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Finite Difference Time Marching in the Frequency Domain: A Parabolic Formulation for the Convective Wave Equation

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An explicit finite difference iteration scheme is developed to study harmonic sound propagation in ducts. To reduce storage requirements for large 3D problems, the time dependent potential form of the acoustic wave equation is used. To insure that the finite difference scheme is both explicit and stable, time is introduced into the Fourier transformed (steady-state) acoustic potential field as a parameter. Under a suitable transformation, the time dependent governing equation in frequency space is simplified to yield a parabolic partial differential equation, which is then marched through time to attain the steady-state solution. The input to the system is the amplitude of an incident harmonic sound source entering a quiescent duct at the input boundary, with standard impedance boundary conditions on the duct walls and duct exit. The introduction of the time parameter eliminates the large matrix storage requirements normally associated with frequency domain solutions, and time marching attains the steady-state quickly enough to make the method favorable when compared to frequency domain methods. For validation, this transient-frequency domain method is applied to sound propagation in a 2D hard wall duct with plug flow.

Introduction

Both steady-state (frequency domain) and transient (time domain) finite difference and finite element techniques have been developed to study sound propagation in ducts. To date, the numerical solutions have generally been limited to moderate frequency sound and mean flow Mach numbers. Wavelength resolution problems have prevented a broader range of applications of the numerical methods. A fine grid is required to resolve the short wavelengths associated with high frequency sound propagation with high Mach numbers.

The present paper focuses on sound propagation in a duct with flow. This is the first step in a larger research effort to develop efficient numerical techniques to predict high frequency sound propagation around 3D aircraft inlet nacelles with large subsonic inlet Mach numbers. The paper begins with a description of the problem, an explanation of why the transient approach is employed, and a description of the governing equations. The bulk of the paper describes the development of a stable, explicit finite difference scheme by a transformation of the governing hyperbolic wave equation. The scheme is iterated in time to converge to the steady-state solution associated with a Fourier transform solution.

Problem Statement

The problem under consideration here is the steady-state propagation of sound, represented by the perturbation acoustic potential, through a 2D rectangular duct with flow. The goal of the paper is to develop a stable, explicit finite difference scheme that incorporates an impedance condition applied at the duct exit and rigid body boundary conditions on the duct walls. For the numerical examples considered here, the grid system shown in Fig. 1 is employed.

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Transient Approach

In frequency domain approaches, pressure and acoustic velocity are assumed to be harmonic functions of time. The matrices associated with numerical solutions of the governing equations in the frequency domain have extremely large storage requirements. In similar 3D electromagnetic applications, frequency domain approaches are limited to several hundred thousand unknowns (still a small problem in 3D). Larger problems encounter roundoff errors and conditioning problems that prevent a reliable solution.

On the other hand, an explicit transient method generates no matrices, and is generally faster than a frequency domain approach (Baumeister, 1980a). Miller has developed explicit relationships showing the time advantage of transient over steady-state techniques (Miller, 1988). Consequently, the method of choice in the present paper is a time dependent method.

Governing Equations

The governing differential equations for studying acoustic propagation in ducts can be formulated in terms of the constitutive continuity and momentum equations or in terms of potential flow. The constitutive equation approach can handle a general 3D sheared flow while the potential approach is limited to 3D inviscid flow. The major advantage of the potential flow approach over the constitutive equation approach comes in decreased storage requirements associated with only one dependent variable.

Fortunately, acoustic propagation can often be reasonably modeled by an inviscid approximation. For single mode JT15D engine inlet nacelles data, a previous finite element study (Baumeister and Horowitz, 1984) employing the potential formulation in the frequency domain showed good agreement with experimental data—in the far field radiation pattern as well as suppressor attenuation. Due to this success, the problem under consideration here is formulated in terms of an acoustic potential.

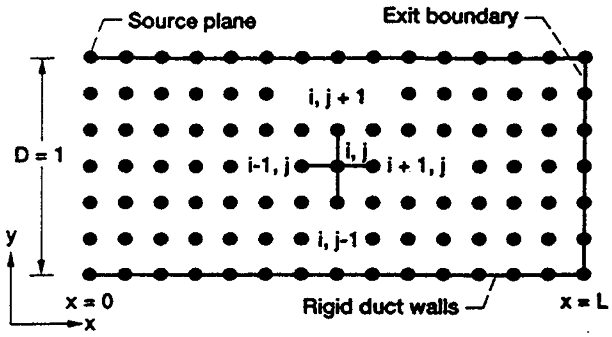

Fig. 1 Structured FD-TD mesh for rectangular duct

For inviscid, nonheat conducting and irrotational flow, the linearized potential wave equation can be written as

$$
\begin{aligned}
0 = {} & f^2\phi'_{tt} - (\bar{c}^2 - \bar{\Phi}_x^2)\phi'_{xx} - (\bar{c}^2 - \bar{\Phi}_y^2)\phi'_{yy} \\
& + 2\bar{\Phi}_x\bar{\Phi}_y\phi'_{xy} + 2f\bar{\Phi}_x\phi'_{xt} + 2f\bar{\Phi}_y\phi'_{yt} \\
& + 2(\bar{\Phi}_x\bar{\Phi}_{xx} + \bar{\Phi}_y\bar{\Phi}_{xy})\phi'_x + 2(\bar{\Phi}_x\bar{\Phi}_{xy} + \bar{\Phi}_y\bar{\Phi}_{yy})\phi'_y \\
& - (\gamma - 1)(f\phi'_t + \bar{\Phi}_x\phi'_x + \bar{\Phi}_y\phi'_y)(\bar{\Phi}_{xx} + \bar{\Phi}_{yy}) \qquad (1)
\end{aligned}
$$

with

$$\bar{c} = \{1 - \tfrac{1}{2}(\gamma - 1)(\bar{\Phi}_x^2 + \bar{\Phi}_y^2)\}^{1/2} \qquad (2)$$

where $\bar{\Phi}(x, y)$ represents the steady mean flow potential and $\phi'(x, y, t)$ the acoustic potential. The relationship between acoustic pressure P' and mean and acoustic potential is

$$\frac{1}{\bar{\rho}} P'_t = -f\phi'_{tt} - (\bar{\Phi}_x\phi'_x + \bar{\Phi}_y\phi'_y)_t \qquad (3)$$

