



# Departure Trajectory Synthesis and the Intercept Problem

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## Abstract

Two areas of the departure problem in air traffic control are discussed. The first topic is the generation of climb-out trajectories to a fix. The trajectories would be utilized by a scheduling algorithm to allocate runways, sequence the proposed departures, and assign a departure time. The second area is concerned with finding horizontal trajectories to merge aircraft from the TRACON to an open slot in the en-route environment. Solutions are presented for the intercept problem for two cases: (1) the aircraft is traveling at the speed of the aircraft in the jetway, (2) the merging aircraft has to accelerate to reach the speed of the aircraft in the en-route stream. An algorithm is given regarding the computation of a solution for the latter case. For the former, a set of equations is given that allows us to numerically solve for the coordinate where the merge will occur.

## Introduction

Departure scheduling is an important area of air traffic control. Currently, there are no automation tools available to assist the tower, departure, and sector controllers with the management of departure traffic. The current procedure of runway assignment and sequencing, administering departure clearances, and the tactical control of aircraft within the TRACON is largely a manual process. By taking advantage of current technology, the development of advanced automation systems that will assist air traffic controllers with the control of departures is being realized. One such system is Expedite Departure Path (EDP). EDP is the component of the Center-TRACON Automation System (CTAS) which will handle the scheduling and flow of departure traffic. EDP will provide departure and en-route controllers with advisories to assist in the movement of aircraft from the runway to the en-route stream.

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The departure problem in air traffic control automation can be separated into three sub-problems. These are runway allocation and sequencing for departure, trajectory synthesis, and the en-route merge problem. The runway allocation and sequencing function will ideally take inputs from a trajectory synthesis program in order to produce an optimal and robust schedule. Miles-in-trail restrictions over a departure gate, runway capacities, and airspace constraints must all be considered in this implementation. The en-route merge problem will ultimately require a trajectory synthesis function to predict aircraft positions along a route, and the time to fly a time-optimal trajectory to a fixed point.

The purpose of this paper is to discuss two parts of the departure problem— the trajectory synthesis problem and the en-route merge or “intercept” problem. Denery and Erzberger<sup>1</sup> state that the management of traffic in constrained airspace requires a capability to accurately predict aircraft trajectories. Due to the abundance of literature which discusses trajectory synthesis, a brief discussion on this topic as it pertains to departures is included for completeness. The intercept problem is addressed since there presently lacks a means to assist controllers in merging aircraft departing the TRACON and entering en-route streams. It will be shown that horizontal trajectories can be computed which deliver an aircraft to a known moving position between aircraft flying in-trail.

## Trajectory Synthesis

The trajectory synthesis portion will make up a large part of the structure of the departure scheduling software. Its importance lies in the fact that accurate departure fix crossing times are needed to produce schedules which are beneficial to the air traffic system. The trajectory synthesis is based entirely upon the trajectory synthesis program (TS) currently within CTAS. Stated simply, the TS uses point mass performance models of the jet and turboprop aircraft employed by the airlines. These models are then used, along with pilot procedures defined by airline operating handbooks, to calculate arrival times to the outer marker for final approach.<sup>2</sup>

Several simplifying assumptions are made which speed-up computation without sacrificing accuracy. Commercial aircraft typically do not partake in large amplitude maneuvers or encounter sudden changes in the flight path (e.g. zoom climbs, maximum performance turns). Therefore, assume that the rate of that the aircraft's flight path angle,  $\gamma_a$ , is “small”, and the acceleration normal to the velocity vector is negligible. It is assumed that the total derivative of the wind velocity field with respect to time is approximately zero. The explicit time dependency of wind is neglected, and it is assumed that the spatial variation changes slowly. It is further assumed that there is no wind component in the vertical direction.

For completeness the simplified equations of motion which are used for determining a trajectory are presented below. The reader is referred to Slattery and Zhao<sup>3</sup> for a complete discussion.

$$\dot{V}_t = \frac{T - D}{W}g - g\gamma_a \quad (1)$$

$$\dot{\psi}_i = \frac{Lg \sin \phi}{WV_g} \quad (2)$$

$$0 = L \cos \phi - W \quad (3)$$

$$\dot{h} = V_t \gamma_a = V_g \gamma_i \quad (4)$$

$$\dot{x} = V_g \sin \psi_i \quad (5)$$

$$\dot{y} = V_g \cos \psi_i \quad (6)$$

Where  $V_t$  is the true airspeed,  $T$  is the thrust and is parallel to the velocity vector,  $D$  is the drag, and  $g$  is the acceleration due to gravity. The flight path angle,  $\gamma_a$ , is measured with respect to the moving air mass, and  $\gamma_i$  is measured with respect to the  $x - y$  plane. The groundspeed,  $\vec{V}_g$ , is the vector sum of the true airspeed and the wind velocity and is given by

$$\vec{V}_g = \vec{V}_t + \vec{V}_w \quad (7)$$

The magnitude of the groundspeed,  $V_g$ , denotes the speed of the aircraft “over the ground.” The groundtrack heading,  $\psi_i$ , is measured positive clockwise from North. The bank angle,  $\phi$ , is positive when the aircraft is turning clockwise. The position of the aircraft is measured on a coordinate system where the  $x$  axis is positive East, and the  $y$  axis is positive North. The altitude is given by  $h$ , and the rate of climb is given by  $\dot{h}$ . The lift generated by the aircraft is  $L$  and the aircraft’s weight is  $W$ .

The departure trajectory is determined by considering the trajectory as two independent components. The first component is the horizontal path. The horizontal path extends from the runway to the final waypoint on the departure route. The second is the vertical profile that is flown to a final or “control” altitude. The aircraft generally follows a pre-determined route through the TRACON, along which the speed profile is known from the vertical profile. Since the aircraft is assumed to have a “small” flight path angle, the equations of motion nearly decouple. This allows the horizontal path to be found independent of the vertical path.

