajectory redesign problems become coupled - this coupled optimization is still an open problem. In contrast, our optimal output trajectory redesign is independent of the particular choice of tracking controller. We can, therefore, decouple the trajectory redesign problem from the tracking-controller design. Note that if a particular closed loop controller is chosen, our approach is still valid. The formulation of our problem not only allows for the minimization of residual vibrations as in available techniques [11], but, additionally, allows for the optimal reduction of vibrations during the maneuver, which is required in maneuvers like the altitude control of flexible spacecraft [17].

We begin by formulating the optimal output trajectory redesign problem, and then solve it in the context of general linear systems. This theory is then applied to an example flexible structure and experimental results are provided.

## 2 Problem Formulation

### System Inversion for Exact Tracking

Consider a square system described by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$  and  $y \in \mathbb{R}^p$ . The inversion approach [4] finds a bounded input-state trajectory that satisfies the above system equations, and yields the exact desired output, i.e.

$$\begin{aligned}\dot{x}_{ref} &= Ax_{ref} + Bu_{ff} \\ y_d &= Cx_{ref}.\end{aligned}$$

The inverse input-state trajectories can be described in terms of Fourier transforms as [1, 2]

$$\begin{aligned}u_{ff}(j\omega) &= [C(j\omega I - A)^{-1}B]^{-1}y_d(j\omega) \\ &:= G_y^{-1}(j\omega)y_d(j\omega) \\ x_{ref}(j\omega) &= [(j\omega I - A)^{-1}B]u_{ff}(j\omega) \\ &:= G_x(j\omega)u_{ff}(j\omega).\end{aligned}\tag{1}$$

This Fourier based inversion approach has been extended to nonlinear time-varying nonminimum phase systems in [4], however, we restrict our discussion to linear time-invariant systems.

**Remark I** We note two results. One, an inverse exists if the output and a certain number of its time-derivatives are bounded. The number of time derivatives of the output that needs to be specified for an inverse to exist is well-defined and depends on the relative degree of the system [19, 4]). Second, for linear systems, if the inverse exists then it is unique [1, 2, 3, 4].

### The Performance Criterion

Trajectory redesign seeks a compromise between the goal of tracking the desired trajectory and other goals like reducing the magnitude of input and vibrations. We formulate this redesign problem as the minimization of a quadratic cost function of the type

$$\int_{-\infty}^{\infty} \left\{ u(t)^T R u(t) + x(t)^T Q_x x(t) + [y(t) - y_d(t)]^T Q_y [y(t) - y_d(t)] \right\} dt \quad (2)$$

where  $R, Q_x$  and  $Q_y$  represents the weight on control input, states, and the error in output tracking respectively.

Using Parseval's theorem we rewrite our optimization problem in frequency domain as the minimization of the cost function

$$J := \int_{-\infty}^{\infty} \left\{ u(j\omega)^* R u(j\omega) + x(j\omega)^* Q_x x(j\omega) + [y(j\omega) - y_d(j\omega)]^* Q_y [y(j\omega) - y_d(j\omega)] \right\} d\omega$$

where the superscript  $*$  denotes complex conjugate transpose.

### Optimal Redesign of the Output

Our main result is given by the following lemma, which shows that the optimal output-trajectory redesign can be described as a pre-filter, which maps desired output trajectories,  $y_d$ , to the redesigned output trajectory,  $y_{opt}$ . This pre-filter,  $G_f$ , doesn't depend on the particular choice of desired trajectory and hence can be pre-computed.

**Lemma** The modified output trajectory,  $y_{opt}$ , is given by

$$y_{opt}(j\omega) = G_f(j\omega) y_d(j\omega)$$

where

$$G_f(j\omega) = 1 - G_y [R + G_x^* Q_x G_x + G_y^* Q_y G_y]^{-1} [R + G_x^* Q_x G_x] G_y^{-1}.$$

**Proof:** See [20] □

**Remark II** We point out two extreme cases. First, if the weight on the tracking error is zero,  $Q_y = 0$ , but  $R$  is positive definite then we obtain  $v = -(G_y^{-1} y_d) = -u_{ff}$ . This implies that the input  $u_{opt} = u_{ff} + v = 0$ , i.e., the best strategy is not to track the desired trajectory at all. It is more optimal to remain at the equilibrium point where the cost is zero. Second, if the weight on the inputs and states are zero, i.e.,  $R = 0$  and  $Q_x = 0$  but with  $Q_y$  positive definite then  $y_{opt} = y_d$ . This implies that exact tracking is optimal, and the cost is again zero.

