GEOPHYSICAL RESEARCH LETTERS, VOL. 21, NO. 25, PAGES 2781-2784, DECEMBER 15, 1994

Towards an MHD theory for the standoff distance of Earth's bow shock

Iver H. Cairns and Crockett L. Grabbe
Department of Physics and Astronomy, University of Iowa

NASA-CR-202739

Abstract. An MHD theory is developed for the standoff distance a_s of the bow shock and the thickness Δ_{ms} of the magnetosheath, using the empirical Spreiter et al. relation $\Delta_{ms} = kX$ and the MHD density ratio X across the shock. The theory includes as special cases the well-known gasdynamic theory and associated phenomenological MHD-like models for Δ_{ms} and a_s. In general, however, MHD effects produce major differences from previous models, especially at low Alfven (M_A) and sonic (M_S) Mach numbers. The magnetic field orientation, M_A, M_S, and the ratio of specific heats γ are all important variables of the theory. In contrast, the fast mode Mach number need play no direct role. Three principal conclusions are reached. First, the gasdynamic and phenomenological models miss important dependances on field orientation and M_S and generally provide poor approximations to the MHD results. Second, changes in field orientation and M_S are predicted to cause factor of ~ 4 changes in Δ_{ms} at low M_A. These effects should be important when predicting the shock's location or calculating γ from observations. Third, using Spreiter et al.'s value for k in the MHD theory leads to maximum a_s values at low M_A and nominal M_S that are much smaller than observations and MHD simulations require. Resolving this problem requires either the modified Spreiter-like relation and larger k found in recent MHD simulations and/or a breakdown in the Spreiter-like relation at very low M_A.

1. Introduction

The location of Earth's bow shock has been actively researched since its prediction and discovery. Subjects of particular interest include the shock's farthest extent sunwards (known as the standoff distance a_s) and the thickness Δ_{ms} of the magnetosheath region separating the shock from the magnetopause, due to their importance in understanding foreshock observations, solar wind-magnetosphere interactions, and the ratio of specific heats γ for the plasma. Figure 1 defines a_s, Δ_{ms}, and the magnetopause standoff distance a_{mp} in the X-Y-Z coordinate system formed by rotating the GSE system so that the shock is symmetric about the solar wind's velocity vector relative to Earth. Clearly $a_s = a_{mp} + \Delta_{ms}$. Balancing the solar wind ram pressure $P = \rho_{sw} v_{sw}^2$ and the magnetostatic pressure of Earth's dipole magnetic field requires $a_{mp} = KP^{-1/6}$, where K is a slowly varying function of the IMF B_z component, the ring current, the magnetopause current system and drag effects for the solar wind-magnetosphere system [*Formisano et al.*, 1971; *Slavin and Holzer*, 1978; *Farris et al.*, 1991; *Sibeck et al.*, 1991]. *Spreiter et al.* [1966, and references therein] developed the first detailed theoretical model for a_s and Δ_{ms},

$$a_s = KP^{-1/6}\left(1 + 1.1\frac{(\gamma-1)M^2+2}{(\gamma+1)M^2}\right) \quad . \tag{1}$$

The ratio Δ_{ms}/a_{mp} is given by the entire second term, with the number 1.1 depending on the obstacle's shape. This equation follows from the empirical linear relation

$$\frac{\Delta_{ms}}{a_{mp}} = kX = k\frac{\rho_{sw}}{\rho_d} \tag{2}$$

obtained from gasdynamic simulations for $M = M_S \gtrsim 5$ [*Spreiter et al.*, 1966] with the ratio ρ_{sw}/ρ_d specified subsequently by the jump conditions for a gasdynamic shock. (Here ρ_d is the mass density downstream from the shock.) An analytic explanation for (2) remains unavailable. *Spreiter et al.* generally identified the (sonic) Mach number M with the Alfven Mach number $M_A = v_{sw}/v_A$ for arbitrary magnetic field orientations and $M_A^2 \gg M_S^2 \gg 1$ (a pseudo Mach number was defined instead for aligned flows with $\mathbf{v}_{sw} \parallel \mathbf{B}_{sw}$). This theory is therefore intrinsically gasdynamic with the subsequent phenomenological replacement of the sonic Mach number $M_S = v_{sw}/c_s$ by M_A. *Spreiter et al.* emphasized the theory's expected limitations at low Mach numbers. More recently *Russell* [1985] suggested that the proper replacement for M in (1) is the fast magnetosonic Mach number M_{ms}, since the bow shock is a fast mode shock. Again, however, this is a phenomenological replacement in a gasdynamic result.