For the special case of plug flow, the mean flow terms in Eq. (1) become

$$\bar{\Phi}_y = \bar{\Phi}_{xy} = \bar{\Phi}_{xx} = 0 \quad \bar{\Phi}_x = M_f \qquad (4)$$

Substituting Eq. (4) into Eqs. (1), (2) and (3) yields

$$0 = f^2\phi'_{tt} - (\bar{c}^2 - M_f^2)\phi'_{xx} - \bar{c}^2\phi'_{yy} + 2fM_f\phi'_{xt} \qquad (5)$$

and

$$\bar{c}^2 = 1 - \tfrac{1}{2}(\gamma - 1)M_f^2 \qquad (6)$$

$$\frac{1}{\bar{\rho}} P' = -f\phi'_t - M_f\phi'_x \qquad (7)$$

Equations (1) and (5) are the basic equations used to establish a finite difference formulation. However, care must be taken when discretizing derivative terms to insure that the resulting scheme is both stable and explicit. Note that it is easy to develop a stable implicit method, but this would yield a matrix formulation that is no better than a frequency domain approach. It turns out that the proper treatment of the mixed time and space derivative terms [which appear on the second line of Eq. (1)] is critical for maintaining stability.

The implicit formulation is obtained by approximating the mixed partials by

$$\phi'_{xt} = \frac{\phi'^{k+1}_{i,j} + \phi'^{k}_{i-1,j} + \phi'^{k}_{i+1,j} + \phi'^{k-1}_{i,j} - 2\phi'^{k}_{i,j} - \phi'^{k+1}_{i-1,j} - \phi'^{k-1}_{i+1,j}}{2\Delta x\Delta t} \qquad (8)$$

$$\phi'_{yt} = \frac{\phi'^{k+1}_{i,j} + \phi'^{k-1}_{i,j} + \phi'^{k}_{i,j+1} + \phi'^{k}_{i,j-1} - 2\phi'^{k}_{i,j} - \phi'^{k+1}_{i,j+1} - \phi'^{k-1}_{i,j-1}}{2\Delta y\Delta t} \qquad (9)$$

where i and j denote the space indices for the nodal system shown in Fig. 1, k is the time index defined by

$$t^{k+1} = t^k + \Delta t \qquad (10)$$

and Δx, Δy, and Δt are the grid spacings.

Baumeister (1980b) showed that Eq. (8) can be used in an explicit fashion for 1D plug flow problems in a duct. However, if the flow is not one dimensional or if the region exterior to the duct is included, the scheme must be implicit. Fortunately, this problem can be circumvented by transforming the potential and consequently modifying the governing equation. The details follow in the next section.

Transformation to Frequency Space

There are several ways to develop a frequency domain formulation for the general 2D acoustic wave Eq. (1) or the plug flow simplification Eq. (5). The Fourier Transform can be applied if the potential has a multi-frequency content. In the monochromatic case, this is equivalent to assuming that

Nomenclature

- $\bar{c}$ = steady dimensionless speed of sound, $\bar{C}^{\#}/C_0^{\#}$, Eq. (2)
- $D^{\#}$ = dimensional duct height or diameter
- D = dimensionless duct height, $D = 1$
- d = parameter, Eq. (33)
- $f^{\#}$ = dimensional frequency
- f = dimensionless frequency, $f^{\#}D^{\#}/C_0^{\#}$, Eq. (38)
- g = parameter, Eq. (33)
- h = parameter, Eq. (33)
- $i = \sqrt{-1}$
- L = length of duct, $L^{\#}/D^{\#}$, Fig. 1
- M_f = Mach number at duct entrance
- n = unit outward normal
- P = dimensionless fluid pressure, $P^{\#}/\rho_0^{\#}C_0^{\#2}$
- P' = acoustic pressure fluctuation, Eq. (3)
- t = dimensionless time, $f^{\#}t^{\#}$
- t_T = total dimensionless calculation time
- Δt = time step
- v_s = dimensionless acoustic velocity, Eq. (37)
- x = dimensionless axial coordinate, $x^{\#}/D^{\#}$
- Δx = axial grid spacing
- y = dimensionless transverse coordinate, $y^{\#}/D^{\#}$
- Δy = transverse grid spacing
- $Z^{\#}$ = impedance, Eq. (26)
- γ = ratio of specific heats
- ζ = dimensionless impedance, $Z^{\#}/\rho_0^{\#}C_0^{\#}$, Eq. (26)
- $\bar{\rho}$ = steady fluid density, $\bar{\rho}^{\#}/\rho_0^{\#}$
- Φ = dimensionless time dependent flow potential, $\Phi^{\#}/C_0^{\#}D^{\#}$
- $\bar{\Phi}$ = steady mean flow potential, Eq. (1)
- ϕ' = transient acoustic potential, Eq. (1)
- ϕ = transient acoustic potential in frequency space, Eq. (13)
- ω = dimensionless frequency, $2\pi f$

Subscripts

- i = axial index
- j = transverse index
- o = ambient or reference condition

Superscripts

- $\#$ = dimensional quantity
- k = time step

$$\phi'(x, y, t) = \psi(x, y)e^{-i\omega^{\#}t^{\#}} = \psi(x, y)e^{-i2\pi t} \quad (11)$$

which, in the case of plug flow [from Eq. (5)], yields

$$0 = (\bar{c}^2 - M_f^2)\psi_{xx} + \bar{c}^2\psi_{yy} + \omega^2\psi + i2\omega M_f\psi_x \quad (12)$$

This equation would be solved numerically using a linear system of equations. However, the associated matrix is not positive definite, which can lead to numerical difficulties, and which precludes the use of iterative techniques (see TRANSIENT APPROACH). Therefore, it is desirable to develop an explicit finite difference scheme to avoid the use of matrices. In time-dependent form, Eq. (1) or (5) cannot easily be discretized in such a way that the resulting finite difference scheme is both stable and explicit in the presence of flow, although it is possible to obtain reasonable results in the no-flow case.