## Horizontal Path

The horizontal path flown will generally be one of the standard departure routes. At those airports located within Class B airspace and some within Class C, Standard Instrument Departure routes (SIDs) are used as the primary routes out of the airspace. The purpose of the SIDs is to separate the departure traffic from the arrivals by using different “corridors.” A SID will typically contain several waypoints for the aircraft to fly until it reaches the VOR for transition to the jetway. The purpose of the horizontal synthesis is then to assign and compute such a route from the runway to the outer departure fix satisfying any necessary constraints along the route.

A typical horizontal trajectory consists of a series of straight line trajectory segments connected at “waypoints” by circular arc transitions. At each waypoint the aircraft undergoes a heading change toward the next point until the aircraft has merged into an en-route trajectory. At the waypoint there are several ways to model the turning segment of the trajectory. The first and most common approach (i.e. that which is used in most on-board flight management systems) is to construct the circular segment such that it is tangent to the two straight line paths, and does not intersect the waypoint (Fig. 1). This is called an “inside turn.” In fact, this is the only type of turn used in our current analysis as other types of turns may force the trajectory to pass through the waypoint.

## Vertical Profile

The vertical profile is the second component of the trajectory. The vertical profile consists of a number of trajectory segments. These segments are (1) an acceleration to a calibrated airspeed (CAS), (2) a climb at a constant CAS, (3) a climb at a constant Mach number, and (4) level flight at a constant CAS or Mach. For simplicity, it is assumed that the bank angle is zero. The vertical trajectory is governed by only two equations of motion, Eqns. 1 and 4. We will consider  $V_i$  and  $h$  to be states and the controls to be  $T$  and  $\gamma_i$ . In order to define a segment, any combination of two controls or states needs to be specified. Alternatively, any two relations that define the controls may be specified. The most commonly used are (1) constant rate-of-climb or flight path angle, (2) constant CAS or Mach, and (3) constant acceleration.

A typical profile that is currently flown in our simulation is shown in Fig. 2. The first segment is for the aircraft to accelerate to a calibrated airspeed of 250 kts (KCAS) while climbing at a fixed rate of climb beginning at approximately 1,000 ft above ground level (AGL). The aircraft then maintains 250 KCAS until 10,000 ft according to Federal Aviation Regulations (FARs). The aircraft then accelerates, again at a fixed climb rate, to a new climb CAS that is dependent upon the aircraft. This CAS is maintained until the aircraft reaches a climb Mach number (a typical altitude for this transition is usually above 27,000 ft). The aircraft continues climbing at a constant Mach until it reaches a cruise altitude.

Taken together, the vertical and horizontal trajectory approximates the aircraft's flight through the TRACON. The trajectory then may be used as an input to a scheduler as well as to provide tactical control once the aircraft is aloft. An example of a climb-out trajectory is shown in Fig. 3. The aircraft departed Denver International on January 8, 1997. The computed climb-out trajectory is also shown along with lines of constant Mach number and constant CAS for comparison. The aircraft was "picked-up" on radar at 12,300 ft at a climb CAS of 275 kts. The "tail" at the bottom of the trajectory was ignored as it is an artifact of the groundspeed filtering algorithm used by the radar. In the simulation, the aircraft climbs at 275 KCAS until it reaches a Mach number of 0.68. It continues at this Mach until it reaches its cruise altitude at flight level 330. The actual climb profile has the aircraft initially at 275 KCAS until it reaches Mach 0.6. The aircraft climbs at Mach 0.6 until it reaches approximately 23,000. At this altitude the aircraft accelerates to Mach 0.68. The new Mach is maintained until the aircraft reached its cruise altitude. The actual time to climb was approximately 14.1 min. The calculated time to climb was 13.8 min, a difference of 21 sec. The simulation assumed a climb rate of 1500 fpm. The actual climb rate started at 2000 fpm and decreased to 1000 fpm as the aircraft reached the end of its climb. The differences in the times to climb can be attributed to the differences in climb rate and the different speed-altitude profiles.

## The Intercept Problem

One problem of interest in departure scheduling is to have an aircraft merge into or intercept a "slot". A slot may be defined to be a gap between two aircraft such that the merging aircraft may fly in-trail to the lead aircraft without violating separation constraints. A slot may also be a position behind a single aircraft such that the mandatory separation requirement is not violated. In either case, the current headings and speeds of all aircraft

are known. The former situation may occur in the en-route environment where the sector controller is trying to merge traffic coming across different fixes. The latter is likely to occur in the terminal area where a departure controller is trying to provide miles-in-trail separation across a departure fix when aircraft are using radar departure procedures. In either situation, we want to be able to find the following: (a) the physical location of the slot at the instant the intercept occurs, (b) the time to intercept the slot, and (c) the horizontal trajectory that satisfies (a) and (b).

The intercept problem is broken down into two different cases. The first case is when the aircraft is traveling at the speed of the traffic. The second case is where the aircraft is traveling at lower speed than the en-route aircraft. The analysis that follows assumes that the intercept problem is independent of the trajectory problem discussed in the previous section. Also, only the horizontal intercept problem is discussed. The “3-D” intercept problem (i.e. the aircraft is merging while climbing to a cruise altitude) is left as the subject of a future paper.

For brevity, the solution for a particular geometry is presented. Using symmetry, the results derived here can be extended to other geometries. The coordinate system is chosen such that the merging aircraft is initially at the origin, with a known heading,  $-180^\circ < \psi_0 \leq 180^\circ$ , and airspeed,  $V_0$ . The route is assumed to be parallel to the  $x$ -axis. Traffic along this route is moving from left to right and has a speed  $V_T$ . The distance from the aircraft to the stream is known and denoted as  $y_f$ , which is assumed positive. Finally, the slot is assumed to have a known initial position on the route,  $x_{i0}$ . The intercept point,  $x_f$ , is where the aircraft merges into the stream. Define the distance traveled by the slot as  $S_s$ , and the distance traveled by the aircraft as  $S_a$ . Likewise, the time for the slot to travel from  $x_{i0}$  to  $x_f$  is  $T_s$  and the time for the aircraft to travel from the origin to the intercept point is  $T_a$ . The problem may be stated as follows: Given the initial conditions ( $\psi_0$ ,  $x_{i0}$ ,  $y_f$ ,  $V_0$  and  $V_T$ ), find  $x_f$ ,  $T_a$ , and the horizontal trajectory to deliver the merging aircraft.

