Towards an MHD theory for the standoff distance of Earth's bow shock

Iver H. Cairns and Crockett L. Grabbe
Department of Physics and Astronomy, University of Iowa

Abstract. An MHD theory is developed for the standoff distance \( a_s \) of the bow shock and the thickness \( \Delta_{ms} \) of the magnetosheath, using the empirical Spreiter et al. relation \( \Delta_{ms} = kX \) and the MHD density ratio \( X \) across the shock. The theory includes as special cases the well-known gasdynamic theory and associated phenomenological MHD-like models for \( \Delta_{ms} \) and \( a_s \). In general, however, MHD effects produce major differences from previous models, especially at low Alfvén \( (M_A) \) and sonic \( (M_S) \) Mach numbers. The magnetic field orientation, \( M_A, M_S \), and the ratio of specific heats \( \gamma \) are all important variables of the theory. In contrast, the fast mode Mach number need play no direct role. Three principal conclusions are reached. First, the gasdynamic and phenomenological models miss important dependences on field orientation and \( M_S \) and generally provide poor approximations to the MHD results. Second, changes in field orientation and \( M_S \) are predicted to cause factor of \( \sim 4 \) changes in \( \Delta_{ms} \) at low \( M_A \). These effects should be important when predicting the shock's location or calculating \( \gamma \) from observations. Third, using Spreiter et al.'s value for \( k \) in the MHD theory leads to maximum \( a_s \) values at low \( M_A \) and nominal \( M_S \) that are much smaller than observations and MHD simulations require. Resolving this problem requires either the modified Spreiter-like relation and larger \( k \) found in recent MHD simulations and/or a breakdown in the Spreiter-like relation at very low \( M_A \).

1. Introduction

The location of Earth's bow shock has been actively researched since its prediction and discovery. Subjects of particular interest include the shock's farthest extent sunwards (known as the standoff distance \( a_s \)) and the thickness \( \Delta_{ms} \) of the magnetosheath region separating the shock from the magnetopause, due to their importance in understanding foreshock observations, solar wind-magnetosphere interactions, and the ratio of specific heats \( \gamma \) for the plasma. Figure 1 defines \( a_s \), \( \Delta_{ms} \), and the magnetopause standoff distance \( a_{mp} \) in the X-Y-Z coordinate system formed by rotating the GSE system so that the shock is symmetric about the solar wind's velocity vector relative to Earth. Clearly \( a_s = a_{mp} + \Delta_{ms} \). Balancing the solar wind ram pressure \( P = \rho_{sw} v_{sw}^2 \) and the magnetostatic pressure of Earth's dipole magnetic field requires \( a_{mp} = KP^{-1/6} \), where \( K \) is a slowly varying function of the IMF \( B_z \) component, the ring current, the magnetopause current system and drag effects for the solar wind-magnetosphere system [Formisano et al., 1971; Slavin and Holzer, 1978; Farris et al., 1991; Sibeck et al., 1991]. Spreiter et al. [1966, and references therein] developed the first detailed theoretical model for \( a_s \) and \( \Delta_{ms} \),

\[
a_s = KP^{-1/6} \left( 1 + \frac{1.1 (\gamma - 1) M_s^2 + 2}{(\gamma + 1) M_s^2} \right)^{1/3} \]  

(1)

The ratio \( \Delta_{ms}/a_{mp} \) is given by the entire second term, with the number 1.1 depending on the obstacle's shape. This equation follows from the empirical linear relation

\[
\frac{\Delta_{ms}}{a_{mp}} = kX = k \frac{\rho_{sw}}{\rho_d} \]  