The resolution of these difficulties is achieved by modifying the monochromatic transformation (11) to

$$\phi'(x, y, t) = \phi(x, y, t)e^{-i\omega^{\#}t^{\#}} = \phi(x, y, t)e^{-i2\pi t} \quad (13)$$

This differs from the classical monochromatic transformation in that the amplitude ϕ (no prime) is no longer assumed to be independent of time, as in Eq. (11). Employing Eq. (13), the time derivatives in Eqs. (1) and (5) can be replaced by the following relationships:

$$\phi'_t = [-i2\pi\phi + \phi_t]e^{-i2\pi t} \quad (14)$$

$$\phi'_{xt} = [-i2\pi\phi_x + \phi_{xt}]e^{-i2\pi t} \quad (15)$$

$$\phi'_{tt} = \frac{\partial}{\partial t}(\phi'_t) = (\phi_{tt} - 2i2\pi\phi_t - (2\pi)^2\phi)e^{-i2\pi t} \quad (16)$$

Under this transformation, the general Eq. (1) and the plug flow Eq. (5) become, respectively

$$\begin{aligned} f^2\phi_{tt} &- [i2f\omega + (\gamma - 1)f(\bar{\Phi}_{xx} + \bar{\Phi}_{yy})]\phi_t \\ &+ 2f\bar{\Phi}_x\phi_{xt} + 2f\bar{\Phi}_y\phi_{yt} = (\bar{c}^2 - \bar{\Phi}_x^2)\phi_{xx} + (\bar{c}^2 - \bar{\Phi}_y^2)\phi_{yy} \\ &- 2\bar{\Phi}_x\bar{\Phi}_y\phi_{xy} + \omega^2\phi + i2\omega\bar{\Phi}_x\phi_x + i2\omega\bar{\Phi}_y\phi_y \\ &- 2(\bar{\Phi}_x\bar{\Phi}_{xx} + \bar{\Phi}_y\bar{\Phi}_{xy})\phi_x - 2(\bar{\Phi}_x\bar{\Phi}_{xy} + \bar{\Phi}_y\bar{\Phi}_{yy})\phi_y \\ &+ (\gamma - 1)(-i\omega\phi + \bar{\Phi}_x\phi_x + \bar{\Phi}_y\phi_y)(\bar{\Phi}_{xx} + \bar{\Phi}_{yy}) \quad (17) \end{aligned}$$

$$\begin{aligned} f^2\phi_{tt} &- 2if\omega\phi_t + 2fM_f\phi_{xt} \\ &= (\bar{c}^2 - M_f^2)\phi_{xx} + \bar{c}^2\phi_{yy} + \omega^2\phi + i2\omega M_f\phi_x \quad (18) \end{aligned}$$

To see the relationship between the two transforms [Eqs. (11) and (13)], consider, for simplicity, the case of plug flow. The monochromatic transform yields Eq. (12), while the transient transformation yields Eq. (18). The only difference is in the presence of the time derivative terms on the left-hand side of Eq. (18). Physically, the time dependence in $\phi(x, y, t)$ is caused by assuming that the duct is quiescent at time 0, and that the source is turned on at that instant.

A series of numerical calculations was performed to investigate the relationship between ψ and ϕ as time progresses. It was verified that ϕ converges to the steady state ψ, so that

$$\lim_{t\to\infty} \phi(x, y, t) = \psi(x, y) \quad (19)$$

At present, a formal proof of convergence is not available.

Preconditioning

At present, transient solutions in frequency space are of no interest. Therefore, approximations to Eqs. (17) or (18) can be made, by modifying the time derivative terms, as long as the following two factors are taken into account: (1) the resulting scheme must be both stable and explicit, and (2) there must be at least one time derivative term in the equation to ensure that the scheme can be marched through time to obtain the steady-state solution. The mixed space-time derivative term in Eq. (18) prevents an explicit finite difference representation of the governing equation, and is therefore dropped. Furthermore, the hyperbolic time term in Eq. (18) is also dropped to further simplify the governing equation. For plug flow, this produces a parabolic "wave" equation of the form:

$$-2if\omega\phi_t = (\bar{c}^2 - M_f^2)\phi_{xx} + \bar{c}^2\phi_{yy} + \omega^2\phi + i2\omega M_f\phi_x \quad (20)$$

This type of differential manipulation is often associated with preconditioning of both time dependent partial differential equations (Turkel, Fiterman and Leer, 1993) and time independent partial differential equations (Turkel and Arnone, 1993) to accelerate convergence to the steady state solution. Here, though, the goal is obtaining a stable, explicit scheme.

In the absence of mean flow, the parabolic wave equation takes on the form

$$-2if\omega\phi_t = \phi_{xx} + \phi_{yy} + \omega^2\phi \quad (21)$$

The right-hand side of Eq. (21) is identified with the Helmholtz equation. In the Fourier transform approach, Bayliss, Goldstein, and Turkel (1982) have developed an iterative approach to its associated matrix; however, the Gaussian elimination approach is still the method of choice. In effect, the time iteration procedure presented in this paper offers a very simple iterative solution to this classic equation.

Initial and Boundary Conditions

The duct is assumed to be quiescent at time 0, so that the initial condition is

$$\phi(x, y, 0) = 0 \quad (22)$$

As the equation is iterated in time, the solution builds up to the steady-state solution.