The distance between the origin and the intercept point,  $(x_f, y_f)$ , is assumed to be large enough such that the optimal horizontal trajectory is a “turn-straight-turn” trajectory. As was shown by Erzberger and Lee,<sup>4</sup> a constant speed, turn-straight-turn trajectory is a “time-optimal” trajectory for aircraft guidance to a point with a specified final heading. Although there are four solutions to such a turn-straight-turn trajectory problem, only two are considered here. The merging aircraft will be required to make a “direct turn” to the route. A “direct turn” is defined to be a turn where the merging aircraft does not “cross over” the route before turning back to complete the merge. This is done in order to prevent conflicts with the en-route traffic. For the geometry presented in this analysis, the final turn will be a clockwise turn, as a counterclockwise final turn will require the aircraft to cross the route and then turn back to intercept the route.

### Case 1: $V_0 = V_T$

The first case presented is the case where the aircraft is traveling at the same speed as the en-route stream. It is the more interesting of the two to analyze since we are able to determine functions which can be solved for the intercept point. The fact that the aircraft are traveling at the same speed means that the aircraft will travel the same distance in the same time. The analysis directly follows from McLean,<sup>5</sup> where the turn-straight-turn trajectories

for minimum time flight to a point with a known heading are solved geometrically.

### *Counterclockwise-Straight-Clockwise*

We will begin our analysis by studying the counterclockwise initial turn, with a goal of finding a function that can be solved, either numerically or analytically for the intercept point. The approach will be to find the total distance traveled by the aircraft, then to equate this to the distance traveled by the slot. The geometry for this problem is shown in Fig. 4.

To begin, some basic relationships are given that describe any general turn-straight-turn trajectory. We start with the relationships for the center of curvature of a turn arc, which is assumed to follow a circular path. McLean <sup>5</sup> defines a parameter  $S_i$  which describes the direction of a turn, where  $S_i = +1$  if the turn is clockwise and  $S_i = -1$  if the turn is counterclockwise. Also,  $R_1$  and  $R_2$  are the turn radii for the first and second turns respectively. The centers of the turns can be given in terms of a position, heading, and turn radius (or air/ground speed) as

$$a_i = x_i + R_i S_i \cos \psi_i \quad (8)$$

$$b_i = y_i - R_i S_i \sin \psi_i \quad (9)$$

where  $x_i$ ,  $y_i$ , and  $\psi_i$  are arbitrary points on the turn circle. Two more parameters to be used are the distance between the centers of the circles,  $Q$ , and the length of the straight line path,  $D$  (see Fig. 4). The distance between the centers of the circles is given as

$$Q = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} \quad (10)$$

McLean shows the tangent line between the circles to have a length of

$$D = \sqrt{Q^2 - (R_2 S_2 - R_1 S_1)^2} \quad (11)$$

For this particular case, where the first turn is counterclockwise and the second turn is clockwise,  $S_1 = -1$ ,  $S_2 = +1$ , and  $R_1 = R_2 = R$ . Therefore, using Eqns. 8 and 9, the initial turn is centered at

$$a_1 = -R \cos \psi_0 \quad (12)$$

$$b_1 = R \sin \psi_0 \quad (13)$$

The center of the final turn circle is a function of the final point,  $x_f$ . The final heading is known (90 deg) as well as the final y coordinate,  $y_f$ . Applying Eqns. 8 and 9

$$a_2 = x_f \quad (14)$$

$$b_2 = y_f - R \quad (15)$$

The length of the straight line segment is found using Eqn. 11

$$D = \sqrt{Q^2 - 4R^2} \quad (16)$$

The tangent points on the initial turn circle are denoted by  $(x_2, y_2)$ , and on the final turn circle by  $(x_3, y_3)$ . The heading of the tangent path is  $\psi_t$ . The tangent points,  $(x_2, y_2)$  and  $(x_3, y_3)$ , can be found from Eqns. 8 and 9 to be

$$x_2 = a_1 + R \cos \psi_t \quad (17)$$

$$y_2 = b_1 - R \sin \psi_t \quad (18)$$

$$x_3 = a_2 - R \cos \psi_t \quad (19)$$

$$y_3 = b_2 + R \sin \psi_t \quad (20)$$

The components of  $D$  are found by subtracting Eqn. 17 from Eqn. 19 and Eqn. 18 from Eqn. 20.

$$x_3 - x_2 = (x_f - a_1) - 2R \cos \psi_t \quad (21)$$

$$y_3 - y_2 = (b_2 - b_1) + 2R \sin \psi_t \quad (22)$$

From Fig. 4 it is seen that

$$x_3 - x_2 = D \sin \psi_t \quad (23)$$

$$y_3 - y_2 = D \cos \psi_t \quad (24)$$

Equating Eqn. 21 with Eqn. 23 and Eqn. 22 with Eqn. 24 gives

$$D \sin \psi_t = x_f - a_1 - 2R \cos \psi_t \quad (25)$$

$$D \cos \psi_t = \Delta b + 2R \sin \psi_t \quad (26)$$

where  $\Delta b = b_2 - b_1$ .

These two equations in turn are solved for  $\sin \psi_t$  and  $\cos \psi_t$ .

$$\sin \psi_t = \frac{D(x_f - a_1) - 2R\Delta b}{D^2 + 4R^2} \quad (27)$$

$$\cos \psi_t = \frac{2R(x_f - a_1) + D\Delta b}{D^2 + 4R^2} \quad (28)$$

To solve for  $\psi_t$ , divide Eqn. 27 by Eqn. 28

$$\tan \psi_t = \frac{D(x_f - a_1) - 2R\Delta b}{2R(x_f - a_1) + D\Delta b} \quad (29)$$

Note the dependence of Eqn. 29 on  $x_f$ . It is this dependence that will make analytical solution of  $x_f$  impossible, and the solution must be done numerically. Also, a four quadrant inverse tangent should be used to find  $\psi_t$ . This is required to insure that the heading is in the correct quadrant.