(2)

obtained from gasdynamic simulations for \( M = M_S > 3 \) [Spreiter et al., 1966] with the ratio \( \rho_{sw}/\rho_d \) specified subsequently by the jump conditions for a gasdynamic shock. (Here \( \rho_d \) is the mass density downstream from the shock.) An analytic explanation for (2) remains unavailable. Spreiter et al. generally identified the (sonic) Mach number \( M \) with the Alfvén Mach number \( M_A = v_{sw}/v_A \) for arbitrary magnetic field orientations and \( M_A^2 \gg M_S^2 \gg 1 \) (a pseudo Mach number was defined instead for aligned flows with \( v_{sw} \parallel B_{sw} \)). This theory is therefore intrinsically gasdynamic with the subsequent phenomenological replacement of the sonic Mach number \( M_S = v_{sw}/c_s \) by \( M_A \). Spreiter et al. emphasized the theory's expected limitations at low Mach numbers. More recently Russell [1985] suggested that the proper replacement for \( M \) in (1) is the fast magnetosonic Mach number \( M_{ms} \), since the bow shock is a fast mode shock. Again, however, this is a phenomenological replacement in a gasdynamic result.

Earth's bow shock is indeed a fast mode shock, not a gasdynamic shock, and so MHD theory is a priori more appropriate. Spreiter et al.'s gasdynamic equation and its phenomenological variants therefore need to be reconsidered and a MHD version of (1) derived more rigorously. Other motivations for studying the bow shock's location include the finding that (1) with \( M \) replaced by \( M_A \) or \( M_{ms} \), predicts \( a_s \) values that are too small for \( M_A + M_{ms} \sim 1 - 3 \) [Russell and Zhang, 1992; Cairns et al., 1994] and the scattered values for \( \gamma \) extracted from the measured \( \Delta_{ms} \), via (1) [Fairfield, 1971; Zhuan9 and Zhang, 1992; Cairns et al., 1991]. Lastly, MHD predictions for \( a_s \) and \( \Delta_{ms} \) are needed now that global MHD simulation codes are available [e.g., Cairns and Lyon, 1994] for studying the Earth-solar wind interaction.
This paper addresses the basis of previous gasdynamic and phenomenological MHD-like models for $a_s$ and $\Delta m_s$, the importance of MHD effects, and current attempts to explain unusually distant shock locations and the plasma's ratio of specific heats $\gamma$. The simplest MHD theory for $a_s$ and $\Delta m_s$ is constructed: Spreiter et al.'s empirical relation between $\Delta m_s$ and $X$ is retained and MHD theory is used to specify the density ratio $X$ across the shock. This MHD theory (Sections 2 and 3) includes the gasdynamic and related phenomenological theories as special cases but in general shows different theoretical dependences. In fact, the MHD theory predicts that $\Delta m_s$ and $a_s$ should depend strongly on the magnetic field orientation, $M_A$ and $M_S$ (and $\gamma$). Zhuang and Russell [1981] previously developed an approximate but not self-consistent MHD expression for $X$ and an unrelated calculation for $a_s$, am. Their paper's theory extends Zhuang and Russell's work on $X$, by retaining all contributions to $X$ and not assuming high $M_A$ & $M_S$ flows, and merges it with Spreiter et al.'s empirical approach. Quantitative comparisons are made between the various theories in Section 3. A discussion and the conclusions are presented in Sections 4 and 5, respectively.

2. Analytic theory

Spreiter et al. [1966] used gasdynamic simulations to show that the linear relation (2) between $\Delta m_s$ and $X$ holds for $M_S \gtrsim 5$. The jump conditions for a gasdynamic shock lead to a quadratic equation for $X$ whence

$$X = \frac{(\gamma - 1)M_S^2 + 2}{(\gamma + 1)M_S^2}, \quad (3)$$

thereby leading to (1) via (2). The transition to the magnetised solar wind was then attempted by phenomenologically replacing $M_S$ by $M_A$ [Spreiter et al., 1966] or $M_{ms}$ [Russell, 1985]. The limiting values of $\Delta m_s$ and $a_s$ for $\gamma = 5/3$ are then: $\Delta m_s/a_{mp} \to 1/4$ and $a_s/a_{mp} \to 1.275$ as $M_A$ and $M_{ms} \to \infty$; $\Delta m_s/a_{mp} \to 1.1$ and $a_s/a_{mp} \to 2.1$ as $M_A$ and $M_{ms} \to 1$. Note that the gasdynamic and phenomenological results have no explicit dependences on magnetic field orientation and $M_S$ except through $M_{ms}$.