Estimating the source mechanism in a duct is a complicated process. For example, duct source conditions have been successfully modeled by a Gutin representation in a wind tunnel (Eversman and Baumeister, 1989), by estimating the measured pressure modal content for a JT15D Turbofan engine (Baumeister and Horowitz, 1984) or most recently by an acoustic analogy for ducted fans (Farassat et al., 1994). For the validation purposes of this paper, however, a simple potential source at the duct entrance ($x = 0$) is assumed:

$$\phi'(0, y, t) = 1e^{-i2\pi t} \quad (23)$$

and through Eq. (13) to

$$\phi(0, y, t) = 1 \quad (24)$$

The hard wall condition, on the duct walls, is

$$\nabla\phi \cdot n = 0 \quad (25)$$

where n is the unit outward normal.

To simulate a nonreflective boundary at the duct exit, the difference equation is expressed in terms of an exit impedance of the form

$$\zeta = \frac{Z^{\#}}{\bar{\rho}^{\#}c_0^{\#}} = \frac{P'}{\dfrac{\partial\phi'}{\partial x}} = \bar{\rho}\bar{c} \quad (26)$$

where $Z^{\#}$ is the acoustic impedance. In ducts, this termination impedance for a reflection free termination is very restrictive (plane waves only). However, this condition is useful as a first order termination in the far field from an open ended duct. Note that an advantage of the transient-frequency formulation presented here is the capability of using the classical impedance concept as developed for the frequency domain.

Finite Difference Equations

The finite difference approximations determine the potential at the spatial grid points at time steps $t^k = k\Delta t$. Starting from the known initial conditions at $t = 0$ and the boundary conditions, the algorithm marches the solution out to later times.

Away from the duct boundaries, each partial derivative in Eq. (20) can be expressed as follows:

$$\phi_t = \frac{\phi_{i,j}^{k+1} - \phi_{i,j}^{k-1}}{2\Delta t} \tag{27}$$

$$\phi_{xx} = \frac{\phi_{i+1,j}^{k} - 2\phi_{i,j}^{k} + \phi_{i-1,j}^{k}}{\Delta x^2} \tag{28}$$

$$\phi_{yy} = \frac{\phi_{i,j+1}^{k} - 2\phi_{i,j}^{k} + \phi_{i,j-1}^{k}}{\Delta y^2} \tag{29}$$

$$\phi_x = \frac{\phi_{i+1,j}^{k} - \phi_{i-1,j}^{k}}{2\Delta x} \tag{30}$$

$$\phi = \phi_{i,j}^{k} \tag{31}$$

Substituting these expressions into Eq. (20) yields

$$\phi_{i,j}^{k+1}\left(\frac{-h}{\Delta t}\right) = \phi_{i,j}^{k}\left(-\frac{2d^2}{\Delta x^2} - \frac{2\bar{c}^2}{\Delta y^2}\right) + \phi_{i+1,j}^{k}\left(\frac{d^2}{\Delta x^2} + \frac{g}{2\Delta x}\right) + \phi_{i-1,j}^{k}\left(\frac{d^2}{\Delta x^2} - \frac{g}{2\Delta x}\right) + \phi_{i,j+1}^{k}\left(\frac{\bar{c}^2}{\Delta y^2}\right) + \phi_{i,j-1}^{k}\left(\frac{\bar{c}^2}{\Delta y^2}\right) + \phi_{i,j}^{k-1}\left(\frac{-h}{\Delta t}\right) + \omega^2\phi_{i,j}^{k} \tag{32}$$

where

$$d^2 = \bar{c}^2 - M_f^2$$
$$h = i\omega f$$
$$g = 2i\omega M_f \tag{33}$$

Equation (32) is an explicit two step scheme. At $t = 0$, field values at t^{k-1} are assumed zero because the initial field is quiescent.

The expressions for the difference equations at the boundaries are complicated somewhat by the impedance condition. However, a simple integration procedure resolves this problem. Baumeister (1980a & 1980b) gives precise details for generating the time difference equations at the boundaries.

Stability

A von Neumann stability analysis indicates that the method is conditionally stable, subject to the condition

$$\Delta t < \frac{1}{\frac{2}{\omega f}\left[\left(\frac{d}{\Delta x}\right)^2 + \left(\frac{\bar{c}}{\Delta y}\right)^2\right] - \pi + \frac{M_f}{f\Delta x}} \tag{34}$$

In a typical application, ω, f, and M_f are set by conditions in the duct. Next, the grid spacing parameters Δx and Δy are set to accurately resolve the estimated spatial harmonic variation of the acoustic field. Finally, Δt is chosen to satisfy Eq. (34).

In the von Neumann analysis, conditional stability means that the amplification factor, which describes how errors propagate from one time step to the next, has magnitude one. Thus, when inequality (34) is satisfied, errors are not magnified or diminished in magnitude. This is a desirable property, since the numerical formulation cannot distinguish between an error and a small acoustic mode.

The von Neumann stability analysis does not take into account boundary conditions. For stability, gradient boundary conditions require the use of smaller Δt than predicted by Eq. (34).