Define the angles,  $\Delta\psi_1$  and  $\Delta\psi_2$  as the initial and final turn respectively:

$$\Delta\psi_1 = \psi_t - \psi_0 + 2\pi S_1 k_1 \quad (30)$$

$$\Delta\psi_2 = \psi_f - \psi_t + 2\pi S_2 k_2 \quad (31)$$



where  $\psi_f$  is the heading of the route, and  $k_1$  and  $k_2$  are constants. The constants,  $k_i$ , are dependent upon the direction of the turn, and are evaluated as follows:<sup>5</sup>

$$k_1 = \begin{cases} 0 & S_1(\psi_t - \psi_0) \geq 0, \\ 1 & S_1(\psi_t - \psi_0) < 0 \end{cases} \quad (32)$$

and

$$k_2 = \begin{cases} 0 & S_2(\psi_f - \psi_t) \geq 0, \\ 1 & S_2(\psi_f - \psi_t) < 0 \end{cases} \quad (33)$$

Eqns. 32 and 33 are needed to obtain the correct “ $2\pi$  multiples” of the turn angles  $\Delta\psi_1$  and  $\Delta\psi_2$ . By definition a counterclockwise turn will result in a final heading that is less than the initial heading. The turn angle in this case is negative. Likewise, a clockwise turn results in a positive turn angle.

The total angle through which the aircraft turns,  $\Delta\psi$ , is simply the sum of the absolute values of the two turn angles

$$\Delta\psi = |\Delta\psi_1| + |\Delta\psi_2| \quad (34)$$

The total distance flown by the aircraft is then  $S_a = R\Delta\psi + D$ . The distance that the slot has moved is  $S_s = x_f - x_{t0}$ . Equating  $S_a$  and  $S_s$  yields the following:

$$x_f - x_{t0} = R\Delta\psi + \sqrt{(x_f - a_1)^2 + (\Delta b)^2 - 4R^2} \quad (35)$$

Solving Eqn. 35 and Eqn. 29 simultaneously will yield a solution to the merge problem for the given initial conditions, given that the solution exists. Typically, a solution for a counterclockwise first turn will not exist for all initial headings. There may be instances where the aircraft will be unable to “catch-up” with the slot, and an intercept will not be possible.

### *Clockwise-Straight-Clockwise*

The second possible intercept path is when the initial turn is a clockwise turn. Again, our goal is to find solutions where the aircraft and the slot travel an equal distance for given initial conditions. The geometry is shown in Fig. 4 for this problem. However, the sense of the velocity vector will be opposite that in the figure, and the line segments  $Q$  and  $D$  will be parallel instead of crossing. Eqns. 8–11 will be used; however, since the initial turn is clockwise,  $S_1 = 1$ .

Using Eqns. 8 and 9, and using  $S_1 = 1$ , the center of the first turn is located at the point

$$a_1 = R \cos \psi_0 \quad (36)$$

$$b_1 = -R \sin \psi_0 \quad (37)$$

The center of the final turn is the same as given in Eqns. 14 and 15. The distance between the centers of the turns is given by the expression for  $Q$  in Eqn. 10. The length of the straight line segment is found from Eqn. 11, with  $S_1 = S_2 = 1$ , which gives

$$D = \sqrt{(x_f - a_1)^2 + (\Delta b)^2} \quad (38)$$

Note that  $D$  has the same length as  $Q$ , which implies that the tangent path is parallel to the line segment connecting the centers of the turns.

The endpoints of the line segment that is tangent to the circles are:

$$x_2 = a_1 - R \cos \psi_t \quad (39)$$

$$y_2 = b_1 + R \sin \psi_t \quad (40)$$

$$x_3 = a_2 - R \cos \psi_t \quad (41)$$

$$y_3 = b_2 + R \sin \psi_t \quad (42)$$

The components of  $D$  are

$$x_3 - x_2 = (x_f - a_1) \quad (43)$$

$$y_3 - y_2 = (b_2 - b_1) \quad (44)$$

From Fig. 4, the components of  $D$  are

$$x_3 - x_2 = D \sin \psi_t \quad (45)$$

$$y_3 - y_2 = D \cos \psi_t \quad (46)$$

So, equating Eqn. 45 with Eqn. 43 and Eqn. 46 with Eqn. 44 will give the sine and cosine of the tangent heading.

$$\sin \psi_t = \frac{x_f - a_1}{D} \quad (47)$$

$$\cos \psi_t = \frac{\Delta b}{D} \quad (48)$$

In turn, dividing these two expressions yields

$$\tan \psi_t = \frac{x_f - a_1}{\Delta b} \quad (49)$$

The heading,  $\psi_t$  is found by taking a four quadrant inverse tangent of Eqn. 49. The turn angle,  $\Delta\psi$ , is defined as before.

The total distance flown by the intercepting aircraft is

$$\begin{aligned} S_a &= R \Delta\psi + D \\ S_a &= R \Delta\psi + \sqrt{(x_f - a_1)^2 + (\Delta b)^2} \end{aligned} \quad (50)$$

Equate Eqn. 50 to the distance traveled by the slot.

$$x_f - x_{t0} = R \Delta\psi + \sqrt{(x_f - a_1)^2 + (\Delta b)^2} \quad (51)$$

Collecting terms and solving for  $x_f$  yields the equation

$$x_f = \frac{a_1 + x_{t0} + R \Delta\psi}{2} - \frac{(\Delta b)^2}{2(x_{t0} + R \Delta\psi - a_1)} \quad (52)$$

Either Eqn. 51 or 52 will yield the intercept point for a clockwise initial turn. As with the counterclockwise initial turn solution, the solution for an initial clockwise turn typically will not exist for all values of  $\psi_0$ . Consider Fig. 5, which shows solutions for counterclockwise and clockwise initial turns. The aircraft and slot have a calibrated airspeed of 280 kts and the slot's initial position is -30 n.mi. Furthermore, the distance from the aircraft to the jetway,  $y_f$ , is 50 n.mi. From an initial heading of 0 to 27 degrees, a solution does not exist for a counterclockwise initial turn. On the other hand the clockwise initial turn solution exists across this range of initial headings. The heading at 27 degrees where the initial turn "switches" from a clockwise turn to a counterclockwise turn is significant. At this particular heading, the initial turn is degenerate (i.e. to complete the intercept no initial turn is required.) The aircraft in this case simply continues along its initial heading until it begins its turn to intercept the route. A discussion on the no turn heading follows in the next section.