## 3 Example

Consider an experimental flexible structure which consists of two discs connected by a thin shaft as shown in figure 1. These two discs can rotate freely.

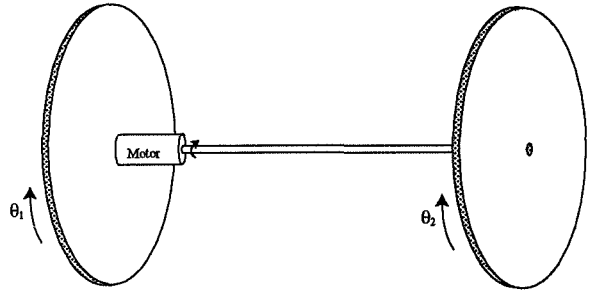


Figure 1 Schematic Experimental Setup

The transfer function of the system, which includes the rigid body mode and one flexible mode, was obtained experimentally using a HP3562A Dynamic Signal Analyzer. Input to the system is torque,  $\tau$ , provided by a DC motor and the outputs are the angular rotations of the two discs  $\theta_1, \theta_2$ . These angular rotations are measured using potentiometers, and the transfer functions are obtained as

$$\begin{aligned} \frac{\theta_1}{\tau} &= \frac{1.0684S^2 + 0.1812S + 3.5953}{S^4 + .2765S^3 + 6.1041S^2} \\ \frac{\theta_2}{\tau} &= \frac{.1590S^2 - 0.0699S + 3.5953}{S^4 + .2765S^3 + 6.1041S^2} \quad (3) \end{aligned}$$

With the state vector  $x$  chosen as

$$x := \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

the system equations can be represented in state-space form as

$$\dot{x} := Ax + Bu$$

$$y := \theta_2 = [0 \ 1 \ 0 \ 0]x$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.2 & -0.16 & 3.2 & 0.38 \\ 0 & 0 & 0 & 1 \\ 2.9 & -0.09 & -2.9 & -0.11 \end{bmatrix},$$

$$B := \begin{bmatrix} 0 \\ 1.81 \\ 0 \\ 0.27 \end{bmatrix}.$$

The control objective is to track the angular rotation,  $\theta_2$  of the disc, that is farthest away from the motor (see figure 1). The desired output trajectory and its time-derivatives as prescribed as shown in figure 2.

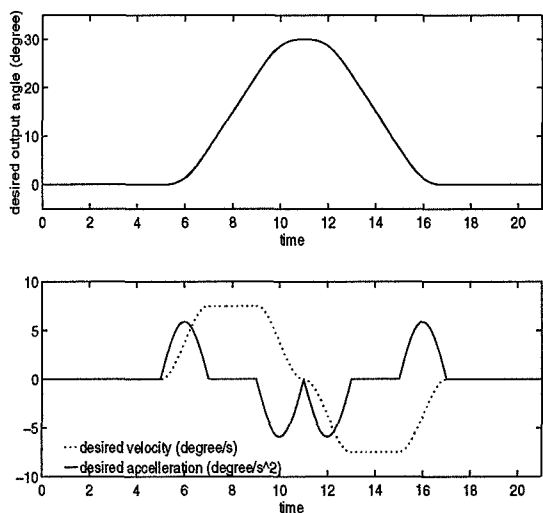


Figure 2 The Desired Output

The relative degree of a single-input single-output linear system is the number of zeros at infinity [19]. For our system, with the torque

as input and  $\theta_2$  as output, the transfer function has four poles and two finite zeros (see equation(3)). Thus the number of zeros at infinity are two and hence the relative degree is two. This implies that the second derivative of the desired output, i.e. the desired angular acceleration profile of the output, uniquely determines the required input-state trajectory, and the resulting structural vibration,  $\theta_1 - \theta_2$  [4]. If the internal vibrations are to be reduced, then we have to relax the exact tracking requirement. Similarly to reduce the required input amplitudes we have to compromise exact tracking. This trade-off can be represented as the minimization of a a general quadratic cost function (section 2) of the form

$$\int_{-\infty}^{\infty} u(t)^T R u(t) + x(t)^T Q_x x(t) + (y(t) - y_d(t))^T Q_y (y(t) - y_d(t)) dt$$

where  $R = r$ ,  $Q_y = q_y$ , and

$$Q_x = q_x \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The scalars  $r, q_x, q_y$  represent the relative weight on the reduction of inputs, structural vibrations and tracking errors respectively. In order to reduce the vibrations and control inputs, we choose  $r = 1, q_x = 5000$ , and  $q_y = 1$  in our simulations. Figure 3 shows the modification of the desired trajectory - about 10% of the final slew angle.