Numerical Examples

The following problem is considered: a plane wave propagates from the left into a quiescent duct of length one, and the acoustic potential field is to be computed in the duct. In the three examples that follow, the parabolic transient-frequency domain results for plane wave propagation with plug flow are compared to the exact results of the steady Fourier transformed solutions, given by

$$\psi(x) = e^{[(i2\pi f)/(\bar{c}+M_f)]x} \tag{35}$$

For the iterates to converge to ψ, a suitable calculation time must be specified. Pearson (1953) has shown that the total time t_T required for the steady-state solution [Eq. (35)] to become established in the duct equals the time for a plane wave propagating at the speed of sound to reach the end of the duct. In terms of the dimensional variables, the distance $x^\#$ a plane wave moves in time $t^\#$ is related to the speed of sound by

$$x^\# = c_0^\# t^\# \tag{36}$$

so that the dimensionless velocity is

$$v_s = \frac{1}{f} \quad (x = v_s t) \tag{37}$$

where the dimensionless frequency f is

$$f = \frac{f^\# D^\#}{c_0^\#} \tag{38}$$

According to Pearson, for the steady-state solution ψ to develop and reach the duct exit at $x = L$, the total calculation time must satisfy

$$t_T > \frac{L}{v_s} \tag{39}$$

For the numerical examples below, f and L are each set to 1, so $t_T > 1$.

Pearson also shows that a decaying transient solution propagates at the speed of sound along with the steady-state solution. This transient solution is reflected back into the duct by the exit condition. Consequently, some slack time ($t_T \gtrsim 3$) was added in the numerical examples to allow this transient to decay and to allow for differences in propagation between the parabolic and hyperbolic equations.

Semi-Infinite Duct. Boundary conditions can introduce instabilities (Baumeister, 1982, and Cabelli, 1982) into otherwise stable finite difference schemes. Therefore, it is important to test the proposed method for convergence in time to the steady-state solution in the absence of the exit boundary condition [Eq. (26)], and to test independently the effect of the exit boundary condition on the solution. Therefore, in this example, the computational boundary is set at $x = 50$, far enough away from the true boundary $x = 1$ that any artifacts arising from imperfections in the exit boundary condition do not affect the solution in [0, 1].

Two cases are considered—no flow, in Fig. 2(a), and flow with Mach number $M_f = -.5$ in Fig. 2(b). For the first case, $\Delta x = 0.05$, so it would take 1980 time steps (1000 forward, 980 backward) for any artifacts to interfere with the solution. Since $t_T = 5$ and $\Delta t = 0.003$, only 1667 steps were run. For the second case, $\Delta x = 0.025$, so it would take 3960 time steps

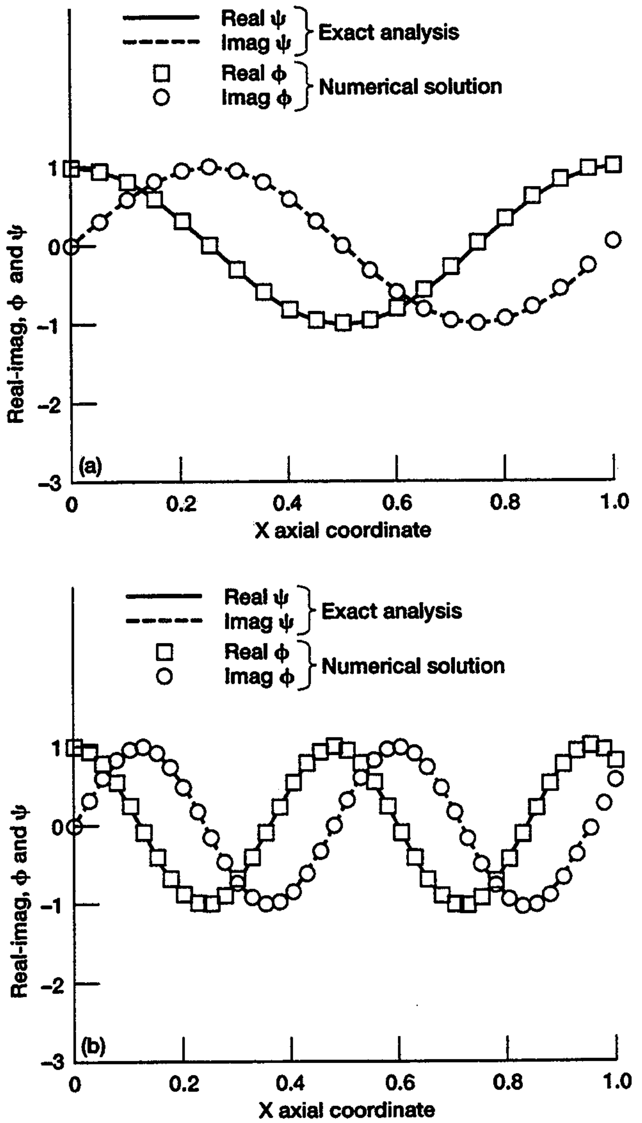

Fig. 2 Analytical and numerical potential profiles along wall for plane wave propagating in a semi-infinite hard wall duct ($f = 1$). (a) $M_f = 0$ ($\Delta x = 0.05$, $\Delta t = 0.003$, $t_T = 5.0$). (b) $M_f = -0.5$ ($\Delta x = 0.025$, $\Delta t = 0.001$, $t_T = 5.0$).

(2000 forward, 1960 backward) for any artifacts to interfere with the solution. Since $t_T = 3$ and $\Delta t = 0.001$, only 3000 steps were run. In both cases, the numerical results show excellent agreement with the exact solution.

Finite Duct, $L = 1$. In this example, the computational boundary is set at the true boundary $x = 1$ to examine the effect of the exit boundary condition [Eq. (26)]. The frequency is normalized to 1. Again, two cases were considered—no flow ($M_f = 0$) and flow with Mach number $M_f = -.5$. The results are shown in Figs. 3(a) and 3(b), respectively. The numerical results again match well with the exact results, but it is clear that the exit boundary condition does have a slight degrading effect on the solution. Notice also that the time step has been decreased here, which increases the execution time; however, the computational domain is much smaller, which decreases the execution time. The total calculation time in this example is $t_T = 4.0$.

Convergence Rate. In this example, the convergence rate is studied for the region $x = 0$ to $x = 1$ using the semi-infinite duct. The results are shown in Fig. 4 for the real and imaginary components of the potential and Fig. 5 for the magnitude of the potential. As seen in these figures, the numerical solution quickly and accurately converges to the exact steady-state solution.