The other boundaries that define whether the initial turn is clockwise or counterclockwise are not as easily determined. Fig. 5 defines one scenario. Note that solutions for each initial turn exist over a common range. The point where the initial turn switches direction in this case will be the heading where the distance traveled along both trajectories is identical. The other case is one where the range of solutions do not overlap. In each case, however, it is not possible to determine *a priori* these particular headings.

### *No First Turn Solution*

The heading where the aircraft does not require an initial turn for the intercept is important because it defines a point on the switching curve boundary. For a given initial slot position, the heading for the no first turn solution is easily obtained. The geometry given in Fig. 6 is used in the following analysis.

Assume that the aircraft is initially at the origin with a heading  $\psi_{nt}$ . Define  $L$  to be the distance from the origin to the point where the aircraft begins its turn to intercept the route. The angle  $\Delta\psi$  is simply  $90^\circ - \psi_{nt}$ . The distance that the aircraft travels will be

$$S_a = L + R \Delta\psi \quad (53)$$

In order to find the length  $L$ , the point,  $(x_3, y_3)$ , where  $L$  is tangent to the final turn circle is needed. Using Eqn. 9 with  $S = 1$  and Eqn. 15 for the center of the turn circle gives

$$y_3 = y_f + R (\sin \psi_{nt} - 1) \quad (54)$$

Equating the expression for  $y_3$  to the vertical distance  $L \cos \psi_{nt}$  and solving for the distance  $L$  yields

$$L = \frac{y_f + R (\sin \psi_{nt} - 1)}{\cos \psi_{nt}} \quad (55)$$

The slot has to travel an identical distance as the aircraft for an intercept to occur (i.e.  $S_s = x_f - x_{t0}$ ). Therefore, the intercept point needs to be determined. This is done solving Eqn. 8 for  $x_f$  where  $a_i = x_f$ ,  $S_i = 1$  and  $x_i = x_3$

$$x_f = x_3 + R \sin \psi_{nt} \quad (56)$$

However, from Fig. 6, it is observed that  $L \cos \psi_t = x_3$ . Substituting this into Eqn. 56 gives us the intercept point as a function of the turn radius, the initial heading, and the length  $L$

$$x_f = L \sin \psi_{nt} + R \sin \psi_{nt} \quad (57)$$

Substituting the expression for  $x_f$  into the distance equation

$$S_s = L \sin \psi_{nt} - x_{t0} + R \cos \psi_{nt} \quad (58)$$

where  $L$  is given by Eqn. 55.

Equate Eqns. 53 and 58 and solve for  $x_{t0}$

$$x_{t0} = \frac{y_f + R (\sin \psi_{nt} - 1)}{\cos \psi_{nt}} (\sin \psi_{nt} - 1) + R (\cos \psi_{nt} - \Delta\psi) \quad (59)$$

Obviously, this equation is only valid for  $-90^\circ < \psi_{nt} < 90^\circ$ . Fig. 7 shows the solution of Eqn. 59. Note that this analysis is done for the aircraft and the slot traveling at the identical speeds. To find  $\psi_{nt}$  given  $x_{t0}$ , Eqn. 59 is solved numerically for the no turn heading given the initial position of the slot, and the speed of the aircraft.

## Case 2: $V_T > V_0$

In this section, an iterative algorithm is described that generates horizontal trajectories for the case where the merging aircraft must increase its airspeed to that of the aircraft in the en-route stream. It is assumed that the merging aircraft makes its initial turn at its initial speed. The aircraft then accelerates only along the straight line segment of the trajectory. This assumption is only valid as long as the distance required to accelerate the aircraft is less than the length of  $D$ . The purpose of flying the trajectory this way is two-fold. First, it is computationally simpler. Second, keeping the aircraft's speed at the lower value during the initial turn means that the aircraft has a higher turn rate, in turn minimizing the time to turn to the heading of the tangent path. The optimality of this method has not been shown, but should be the subject of future research.

The trajectory is broken up into four parts: (1) the initial turn, which is made at the aircraft's initial speed; (2) an acceleration along the beginning of the straight line segment to the desired speed; (3) travel along the straight line segment at the new speed; and (4) a turn to the final heading. The algorithm uses the "newpsi" subroutine found in McLean<sup>5</sup> to get the total path length for a given intercept point, initial and final heading, and initial and final speed. The time to traverse the trajectory is then found by calculating the time to traverse each segment:

$$\begin{aligned} T_1 &= \frac{V_0 \Delta\psi_1}{R_1} \\ T_2 &= \frac{V_T - V_0}{a} \\ T_3 &= \frac{D}{V_T} - \frac{V_T^2 - V_0^2}{2aV_T} \\ T_4 &= \frac{V_T \Delta\psi_2}{R_2} \\ T_a &= T_1 + T_2 + T_3 + T_4 \end{aligned} \quad (60)$$

Where  $T_1$  is the time to turn through the angle  $\Delta\psi_1$ ,  $T_2$  is the time to accelerate to the new airspeed,  $T_3$  is the time to traverse the remainder of the straight line segment, and  $T_4$  is the time to turn through an angle  $\Delta\psi_2$ . Note that if the aircraft cannot reach the new airspeed along the straight line segment, this algorithm is not valid.