Earth's bow shock is indeed a fast mode shock, not a gasdynamic shock, and so MHD theory is a priori more appropriate. *Spreiter et al.'s* gasdynamic equation and its phenomenological variants therefore need to be reconsidered and a MHD version of (1) derived more rigorously. Other motivations for studying the bow shock's location include the finding that (1) with M replaced by M_A or M_{ms} predicts a_s values that are too small for M_A & $M_{ms} \sim 1-3$ [*Russell and Zhang*, 1992; *Cairns et al.*, 1994] and the scattered values for γ extracted from the measured Δ_{ms} via (1) [*Fairfield*, 1971; *Zhuang and Russell*, 1981; *Farris et al.*, 1991]. Lastly, MHD predictions for a_s and Δ_{ms} are needed now that global MHD simulation codes are available [e.g., *Cairns and Lyon*, 1994] for studying the Earth-solar wind interaction.

Paper number 94GL02551
0094-8534/94/94GL-02551$03.00

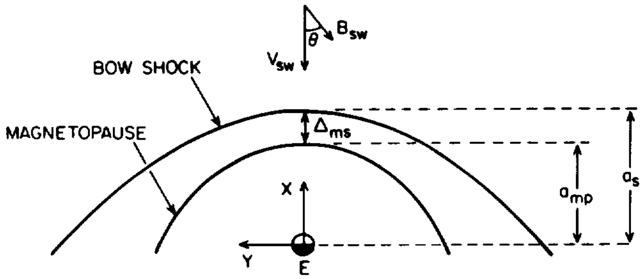

Figure 1. Definitions of a_s, a_{mp}, Δ_{ms}, the angle θ, and the X-Y-Z coordinate system.

This paper addresses the basis of previous gasdynamic and phenomenological MHD-like models for a_s and Δ_{ms}, the importance of MHD effects, and current attempts to explain unusually distant shock locations and the plasma's ratio of specific heats γ. The simplest MHD theory for a_s and Δ_{ms} is constructed: *Spreiter et al.'s* empirical relation between Δ_{ms} and X is retained and MHD theory is used to specify the density ratio X across the shock. This MHD theory (Sections 2 and 3) includes the gasdynamic and related phenomenological theories as special cases but in general shows different theoretical dependances. In fact, the MHD theory predicts that Δ_{ms} and a_s should depend strongly on the magnetic field orientation, M_A and M_S (and γ). *Zhuang and Russell* [1981] previously developed an approximate but not self-consistent MHD expression for X and an unrelated calculation for Δ_{ms} when M_A & $M_S \gg 1$. This paper's theory extends *Zhuang and Russell's* work on X, by retaining all contributions to X and not assuming high M_A & M_S flows, and merges it with *Spreiter et al.'s* empirical approach. Quantitative comparisons are made between the various theories in Section 3. A discussion and the conclusions are presented in Sections 4 and 5, respectively.

2. Analytic theory

Spreiter et al. [1966] used gasdynamic simulations to show that the linear relation (2) between Δ_{ms} and X holds for $M_S \gtrsim 5$. The jump conditions for a gasdynamic shock lead to a quadratic equation for X whence

$$X = \frac{(\gamma-1)M_S^2+2}{(\gamma+1)M_S^2} \quad , \qquad (3)$$

thereby leading to (1) via (2). The transition to the magnetised solar wind was then attempted by phenomenologically replacing M_S by M_A [*Spreiter et al.*, 1966] or M_{ms} [*Russell*, 1985]. The limiting values of Δ_{ms} and a_s for $\gamma = 5/3$ are then: $\Delta_{ms}/a_{mp} \rightarrow 1.1/4$ and $a_s/a_{mp} \rightarrow 1.275$ as M_A and $M_{ms} \rightarrow \infty$; $\Delta_{ms}/a_{mp} \rightarrow 1.1$ and $a_s/a_{mp} \rightarrow 2.1$ as M_A and $M_{ms} \rightarrow 1$. Note that the gasdynamic and phenomenological results have no explicit dependences on magnetic field orientation and M_S except through M_{ms}.