For plane wave propagation, the exact solution to the hyperbolic governing equations indicates the arrival of the "steady state" solution at $t = 1$ [Pearson, 1953, or Baumeister, 1983, Eq. (28) renormalized]. As seen in Figs. 4 and 5, the parabolic solution does not converge to the steady-state until about time $t = 2$. Thus, the parabolic marching scheme requires more time steps to converge than a hyperbolic method.

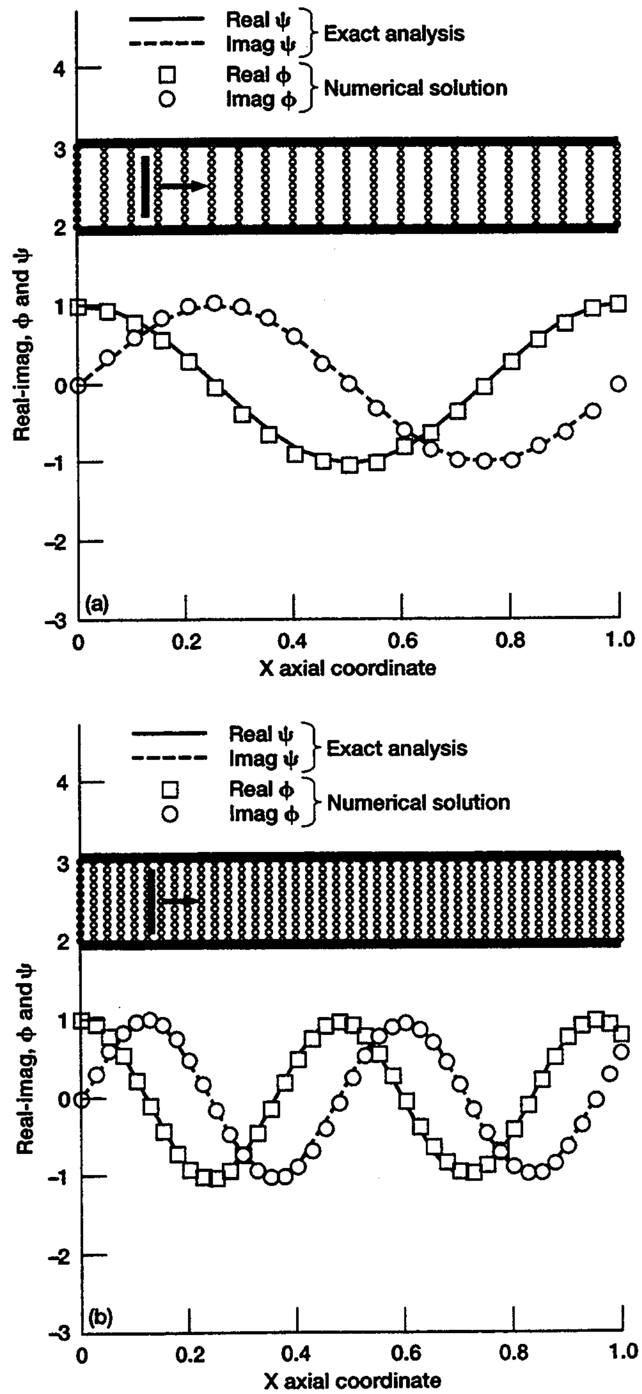

Fig. 3 Analytical and numerical potential profiles along wall for plane wave propagating in a hard wall duct of unit length and a non-reflecting exit condition ($f = 1$). (a) $M_f = 0$ ($\Delta x = 0.05$, $\Delta t = 0.0001$, $t_T = 4.0$). (b) $M_f = -0.5$ ($\Delta x = 0.025$, $\Delta t = 0.00001$, $t_T = 4.0$).