Using the final position,  $x_f$ , the time for the slot to reach the intercept point is then

$$T_s = \frac{x_f - x_{t0}}{V_T} \quad (61)$$

Eqn. 60 and Eqn. 61 must be solved iteratively for  $x_f$  such that  $T_a = T_s$ . At the  $i_{th}$  iteration, let

$$T_{error} = T_a - T_s \quad (62)$$

If the absolute value of Eqn. 62 is greater than some tolerance, a new intercept point is calculated

$$x_f(i+1) = x_f(i) + V_T T_{error} \quad (63)$$

and the solutions repeated. Convergence is fairly rapid, when a solution exists. If the iteration reaches an upper limit, or the final point moves "too far" downrange, it is assumed that there is no solution for this problem.

The difficulty with this algorithm lies in the determination of the direction of the initial turn *a priori*. Depending upon the initial conditions, there are either two or three headings that determine the "switching curve." The most important heading, the no turn heading, is found using the method shown in the section above. The only difference will be that the aircraft and slot must fly paths that have equal times. The time is computed as described for the turn-straight-turn case, with  $\Delta\psi_1$  set to zero. The initial heading is then found by searching over a range of slot initial positions. The heading for the "no turn" solution lies between -90 and 90 degrees.

The importance of the no turn heading is dependent upon the observation that any initial aircraft heading will turn towards the no turn heading, as long as the initial heading is between -90 and 90 degrees. However, the switching curve has either one or two additional boundaries, depending upon the initial conditions. For example, if at least one solution (either clockwise or counterclockwise initial turn) exists for all headings, for fixed initial conditions, there is an additional heading, which lies between 90 and 270 degrees, that defines the second half of the switching curve. Physically, this heading represents the point where the time to intercept for a clockwise initial turn and a counterclockwise initial turn are equal. Unfortunately, there is no way to determine this heading without examining all solutions for the given initial conditions.

The second situation is where there would be two headings in addition to the no turn heading that define the switching curve. The additional headings represent a range of headings for which no intercept is possible. Again, this is not something that can be predicted *a priori*, and the best practical solution is to compute both candidate trajectories to see if a solution exists.

Figure 8 shows representative trajectories for the case where there are two headings that define the switching curve. The initial conditions for this particular problem are  $x_{t0} = -50$  n.mi.,  $y_f = 50$  n.mi.,  $V_0 = 280$  KCAS at 30,000 feet, and  $V_T = 330$  KCAS at 30,000 feet.

The aircraft is assumed to be able to accelerate at  $0.1g$ . The heading where there is a no first turn solution is  $-6$  degrees. Likewise, the heading where the times are equal is  $243$  degrees. One can see that as the initial heading approaches the bifurcation heading, the trajectory length increases. As the initial heading passes through the bifurcation point and continues on towards  $360$  degrees, the trajectories become shorter.

## Conclusions

The intercept problem and the departure trajectory problem have been addressed. Solutions have been found for the intercept problem for the case where the aircraft is traveling at the speed of the aircraft in the jetway, as well as the case where the aircraft has to accelerate to reach the velocity of the stream. An algorithm is given regarding the computation of a solution for the latter case. For the former, a set of equations is given that allows us to numerically solve for the coordinate where the intercept will occur. Currently, validation of the code that calculates the departure trajectory is in progress. Results are being compared for a data set taken from Denver International Airport on Jan. 8, 1997.

## Acknowledgement

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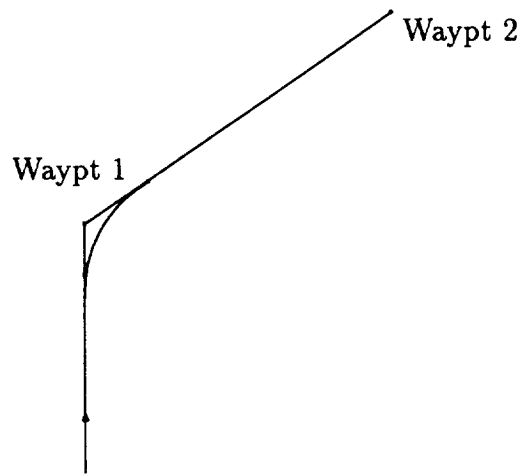