The theory developed here assumes that (2) remains valid, as supported by Cairns and Lyon's [1994] MHD simulations for $M_A \gtrsim 1.5$, and uses MHD theory to specify the density jump $X$. Factoring out the solution $X = 1$, the MHD jump conditions lead to a cubic equation for $X$ [e.g., Zhuang and Russell, 1981]:

$$AX^3 + BX^2 + CX + D = 0 \quad (4)$$

$$A = (\gamma + 1)M_A^6$$
$$B = (\gamma - 1)M_A^4 + (\gamma + 2)\cos^2 \theta M_A^4 + (\gamma + \beta)M_A^4$$
$$C = (\gamma - 2 + \gamma \cos^2 \theta)M_A^4 + (\gamma + 1 + 2\beta)\cos^2 \theta M_A^4$$
$$D = (\gamma - 1)\cos^2 \theta M_A^4 + \gamma \beta \cos^4 \theta \quad (5)$$

The new variables introduced are $\theta \in [0, \pi)$, the angle between $v_{sw}$ and $B_{sw}$ (the shock normal is antiparallel to $v_{sw}$), and the upstream plasma $\beta$ is defined by $\gamma \beta = 2c_s^2/v_A^2 = 2M_A^2/M_S^2$. The natural Mach numbers in the theory are then $M_A$ and $M_S$; $M_{ms}$ need play no role. In comparison, neither $\theta$ nor $\beta$ play roles in the gasdynamic or phenomenological theories (except through $M_{ms}$). The standard cubic analysis provides general solutions to (4), although these typically provide little insight. Consider, however, the special cases of parallel ($\theta = 0^\circ$) and perpendicular ($\theta = 90^\circ$) flows.

The cubic is easily factored when $B_{sw}$ is parallel to the shock normal ($\cos \theta = 1$). Ignoring the two ‘switch-on’ shock solutions $X = M_A^{-2}$, (2) and (4) yield

$$a_s = 1 + k\frac{(\gamma - 1)M_A^2 + \gamma \beta}{(\gamma + 1)M_A^2} = 1 + k\frac{(\gamma - 1)M_S^2 + 2}{(\gamma + 1)M_S^2} \quad (5)$$

The rightmost form reveals that the gasdynamic expressions for $X$ and $a_s$ are recovered, cf. (1) and (3), as expected since the magnetic field essentially drops out of the problem in the parallel case. This equation implies two important results for $\theta = 0^\circ$. First, phenomenological replacement of $M_S$ by $M_A$ or $M_{ms}$ ($= M_A$ for $\theta = 0^\circ$) in (1), as proposed by Spreiter et al. and Russell, is incorrect except in the special case $\gamma \beta = 2$. The phenomenological theories are therefore restricted special cases of the MHD theory. Second, $\Delta m_s, a_s$ and $X$ are independent of $M_A$ and $M_{ms}$, and depend only on $\gamma$ and $M_S$ (with the intuitive caveats that $M_A$ & $M_S \geq 1$). This is a major difference from (1) with the replacements $M \to M_A$ or $M_{ms}$, which predict $\Delta m_s/a_{mp} \to k$ as $M_A \to 1$ instead of the correct result $\Delta m_s/a_{mp} \to k/4$ for $\gamma = 5/3$ and $M_S \geq 1$.