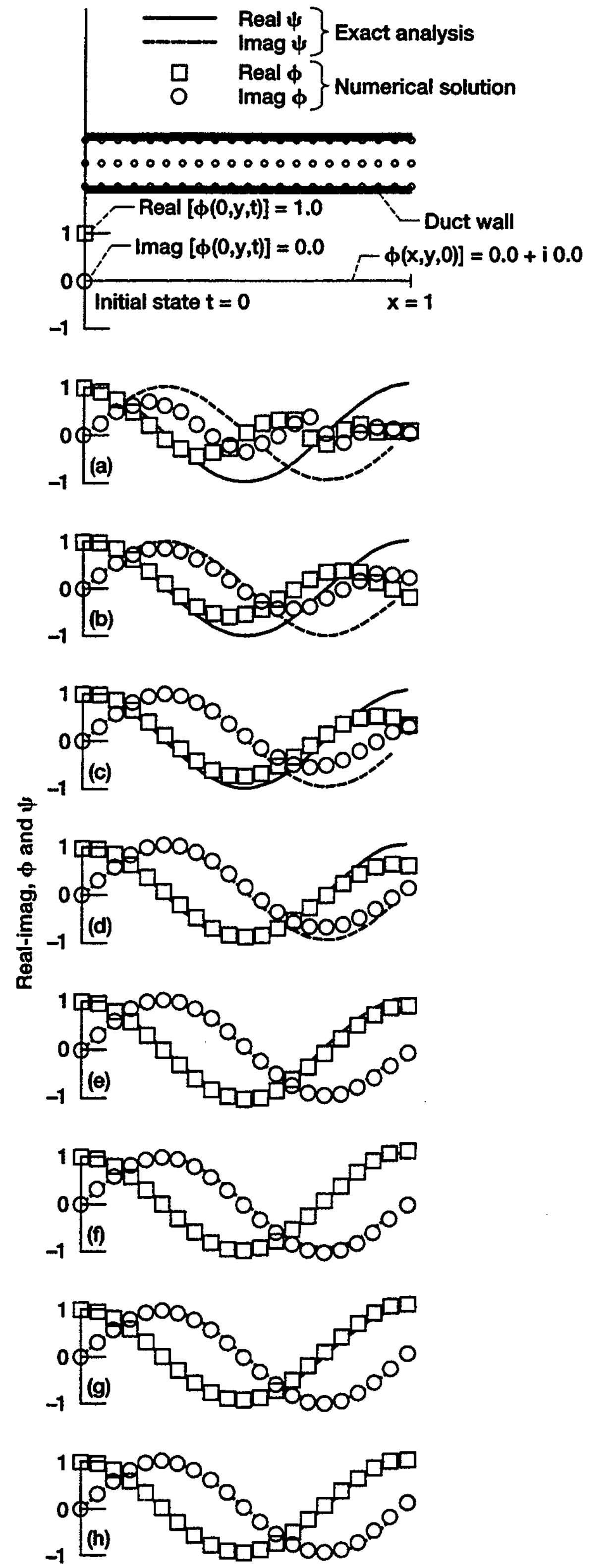

Fig. 4 Developing history of disturbance propagation in Fourier transformed domain as a function of time. (M_f = 0, Δx = 0.05, Δt = 0.003). (*a*) t = 0.25. (*b*) t = 0.5. (*c*) t = 0.75. (*d*) t = 1.00. (*e*) t = 1.5. (*f*) t = 2.0. (*g*) t = 2.5. (*h*) t = 3.0.

For parabolic partial differential equations with real coefficients, disturbances propagate with infinite speed. In the present calculations, numerical solutions to the parabolic Eq. (20) have this trait. The finite difference values of the potential propagate throughout the domain at the numerical velocity $\Delta x/\Delta t$ rather than the speed of sound. This property may account for the slower convergence time for the parabolic formulation as com-

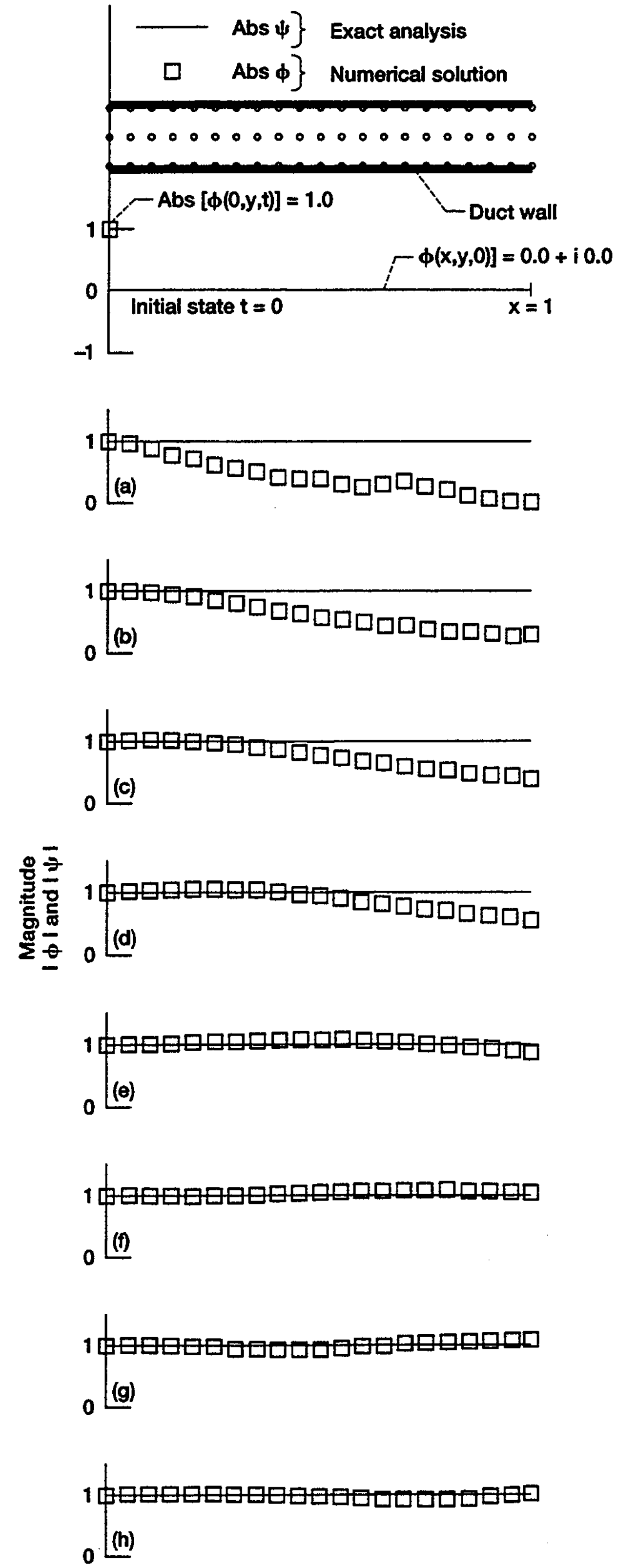

Fig. 5 Developing history of magnitude of disturbance propagation in Fourier transformed domain as a function of time. (M_f = 0, Δx = 0.05, Δt = 0.003). (*a*) t = 0.25. (*b*) t = 0.5. (*c*) t = 0.75. (*d*) t = 1.00. (*e*) t = 1.5. (*f*) t = 2.0. (*g*) t = 2.5. (*h*) t = 3.0.