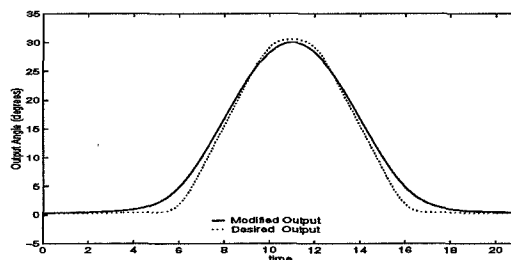


Figure 3 Simulation: Re-Designed Output

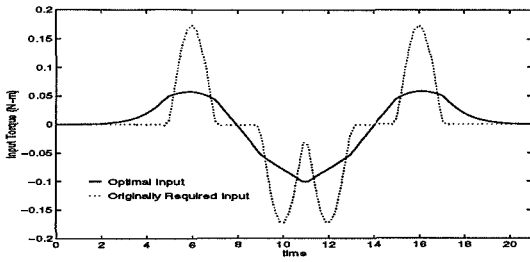


Figure 4 Optimal Input

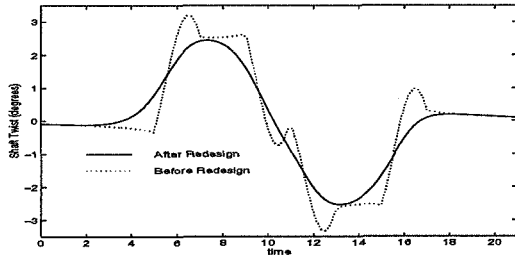


Figure 5 Simulation: Twisting of the Flexible Shaft

The maximum magnitude of the required input, however, is reduced by 60% and the structural vibrations,  $\theta_1 - \theta_2$ , is reduced by 20% as shown in figures 4 and 5. The input-state trajectories found through inversion as described in equation 1 : two simulations are performed, first with the original desired trajectory,  $y_d$  and second with the modified output trajectory,  $y_{opt}$ . These state trajectories are stabilized through feedback (see control scheme in figure 6), and preliminary experimental results are presented in figures 7 and 8.

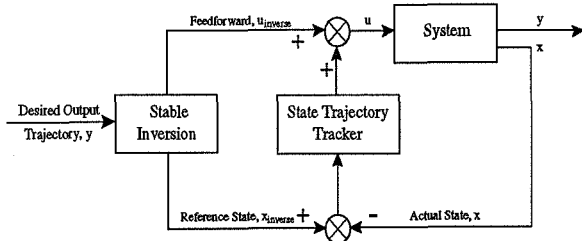


Figure 6 The Control Scheme

With the reference feedforward inputs as the original inversion input  $u_{ff}$  and the modified

(reduced) optimal input  $u_{opt}$  (figure 4), the experimental results verify that the small modification of the output trajectory (figure 7) reduces the maximum internal vibrations (50%, figure 8). Further reduction in the vibrations are possible if the tracking requirement ( $Q_y$ ) is relaxed. Also, it is possible to stabilize the system first through the feedback loop and then apply the trajectory redesign scheme to the closed loop system - but the redesign for our control scheme will not change since the input state trajectories always satisfy the state equations and hence in the redesign the stabilizing feedback is zero. For example,  $(u_{ff}, x_{ref})$  satisfies the state equations and thus the stabilizing feedback is zero if  $x = x_{ref}$  and does not affect the output redesign.

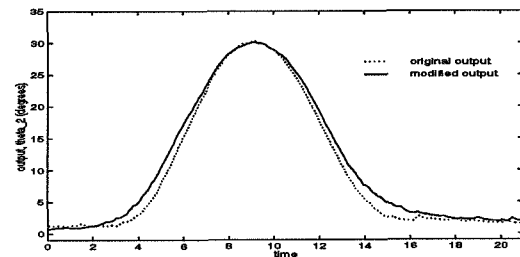


Figure 7 Experimental Results: The Tracking

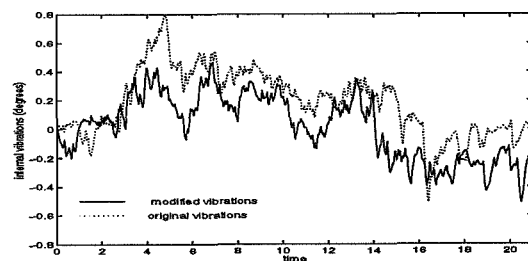


Figure 8 Experimental Results: The Internal Vibrations

## 4 Conclusion

We formulated and solved the trajectory redesign problem in the context of linear invertible systems - including nonminimum phase