For perpendicular flows ($\cos \theta = 0$) the cubic equation for $X$ collapses to a quadratic, whence

$$a_s = 1 + \frac{k}{2(\gamma + 1)} \left( A \pm \sqrt{A^2 - 4(\gamma - 2)(\gamma + 1)M_A^2} \right) \quad (6)$$

with $A = (\gamma - 1 + \gamma/M_A^2 + 2/M_S^2)$. For $\gamma \leq 2$ only the solution $a_s = a_{+}$ is relevant (only $X = X_+ > 0$). In general, (6) is not equivalent to the gasdynamic solution (1), the Spreiter et al. and Russell variants of (1) with $M \to M_A$ and $M_{ms}$, respectively, or (5)'s MHD solution for $\theta = 0^\circ$. However, in the special case $\gamma = 2$ (6) reduces to Russell's form for arbitrary $\beta$ (writing $M_A$ and $M_S$ in terms of $M_{ms}$ and $\beta$) and to Spreiter et al.'s form in the limit $\beta \to \infty$. Furthermore, the gasdynamic result (1) is recovered in the limit $M_A \to \infty$ (or $\beta \to 0$) for arbitrary $\gamma$. (This is true for all $\theta$.) Equation (6) also implies three corollary results.
First, $a_\ast$, $\Delta m_\ast$ and $X$ vary strongly with the angle $\theta$. Second, in general the solution depends intrinsically on both $M_A$ and $M_S$; while (6) can easily be rewritten in terms of $M_{ms}$ and $\beta$, only for $\gamma = 2$ does the explicit $\beta$ dependence disappear and the solution depend solely on $M_{ms}$. Last, the behavior as $M_A \to 1$ for $\theta = 90^\circ$ differs greatly from the MHD result (5) for $\theta = 0$: for $\gamma = 5/3$, (5) implies $X \to 1/4$ for $\beta = 0$ ($M_S = \infty$) while (6) implies $X_\ast \to 1$ for $\beta = 0$ and $X_\ast \to 1.7$ for $\gamma \beta = 1$. Thus, factor of $\gtrsim 4$ changes in $\Delta m_\ast$ should exist at low $M_A$ for different $\theta$ and $M_S$.

3. Numerical Analyses

When the magnetic field is neither parallel nor perpendicular to the flow direction and shock normal, (4)'s solutions are most instructive when presented graphically. Figure 2 shows the MHD theory’s predictions for the ratio $a_\ast/a_{mp} = 1 + 1.1X$, where Spreiter et al.’s empirical value for $k$ in (2) is used (see Section 4) and (4) is solved numerically, as a function of $M_A$ for $\theta = 0$, 45, and 90° and $M_S = 8$. Figure 3 is an analogous plot for $M_A = 2$. The strong dependences on $\theta$, $M_A$ and $M_S$ implied by (4) - (6) are clearly evident. Note that the $\theta = 0$° curves are all flat, and so independent of $M_A$ as in (5), with a level that depends on $M_S$. The curves for $\theta = 45^\circ$ lie below the $\theta = 90^\circ$ curves. Indeed, it may be shown analytically that the maximum allowed values of $a_\ast$ (and $X$) for given $M_S$ occur at $\theta = 90^\circ$ and $M_A = 1^+$, with smaller $M_S$ leading to larger $X$ and $a_\ast$.

Figure 4 shows how $X$ and $a_\ast$ depend on $\gamma$: for a given $\theta$ and $M_S$, a larger $\gamma$ leads to a higher curve for $a_\ast$ versus $M_A$. However, it can be shown analytically from (4) that the maximum values of $X$ and $a_\ast$ are independent of $\gamma$: curves for different $\gamma$ therefore all converge to the same maximum at $M_A = 1$ and $\theta = 90^\circ$.

Quantitative comparisons between the MHD theory developed here, Spreiter et al.’s model given by (1) with $M = M_A$, and Russell’s model are shown for various $\theta$ and $M_A$ in Figure 5. For all $\theta$ the gasdynamic theory with $M = M_S$ coincides with the $\theta = 0^\circ$ MHD solution. In general significant differences between the MHD predictions and the phenomenological models are apparent, except in the limited region $M_A \approx 1.5$ and $\theta \gtrsim 30^\circ$ (decreasing $M_S$ shrinks this region further). Note that the phenomenological models have no $\theta$ or (direct) $M_S$ dependences, whereas the MHD theory shows these dependences to be very important. The maximum $a_\ast$ values predicted by the MHD and phenomenological models ($M_A \sim 1$) are, however, almost identical for large $M_S \gtrsim 5$. When $M_S$ is small the MHD theory predicts larger $a_\ast$ values. However, the maximum difference in $a_\ast$ between the MHD and ‘gasdynamic’ theories remains less than 50% for $M_S > 1$.