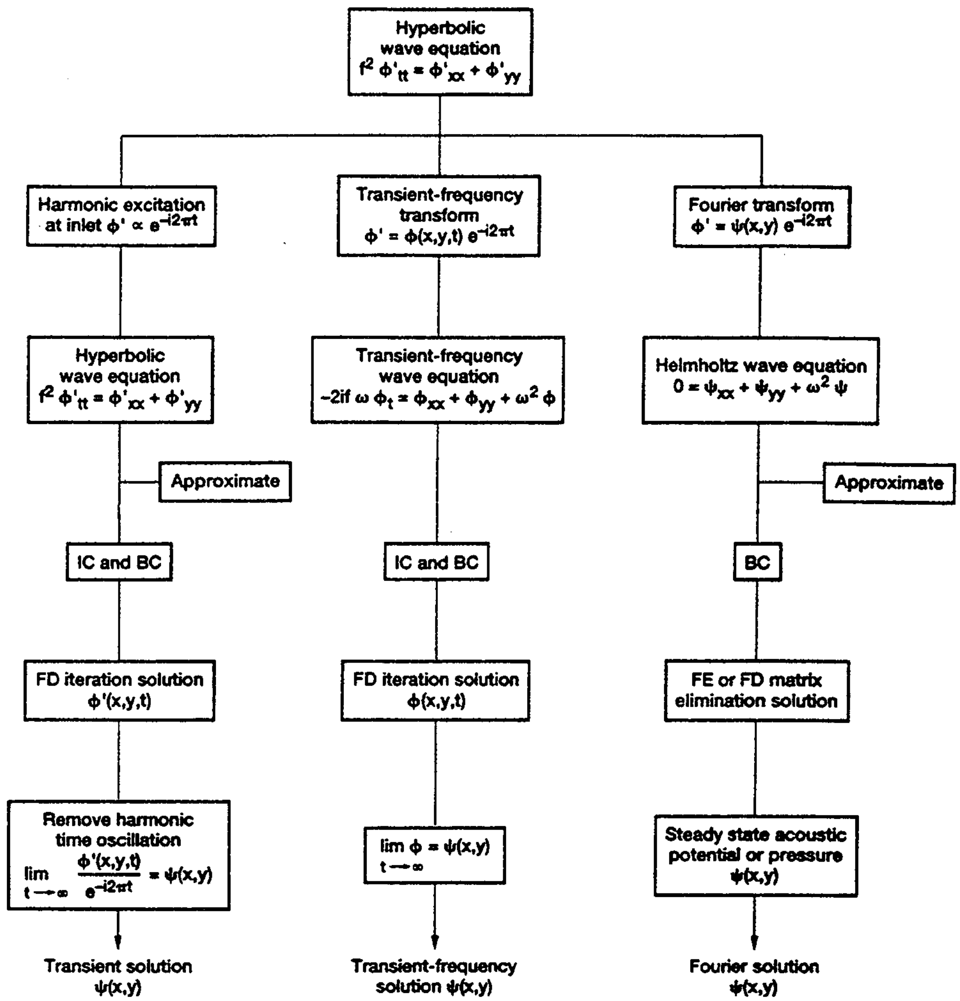

Fig. 6 Alternate finite difference/element methods in solving wave equations

pared to the hyperbolic formulation of the appropriate governing equations. In the parabolic scheme, acoustic energy is numerically transferred ahead of the true wave front. As the algorithm proceeds, this transience dies down and the correct steady-state solution is attained, using minimal computer storage with run times comparable to other methods.

Historical Perspective

With the transient-frequency approach developed in this paper, three different solution techniques are now available to solve the wave equation. This is outlined in Fig. 6 for the zero mean flow case.

The Fourier transform of the wave equation was the first numerical approach used to study sound propagation in jet engine ducts (Baumeister and Bittner, 1973). This steady-state approach is outlined on the right side of Fig. 6. The governing hyperbolic wave equation is transformed to the elliptic Helmholtz equation. Finite difference (FD) and finite element (FE) numerical formulations have been employed to solve this equation. After the application of the boundary conditions (Fig. 6; [BC]), the associated finite difference or finite element global matrix is solved for the velocity potential (or pressure). Because the matrix form of the Helmholtz equation is not positive definite, matrix elimination solutions are generally employed. This requires extensive storage, as discussed in Transient Approach. Conveniently, the steady-state approach allows the direct calculation of the potential (or pressure) fields.

To make the numerical solutions more cost effective, grid saving approximations to the governing Helmholtz equation have been used (Fig. 6; [Approx.]). Baumeister (1974) employed the wave envelope theory. Candel (1986) presents an extensive discussion of research accomplishments in this area.

The transient solution to the wave equation was the second numerical approach used to study propagation in ducts, to eliminate matrix storage requirements, as shown in the left-hand column of Fig. 6. Harmonic sound is introduced as a boundary condition at the duct entrance into a quiescent duct. FD approximations to the wave equation are then solved by an iteration process. The calculations are run until the initial transience dies out and steady harmonic oscillations are established. Finally, the transient variable ϕ' is transformed into the steady-state variable ψ associated with the solution of the Helmholtz equation. As with the steady-state approach, simplifications to the governing equations can reduce computer storage and run times (Baumeister, 1986).

The third option, the transient-frequency technique, is illustrated in the central column of Fig. 6. This explicit iteration method eliminates the large matrix storage requirements of steady-state techniques and allows the use of conventional impedance conditions. As time increases, the iteration process directly computes the steady-state variable ψ.

Concluding Remarks

A transient-frequency domain numerical solution of the potential acoustic equation has been developed. The potential form of the governing equations has been employed to reduce the number of dependent variables and their associated storage requirements. This explicit iteration method represents a significant advance over previous steady-state and transient techniques. Time is introduced into the steady-state formulation to form a hyperbolic equation. A parabolic approximation (in time) to this hyperbolic equation is employed in this paper. The field is iterated in time from an initial value of 0 to attain the steady-state solution.

The method eliminates the large matrix storage requirements of steady state techniques in the frequency domain but still allows the use of conventional impedance conditions. Most importantly, the formulation is explicit under flow conditions. In each example provided, the numerical solution quickly and accurately converges to the exact steady-state solution.

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