Figure 1: Route Intercept Geometry

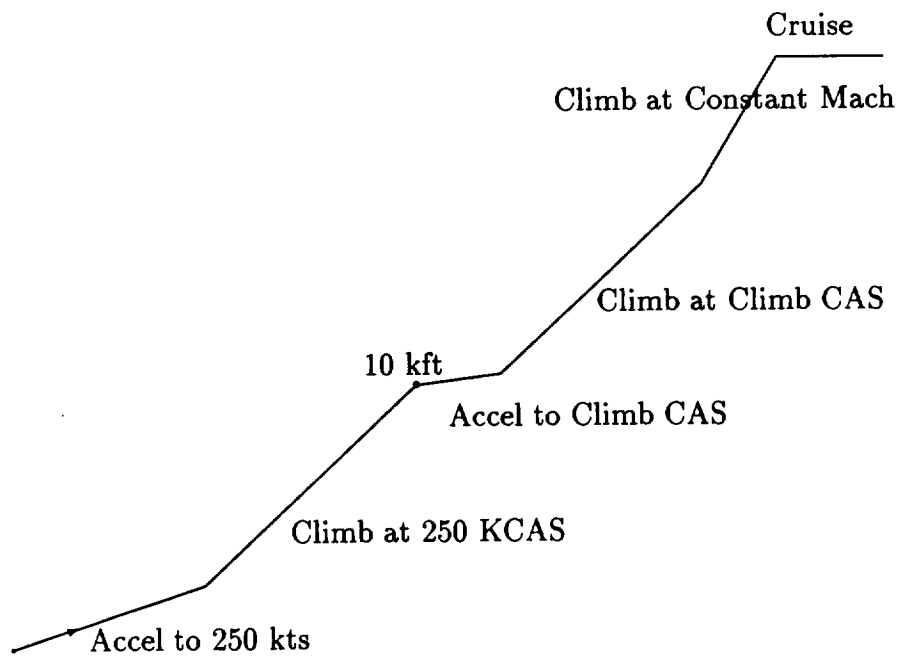


Figure 2: Climb Trajectory

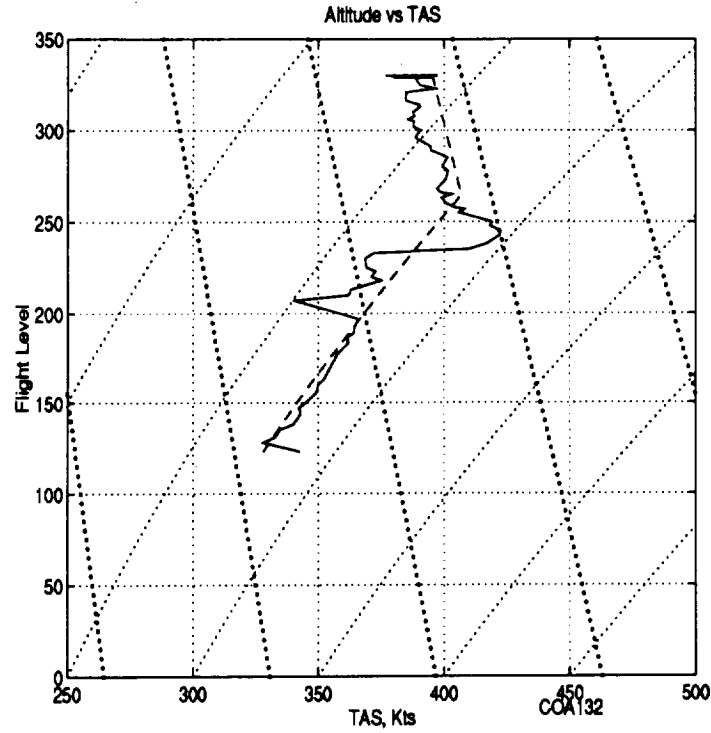


Figure 3: Example Climb-Out Trajectory

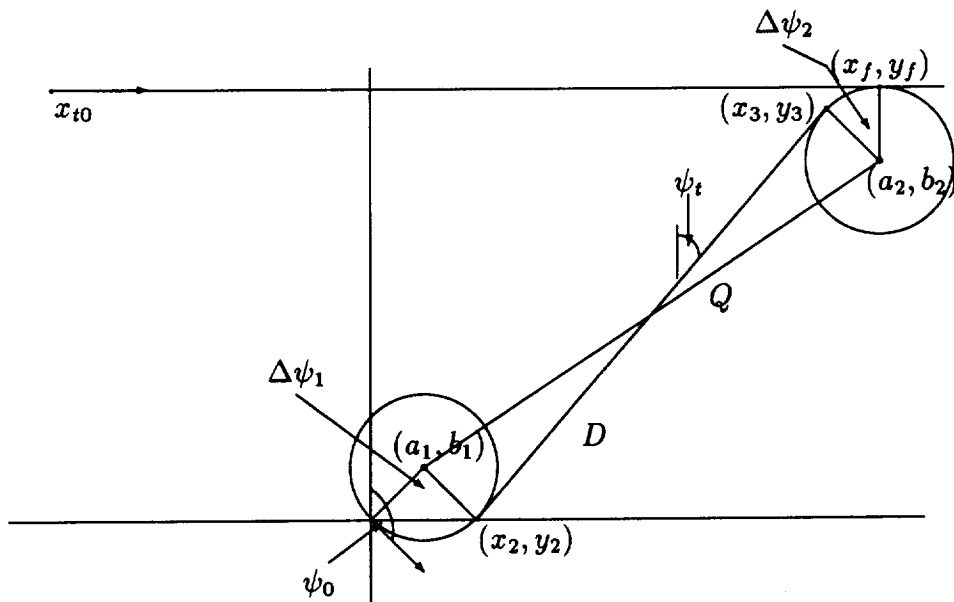


Figure 4: Geometry of a Horizontal Intercept Trajectory



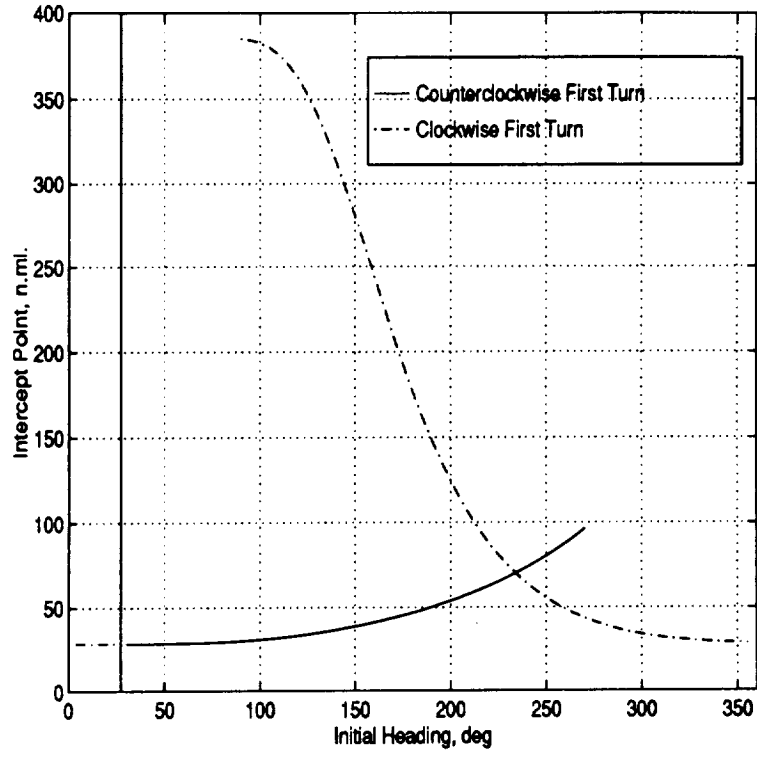