systems. Thus we provide a systematic approach to an optimal tradeoff between tracking desired trajectory and other goals like vibration reduction and reduction of required inputs. The approach was applied to an example flexible structure and experimental results were presented. Future work will address trajectory redesign for nonlinear systems.

## References

- [1] E. Bayo. A finite-element approach to control the end-point motion of single-link flexible robot. *J. of Robotic Systems*, 4,(1):63–75, 1987.
- [2] D. Kwon and W. J. Book. An inverse dynamic method yielding flexible manipulator state trajectories. *Proceedings of the American Control Conference*, 1990.
- [3] R. Ledesma, S. Devasia, and E. Bayo. Inverse dynamics of spatial open-chain flexible manipulators with lumped and distributed actuators. *Journal of Robotic Systems*, 11(4):327–338, 1994.
- [4] S. Devasia and B. Paden. Exact output tracking for nonlinear time-varying systems. *Proceedings of the 33rd CDC, Lake Buena Vista, Florida*, pages 2346–2355, 1994.
- [5] H. Moulin and E. Bayo. On the end-point trajectory tracking for flexible manipulator through non-causal inverse dynamics. *ASME Journal of Dynamic Systems, Measurement and Control*, 113(2):320–324, 1991.
- [6] B. Paden, D. Chen, R. Ledesma, and E. Bayo. Exponentially stable tracking control for multi-joint flexible manipulators. *ASME JDSMC*, 115:53–59, 1993.
- [7] G. Meyer, L. R. Hunt, and R. Su. Nonlinear system guidance in the presence of transmission zero dynamics. *NASA Technical Memorandum 4661*, January, 1995.
- [8] P. Martin, S. Devasia, and B. Paden. A different look at output tracking: Control of a vtol aircraft. *CDC, Florida*, pages 2376–2381, 1994.
- [9] C. Tomlin, J. Lygeros, and S. Sastri. Output tracking for a nonminimum phase dynamic ctol aircraft model. *CDC*, pages 1867–1872, 1995.
- [10] N. C. Singer and W. P. Seering. Using acausal shaping techniques to reduce robot vibration. *Proc. of the IEEE Int. Conf. on Robotics and Automation, PA*, 3:1434–1439, 1988.
- [11] N. C. Singer and W. P. Seering. Preshaping command inputs to reduce system vibration. *ASME J. of Dynamic Systems, Measurement and Control*, 112:76–82, 1990.
- [12] D. M. Aspinwall. Acceleration profiles for minimizing residual response. *ASME Journal of Dynamic Systems, Measurement, and Control*, 102(1):3–6, March 1980.
- [13] C. J. Swigert. Shaped torque techniques. *Journal of Guidance and Control*, 3(5):460–467, Sept-Oct 1980.
- [14] R. L. Farrenkopf. Optimal open-loop maneuver profiles for flexible spacecraft. *Journal of Guidance and Control*, 2(6):491–498, Nov-Dec 1979.
- [15] S. P. Bhat and D. K. Miu. Precise point-to-point positioning of flexible structures. *ASME Journal of Dynamic Systems, Measurement, and Control*, 112:667–674, 1990.
- [16] O. J. M. Smith. *Feedback Control Systems*. McGraw-Hill, New York, 1958.
- [17] H. M. Chun, J. D. Turner, and J. Juang. Disturbance-accommodating tracking maneuvers of flexible spacecraft. *The Journal of the Astronautical Sciences*, 33:197–216, April-June 1985.
- [18] G. Cook. Discussion: Preshaping command inputs to reduce system vibration. *ASME Journal of Dynamic Systems, Measurement, and Control*, 115:309–310, June 1993.
- [19] A. Isidori. *Nonlinear Control Systems: An Introduction*. Springer-Verlag, 1989.
- [20] S. Devasia. Optimal output trajectory redesign for invertible systems. *Accepted, AIAA Journal of Guidance, Control and Dynamics*, 1996.