4. Discussion

Differences of 50% or more in $a_\ast$ and 400% in $\Delta m_\ast$, due to $\theta$ and $M_S$ effects when $M_A \lesssim 5$, are easily discerned in Figures 2 - 5. Differences occur both between the MHD, gasdynamic and phenomenological theories and between different parameter sets for the MHD theory. That $\theta$ effects should be very important, as well as $M_A$ and $M_S$ variations, in determining $a_\ast$ and $\Delta m_\ast$ is one of this paper’s important predictions. Refinements to the present ‘local’ MHD theory will undoubtedly occur when global MHD effects are considered.

The above results argue that values for $\gamma$ derived from measurements of $\Delta m_\ast$ [Fairfield, 1971; Zhuang and Russell, 1981; Farris et al., 1991] should depend on the formulae used for $\Delta m_\ast$ and on whether $M_A$, $\theta$ and $M_S$ variations are all considered. Finite $M_A$, $\theta$ and $M_S$ effects may explain the wide scatter in the published $\gamma$ values. Analyses to measure $\gamma$ should be redone using (4)’s explicit MHD solutions (or successors thereof).

Near 1 AU the solar wind speed and electron temperature vary relatively little, whence $M_S$ lies within a factor of 2 of the nominal value $M_S \sim 7$. For $M_A \sim 1 - 3$,
then, Figures 2-5 predict that \( a_s \) will be within 10\% of (1)'s prediction (for \( M = M_A \)). This paper's MHD theory, using Spreiter et al.'s empirical value for \( k \), reduces to the gasdynamic theory in number and no explicit logical models involve only a single (differing) Mach

5. Conclusions

The analyses above show that more detailed consideration of MHD effects results in major differences from the well-known gasdynamic theory, and its phenomenological variants, for \( \Delta_{ms} \) and \( a_s \). The MHD model depends on \( M_A, M_S, \theta, \) and \( \gamma \), all of which have important effects; in comparison the gasdynamic and phenomenological models involve only a single (differing) Mach number and no explicit \( \theta \) effects (save through \( M_{ms} \)). The MHD theory reduces to the gasdynamic theory in the special cases \( \theta = 0^\circ \) or \( M_A \gg M_S \gg 1 \). The Spreiter et al. [1966] and Russell [1985] phenomenological theories reappear as the special cases of the MHD theory for (i) \( \theta = 0^\circ \) with \( \gamma \beta = 2 \), and (ii) \( \theta = 90^\circ \) with \( \gamma = 2 \) and \( \beta = 0 \) or arbitrary \( \beta \), respectively. The MHD theory predicts that \( \Delta_{ms} \) and \( a_s \) should depend strongly on \( \theta, M_A \) and \( M_S \) when \( M_A \) and/or \( M_S \leq 5 \), with the functional form of the \( M_A \) and \( M_S \) dependence varying with \( \theta \). In particular, varying \( \theta \) from 0\(^\circ\) to 90\(^\circ\) at low \( M_A \) should cause \( \Delta_{ms} \) to vary by \( \sim 400\% \). Since \( M_A \) and \( M_S \) are frequently of this order out to 1 AU, the predicted MHD effects should have widespread applicability; for instance, in calculations of \( \gamma \) from measurements of \( \Delta_{ms} \). It is found that the MHD theory with Spreiter et al.'s value for \( k \) cannot explain observations and MHD simulations of distant, very low \( M_A \) bow shocks. Cairns and Lyon [1994] can explain the simulation results by inserting the larger, MHD value for \( k \) into the present paper's MHD theory. It remains possible, nevertheless, that a nonlinear model for \( \Delta_{ms} = \Delta_{ms}(X) \) is necessary at very low \( M_A \).

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References


I. H. Cairns and C. L. Grabbe, Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242.

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