The theory developed here assumes that (2) remains valid, as supported by *Cairns and Lyon's* [1994] MHD simulations for $M_A \gtrsim 1.5$, and uses MHD theory to specify the density jump X. Factoring out the solution $X = 1$, the MHD jump conditions lead to a cubic equation for X [e.g., *Zhuang and Russell*, 1981]:

$$AX^3 + BX^2 + CX + D = 0 \qquad (4)$$

$$\begin{aligned} A &= (\gamma+1)M_A^6 \\ -B &= (\gamma-1)M_A^6 + (\gamma+2)\cos^2\theta M_A^4 + (\gamma+\gamma\beta)M_A^4 \\ C &= (\gamma-2+\gamma\cos^2\theta)M_A^4 + (\gamma+1+2\gamma\beta)\cos^2\theta M_A^2 \\ -D &= (\gamma-1)\cos^2\theta M_A^2 + \gamma\beta\cos^4\theta \ . \end{aligned}$$

The new variables introduced are $\theta \subset [0, 90°]$, the angle between $\mathbf{v}_{sw}$ and $\mathbf{B}_{sw}$ (the shock normal is antiparallel to $\mathbf{v}_{sw}$), and the upstream plasma β is defined by $\gamma\beta = 2c_s^2/v_A^2 = 2M_A^2/M_S^2$. The natural Mach numbers in the theory are then M_A and M_S; M_{ms} need play no role. In comparison, neither θ nor β play roles in the gasdynamic or phenomenological theories (except through M_{ms}). The standard cubic analysis provides general solutions to (4), although these typically provide little insight. Consider, however, the special cases of parallel ($\theta = 0°$) and perpendicular ($\theta = 90°$) flows.

The cubic is easily factored when $\mathbf{B}_{sw}$ is parallel to the shock normal ($\cos\theta = 1$). Ignoring the two 'switch-on' shock solutions $X = M_A^{-2}$, (2) and (4) yield

$$\frac{a_s}{a_{mp}} = 1 + k\frac{(\gamma-1)M_A^2+\gamma\beta}{(\gamma+1)M_A^2} = 1 + k\frac{(\gamma-1)M_S^2+2}{(\gamma+1)M_S^2} \ . \qquad (5)$$

The rightmost form reveals that the gasdynamic expressions for X and a_s are recovered, cf. (1) and (3), as expected since the magnetic field essentially drops out of the problem in the parallel case. This equation implies two important results for $\theta = 0°$. First, phenomenological replacement of M_S by M_A or M_{ms} ($= M_A$ for $\theta = 0°$) in (1), as proposed by *Spreiter et al.* and *Russell*, is incorrect except in the special case $\gamma\beta = 2$. The phenomenological theories are therefore restricted special cases of the MHD theory. Second, Δ_{ms}, a_s and X are independent of M_A and M_{ms}, and depend only on γ and M_S (with the intuitive caveats that M_A & $M_S \geq 1$). This is a major difference from (1) with the replacements $M \rightarrow M_A$ or M_{ms}, which predict $\Delta_{ms}/a_{mp} \rightarrow k$ as $M_A \rightarrow 1$ instead of the correct result $\Delta_{ms}/a_{mp} \rightarrow k/4$ for $\gamma = 5/3$ and $M_S \gg 1$.

For perpendicular flows ($\cos\theta = 0$) the cubic equation for X collapses to a quadratic, whence

$$\frac{a_s}{a_{mp}} = 1 + \frac{k}{2(\gamma+1)}\left(A \pm \sqrt{A^2 - 4(\gamma-2)(\gamma+1)M_A^{-2}}\right) \qquad (6)$$

with $A = (\gamma-1) + \gamma/M_A^2 + 2/M_S^2$. For $\gamma \leq 2$ only the solution $a_s = a_{s+}$ is relevant (only $X = X_+ > 0$). In general, (6) is not equivalent to the gasdynamic solution (1), the *Spreiter et al.* and *Russell* variants of (1) with $M \rightarrow M_A$ and M_{ms}, respectively, or (5)'s MHD solution for $\theta = 0°$. However, in the special case $\gamma = 2$ (6) reduces to *Russell's* form for arbitrary β (writing M_A and M_S in terms of M_{ms} and β) and to *Spreiter et al.'s* form in the limit $\beta \rightarrow \infty$. Furthermore, the gasdynamic result (1) is recovered in the limit $M_A \rightarrow \infty$ (or $\beta \rightarrow 0$) for arbitrary γ. (This is true for all θ.) Equation (6) also implies three corollary results.