Figure 5: Intercept Points vs Initial Heading

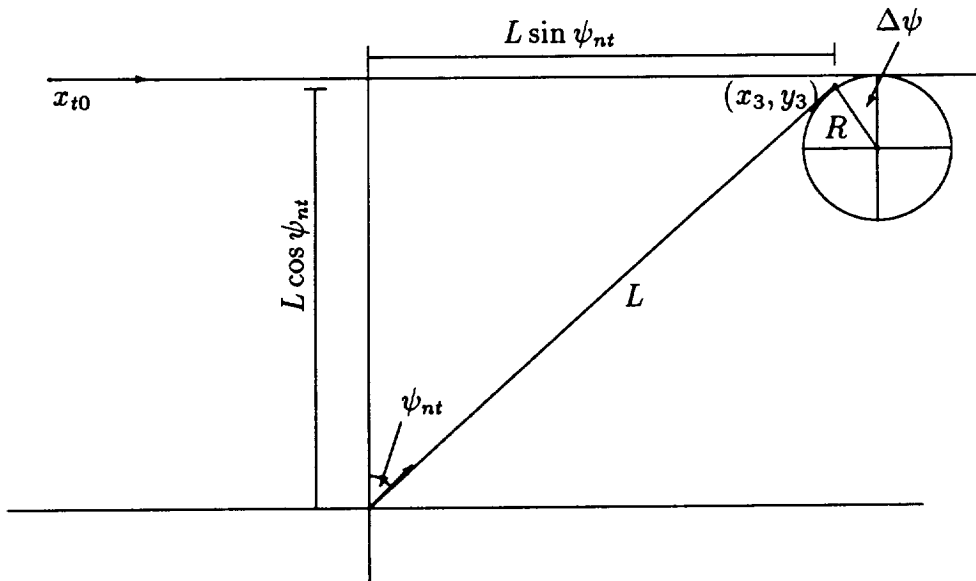


Figure 6: No First Turn Horizontal Trajectory

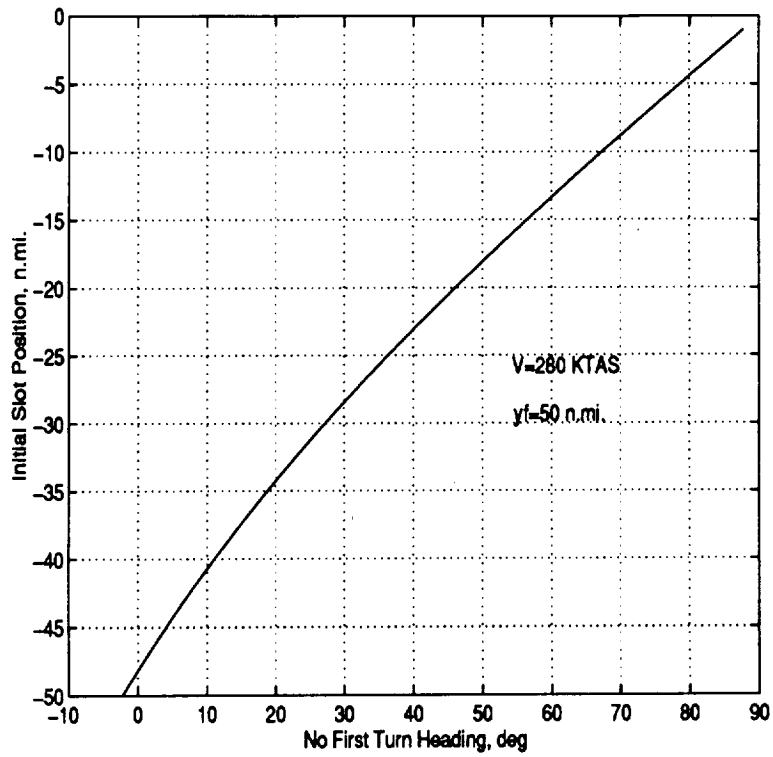


Figure 7: Initial Slot Position vs No First Turn Heading for Same Speed Case

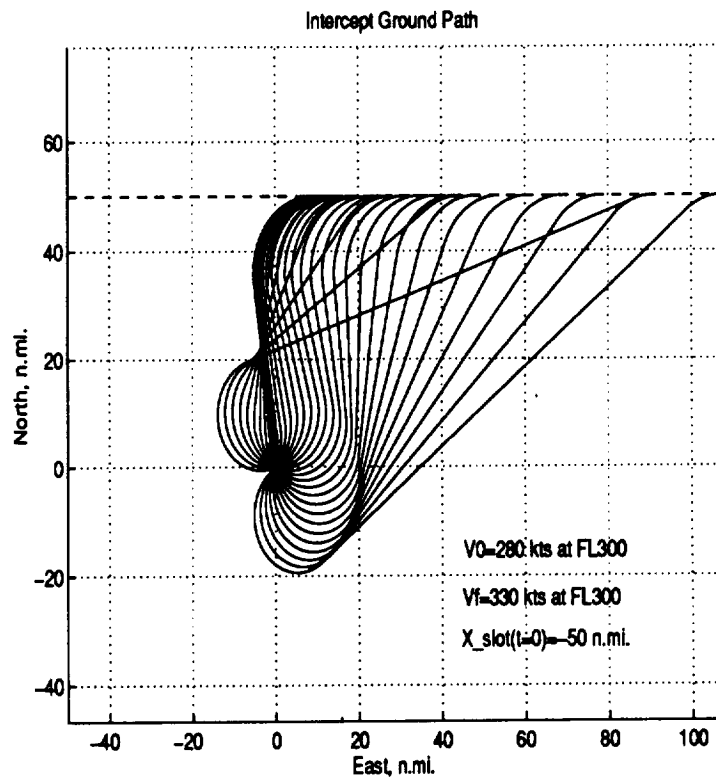


Figure 8: Variable Speed Horizontal Intercept Trajectories