First, a_s, Δ_{ms} and X vary strongly with the angle θ. Second, in general the solution depends intrinsically on both M_A and M_S; while (6) can easily be rewritten in terms of M_{ms} and β, only for $\gamma = 2$ does the explicit β dependance disappear and the solution depend solely on M_{ms}. Last, the behavior as $M_A \to 1$ for $\theta = 90°$ differs greatly from the MHD result (5) for $\theta = 0$: for $\gamma = 5/3$, (5) implies $X \to 1/4$ for $\beta = 0$ ($M_S = \infty$) while (6) implies $X_+ \to 1$ for $\beta = 0$ and $X_+ \to 1.7$ for $\gamma\beta = 1$. Thus, factor of $\gtrsim 4$ changes in Δ_{ms} should exist at low M_A for different θ and M_S.

3. Numerical Analyses

When the magnetic field is neither parallel nor perpendicular to the flow direction and shock normal, (4)'s solutions are most instructive when presented graphically. Figure 2 shows the MHD theory's predictions for the ratio $a_s/a_{mp} = 1 + 1.1X$, where Spreiter et al.'s empirical value for k in (2) is used (see Section 4) and (4) is solved numerically, as a function of M_A for $\theta = 0$, 45, and 90° and $M_S = 8$. Figure 3 is an analogous plot for $M_S = 2$. The strong dependences on θ, M_A and M_S implied by (4) - (6) are clearly evident. Note that the $\theta = 0°$ curves are all flat, and so independent of M_A as in (5), with a level that depends on M_S. The curves for $\theta = 45°$ lie below the $\theta = 90°$ curves. Indeed, it may be shown analytically that the maximum allowed values of a_s (and X) for given M_S occur at $\theta = 90°$ and $M_A = 1^+$, with smaller M_S leading to larger X and a_s.

Figure 4 shows how X and a_s depend on γ: for a given θ and M_S, a larger γ leads to a higher curve for a_s versus M_A. However, it can be shown analytically from (4) that the maximum values of X and a_s are independent of γ: curves for different γ therefore all converge to the same maximum at $M_A = 1$ and $\theta = 90°$.

Quantitative comparisons between the MHD theory developed here, *Spreiter et al.'s* model given by (1) with $M = M_A$, and *Russell's* model are shown for various θ and M_A in Figure 5. For all θ the gasdynamic theory with $M = M_S$ coincides with the $\theta = 0°$ MHD solution. In general significant differences between the MHD predictions and the phenomenological models are apparent, except in the limited region $M_A \lesssim 1.5$ and $\theta \gtrsim 30°$ (decreasing M_S shrinks this region further). Note that the phenomenological models have no θ or (direct) M_S

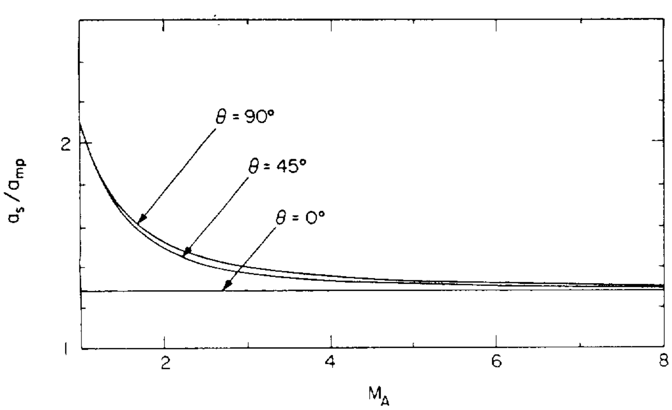

Figure 2. Ratio a_s/a_{mp} as a function of M_A and θ for $\gamma = 5/3$, $M_S = 8$ and $k = 1.1$. The ratio is independent of M_A for $\theta = 0°$ and is maximum at $\theta = 90°$ & $M_A = 1$.

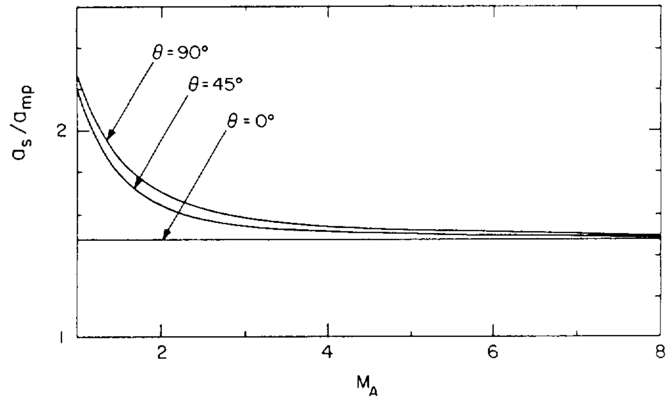

Figure 3. Similar to Figure 2 but with $M_S = 2$. Note that a_s increases as M_S decreases (for all θ).

dependences, whereas the MHD theory shows these dependences to be very important. The maximum a_s values predicted by the MHD and phenomenological models ($M_A \sim 1$) are, however, almost identical for large $M_S \gtrsim 5$. When M_S is small the MHD theory predicts larger a_s values. However, the maximum difference in a_s between the MHD and 'gasdynamic' theories remains less than 50% for $M_S > 1$.

4. Discussion

Differences of 50% or more in a_s and 400% in Δ_{ms}, due to θ and M_S effects when $M_A \lesssim 5$, are easily discerned in Figures 2 - 5. Differences occur both between the MHD, gasdynamic and phenomenological theories and between different parameter sets for the MHD theory. That θ effects should be very important, as well as M_A and M_S variations, in determining a_s and Δ_{ms} is one of this paper's important predictions. Refinements to the present 'local' MHD theory will undoubtedly occur when global MHD effects are considered.

The above results argue that values for γ derived from measurements of Δ_{ms} [*Fairfield*, 1971; *Zhuang and Russell*, 1981; *Farris et al.*, 1991] should depend on the formulae used for Δ_{ms} and on whether M_A, θ and M_S variations are all considered. Finite M_A, θ and M_S effects may explain the wide scatter in the published γ values. Analyses to measure γ should be redone using (4)'s explicit MHD solutions (or successors thereof).

Near 1 AU the solar wind speed and electron temperature vary relatively little, whence M_S lies within a factor of 2 of the nominal value $M_S \sim 7$. For $M_A \sim 1 - 3$,

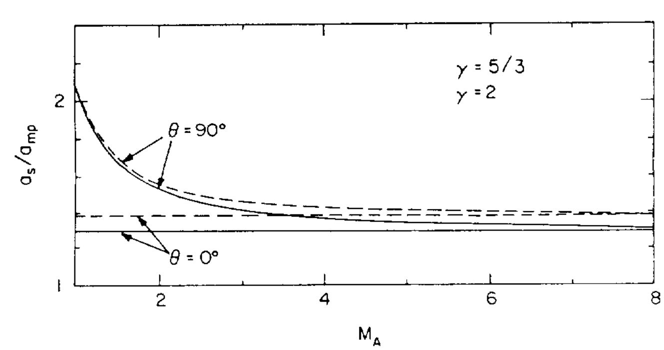

Figure 4. Ratio a_s/a_{mp} versus M_A for $\gamma = 5/3$ (full lines) and $\gamma = 2$ (dashed), $M_S = 8$, and $\theta = 0$ & 90°.

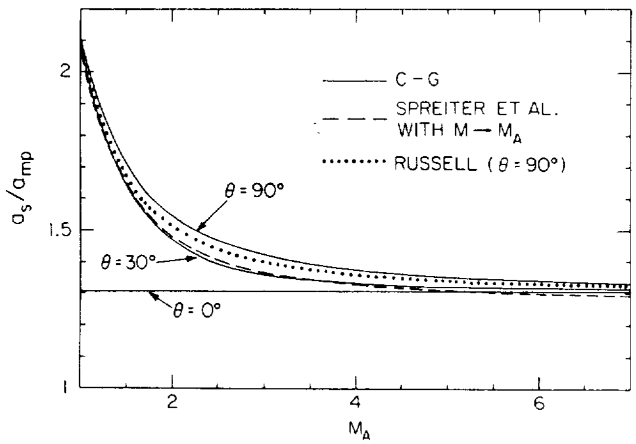

Figure 5. Comparison between the MHD theory given by (2) and (4) (full lines) for various θ, and the phenomenological *Spreiter et al.* (dashed) and *Russell* (dotted for $\theta = 90°$) models for $M_S = 5$ and $\gamma = 5/3$. For arbitrary θ the *Russell* curve lies between the dotted and dashed curves.

then, Figures 2-5 predict that a_s will be within 10% of (1)'s prediction (for $M = M_A$). This paper's MHD theory, using Spreiter et al.'s empirical value for k in (2) and MHD theory to determine X, can therefore not explain unusually distant bow shock observations at low M_A [*Russell and Zhang,* 1992; *Cairns et al.,* 1994]. *Cairns and Lyon's* [1994] MHD simulations show, however, that intrinsically MHD effects modify (2) and increase k by a factor $\sim$ 3 over *Spreiter et al.'s* value. Incorporating the modifications into this paper's MHD theory they can explain the large a_s values at low M_A in the simulations. Comparisons with observational data still need to be done. It remains possible that the MHD variant of (2) breaks down at very low M_A.

5. Conclusions

The analyses above show that more detailed consideration of MHD effects results in major differences from the well-known gasdynamic theory, and its phenomenological variants, for Δ_{ms} and a_s. The MHD model depends on M_A, M_S, θ, and γ, all of which have important effects; in comparison the gasdynamic and phenomenological models involve only a single (differing) Mach number and no explicit θ effects (save through M_{ms}). The MHD theory reduces to the gasdynamic theory in the special cases $\theta = 0°$ or $M_A \gg M_S \gg 1$. The *Spreiter et al.* [1966] and *Russell* [1985] phenomenological theories reappear as the special cases of the MHD theory for (i) $\theta = 0$ with $\gamma\beta = 2$, and (ii) $\theta = 90°$ with $\gamma = 2$ and $\beta = 0$ or arbitrary β, respectively. The MHD theory predicts that Δ_{ms} and a_s should depend strongly on θ, M_A and M_S when M_A and/or $M_S \lesssim 5$, with the functional form of the M_A and M_S dependence varying with θ.In particular, varying θ from 0° to 90° at low M_A should cause Δ_{ms} to vary by $\sim$ 400%. Since M_A and M_S are frequently of this order out to 1 AU, the predicted MHD effects should have widespread applicability; for instance, in calculations of γ from measurements of Δ_{ms}. It is found that the MHD theory with *Spreiter et al.'s* value for k cannot explain observations and MHD simulations of distant, very low M_A bow shocks. *Cairns and Lyon* [1994] can explain the simulation results by inserting the larger, MHD value for k into the present paper's MHD theory. It remains possible, nevertheless, that a nonlinear model for $\Delta_{ms} = \Delta_{ms}(X)$ is necessary at very low M_A.

Acknowledgments. Grants NAGW-2040 and ATM-9312263 supported this research.

References

Cairns, I.H., and J.G. Lyon, MHD simulations of Earth's bow shock at low Mach numbers: standoff distances, Geophys. Res. Lett., submitted, 1994.

Cairns, I.H., D.H. Fairfield, R.R. Anderson, V.E.H. Carlton, K.I. Paularena, and A.J. Lazarus, Unusual locations of Earth's bow shock on 24-25 September 1987: Mach number effects, J. Geophys. Res., in press, 1994.

Fairfield, D.H., Average and unusual locations of the Earth's magnetopause and bow shock, J. Geophys. Res., *76*, 6700, 1971.

Farris, M.H., S.M. Petrinec, and C.T. Russell, The thickness of the magnetosheath: constraints on the polytropic index, Geophys. Res. Lett., *18*, 1821, 1991.

Formisano, V., P.C. Hedgecock, G. Moreno, J. Sear, and D. Bollea, Observations of Earth's bow shock for low mach numbers, *Planet. Space Sci.*, *19*, 1519, 1971.

Holzer, R.E., and J.A. Slavin, Magnetic flux transfer associated with expansions and contractions of the dayside magnetosphere, J. Geophys. Res., *83*, 3831, 1978.

Russell, C.T., Planetary bow shocks, in *Collisionless Shocks in the Heliosphere: Reviews of Current Research*, edited by B.T. Tsurutani and R.G. Stone, p. 109, Amer. Geophys. Union, Washington, D.C., 1985.

Russell, C.T., and T.-L. Zhang, Unusually distant bow shock encounters at Venus, Geophys. Res. Lett., *19*, 833, 1992.

Sibeck, D.G., R.E. Lopez, and E.C. Roelof, Solar wind control of the magnetopause shape, location, and motion, J. Geophys. Res., *96*, 5489, 1991.

Spreiter, J.R., A.L. Summers, and A.Y. Alksne, Hydromagnetic flow around the magnetosphere, *Planet. Space Sci.*, *14*, 223, 1966.

Zhuang, H.C., and C.T. Russell, An analytic treatment of the structure of the bow shock and magnetosheath, J. Geophys. Res., *86*, 2191, 1981.

I. H. Cairns and C. L. Grabbe, Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242.

(received February 3, 1994; revised June 14, 1994; accepted September 16, 1994.)